ON FRACTIONAL DIFFERENCE LANGEVIN EQUATIONS INVOLVING NON–LOCAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study a fractional difference Langevin equation within nabla Caputo fractional difference and subject to non–local boundary conditions. The main results are proved by accommodating the newly defined discrete fractional calculus. The existence–uniqueness of solution is proved by Banach contraction principle. Besides, the existence of solutions is proved via Krasnoselskii and the nonlinear alternative Leray–Schauder fixed point theorems. The stability of solutions in sense of Ulam–Hyers is also established. We present a specific example to illustrate the applicability of our theoretical results.

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1. Introduction

Fractional calculus (FC) has a long and distinguished history as a major area of mathematical analysis. A strong argument can be made that FC has become one of the most important areas in mathematics. Due to its nonlocal character, FC has applications in many problems of science and engineering with particular emphasize on modelling physical phenomena. Several monographs have been written on the theory and applications of FC; see the remarkable monographs [1, 2, 3, 4, 5, 6] and the references cited therein.

Fractional differential equations (FDEs) are considered as a vital tool to understand complex systems. The qualitative theory of FDEs have been the object of interested researchers over the last years. We claim that the study of the stability
analysis for ODEs as well as FDEs is a centre-piece of qualitative theory. There have appeared many papers on this subject, with particular emphasize on the applications of Ulam–Hyres stability analysis [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The authors in [18, 19] initiated the theory of discrete fractional calculus (DFC) which deals with sums and differences of arbitrary orders. Latter in [20, 21], Atici and Eloe have worked on some basic operational properties of DFC. This promising branch of FC is still in its infancy and thus researchers have started contributing to its development. Two common approaches to DFC are using the delta fractional difference approach and another using the nabla fractional difference approach. The development of the theory of nabla DFC is attributed to Gray and Zhang [22], Atici and Eloe [23] and Anastassiou [24]. Currently, the research in this field is mainly concentrated on the development of operational properties, existence theory and stability analysis [25, 26, 27, 28, 29, 30, 31, 32].

In 1940, Ulam and Hyers have given a well-known stability criterion for functional equations. Onwards, various generalizations of concept of Ulam-Hyers stability have appeared in literature. Ulam–Hyers stability for first-order difference equations has been investigated in [33, 34]. In [35], the authors used fixed point methods along with Picard operator to obtain Ulam-Hyers stability results of the initial value problem in quaternionic analysis. On other hand, the Ulam stability for partial differential inclusions of fractional order was investigated by applying the Picard operators theory [36]. A new stability concept, which is referred to as Ulam–Hyers–Rassias, was introduced to discuss the stability for nonlinear equations in Banach spaces [37].

The Langevin equation, introduced by Langevin in 1908, is widely used to describe the evolution of physical phenomena in fluctuating environments [38, 39]. This equation has been investigated for existence and stability of solutions by many researchers. Ahmad et al. discussed the existence of solutions for a three-point boundary value problem of a fractional Langevin equation involving two different fractional orders [40]. In [41], however, Li et al., studied a certain class of Langevin equation with two fractional orders involving infinite-point boundary conditions. Wang et al derived existence of solutions by taking the advantage of boundedness, monotonicity and continuity of Mittag–Leffler functions and some famous fixed point methods [42]. Yukunthorn et al. studied the uniqueness of solution for a sequential Caputo-Langevin equation involving Riemann-Liouville integral conditions [43]. The existence of non-negative solutions of certain Langevin equations is discussed in [44]. For some relevant works on fractional Langevin equation, we refer the reader to [45, 46, 47, 48, 49, 50, 51]. To the best of authors observation, however, the consideration of fractional difference Langevin equation is comparably seldom.

In this paper, we will investigate a new variant of Langevin equation within fractional difference operators given as
\begin{equation}
\begin{cases}
\DDD_0^\beta (\DDD_0^\alpha + \lambda(t))x(t) = \mathcal{F}(t,x(t)), \ t \in \mathbb{N}_0^b, \ b \in \mathbb{N}_1, \\
x(0) = 0 = x(b), \ \DDD_0^\alpha x(0) + \DDD_0^\alpha x(b) = \mu \sum_{s=1}^{\eta} x(s),
\end{cases}
\end{equation}

where $0 < \alpha < 1$, $1 < \beta \leq 2$, $\mu \in \mathbb{R}$, $\lambda : \mathbb{N}_0^b \to \mathbb{R}$, $0 < \eta < 1$, $\DDD_0^\nu$ is the nabla fractional difference of order $\nu \in \{\alpha, \beta\}$ and $\mathcal{F} : \mathbb{N}_0^b \times \mathbb{R} \to \mathbb{R}$. It is to be noted that problem (1.1) involves two fractional order defined on two different intervals and associated with boundary conditions of nonlocal character which also incorporates fractional difference operators. The main results are proved by accommodating the newly defined discrete fractional calculus. The existence, uniqueness and stability of solutions will be studied for problem (1.1). This work is motivated by the papers cited above and particularly the work of Salem et al. [52].

Current paper is organized as follows: In Section 2, we assemble some basic definitions, lemmas and theorems which are essential in the remaining part of the paper. Section 3 contains the main results in which we study the existence and uniqueness of solutions for problem (1.1). To prove this, we use Banach contraction principle, Krasnoselskiis and nonlinear alternative Leray Schauder fixed point theorems. In Section 4, we discuss the Ulam–Hyers stability of the above mentioned problem. Example with specific parameters is provided in Section 5 to illustrate the validity of the theoretical results.

2. Preliminaries

In this section, we recall some essential preliminaries that operate as prerequisites for our main results. Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+$ the set of all positive real numbers, $\mathbb{N}_a = \{a, a+1, \ldots\}$ and $\mathbb{N}_0^b = \{0, 1, 2, \ldots, b\}$.

**Definition 2.1.** [53] The (generalized) rising function is defined as

$$t^r = \frac{\Gamma(t + r)}{\Gamma(t)}, \quad t \notin \{0, -1, -2, \ldots\},$$

for those values of $t$ and $r$ so that the right-hand side of the above equation is sensible.

**Definition 2.2.** [53] For $x : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$, the $\nu$-th order nabla fractional sum of $x$ is given by

$$\nabla_a^{-\nu} x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\nu-1} x(s), \quad t \in \mathbb{N}_a,$$

where $\rho(s) = s - 1$ and $\nabla_a^{-0}$ is defined as the identity operator.
Definition 2.5. [53] For \( x : \mathbb{N}_a \to \mathbb{R}, \nu > 0, \nu \notin \mathbb{N}_1, \) \( n \in \mathbb{N}_1 \) such that \( n - 1 < \nu < n, \) then the \( \nu \)-th order nabla fractional difference of \( x \) can be written as

\[
\nabla_\nu^a x(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t - \rho(s))^{-\nu-1} x(s), \quad t \in \mathbb{N}_{a+1},
\]

where \( \nabla_0^0 \) is the identity operator.

Definition 2.6. [53] For \( x : \mathbb{N}_{a-n+1} \to \mathbb{R} \) and \( \nu > 0, \) the \( \nu \)-th order Caputo nabla fractional difference of \( x \) is given by

\[
c_nabla_\nu^a x(t) = \nabla_a^{-(n-\nu)} \left[ \nabla_\nu^n x(t) \right], \quad t \in \mathbb{N}_{a+1}, \ n = \lceil \nu \rceil.
\]

Here \( c_nabla_0^0 \) is the identity operator.

Theorem 2.7. [53] Let \( \nu \in \mathbb{R}^+ \) and \( \mu \in \mathbb{R} \) such that \( \mu, \nu + \mu, \mu - \nu \) are nonnegative integers. Then we have that

(i) \( \nabla_\nu^\mu(t - a)^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} (t - a)^{\mu+\nu} \).

(ii) \( \nabla_\nu^\nu(t - a)^{\nu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} (t - a)^{\mu+\nu} \).

Lemma 2.8. [52] (Krasnoselskiis Fixed Point Theorem) Let \( \mathcal{N} \) be a closed convex and nonempty subset of a Banach space \( \mathcal{S} \). Let \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) be two operators such that:

(i) \( \mathcal{T}_1 x + \mathcal{T}_2 y \in \mathcal{N}, \ x, y \in \mathcal{N}, \)

(ii) \( \mathcal{T}_1 \) is compact and continuous on \( \mathcal{N}, \)

(iii) \( \mathcal{T}_2 \) is a contraction mapping on \( \mathcal{N}. \)

Then, there exits \( z \in \mathcal{N} \) such that \( z = \mathcal{T}_1 z + \mathcal{T}_2 z. \)

Lemma 2.9. [52] (Nonlinear Alternative Leray Schauder Theorem) Let \( \mathcal{S} \) be a Banach space, \( \mathcal{C} \) be a closed and convex subset of \( \mathcal{S}, \) \( \mathcal{U} \) be an open subset of \( \mathcal{C} \) and \( 0 \in \mathcal{U}. \) Suppose that the operator \( \mathcal{T} : \overline{\mathcal{U}} \to \mathcal{C} \) is a continuous and compact map. Then, either

(i) \( \mathcal{T} \) has a fixed point \( x^* \in \overline{\mathcal{U}}, \)

(ii) there exists \( x \in \partial \mathcal{U} \) and \( \delta > 1 \) such that \( \delta x = \mathcal{T}(x). \)

For \( \epsilon > 0, \) consider problem (1.1) with the following inequalities:

\begin{align*}
(2.1) & \quad \left| \nabla_0^\alpha \left( c_nabla_0^\beta + \lambda(t) \right) y(t) - \mathcal{F}(t, y(t)) \right| \leq \epsilon, \quad t \in \mathbb{N}_0^b. \\
(2.2) & \quad \left| c_nabla_0^\beta \left( c_nabla_0^\alpha + \lambda(t) \right) y(t) - \mathcal{F}(t, y(t)) \right| \leq \psi(t), \quad t \in \mathbb{N}_0^b. \\
(2.3) & \quad \left| c_nabla_0^\beta \left( c_nabla_0^\alpha + \lambda(t) \right) y(t) - \mathcal{F}(t, y(t)) \right| \leq \epsilon \psi(t), \quad t \in \mathbb{N}_0^b.
\end{align*}
Here $\psi : \mathbb{N}_0^b \to \mathbb{R}^+$ is a continuous function. We state herein all relevant stability definitions.

**Definition 2.9.** Problem (1.1) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each solution $y$ of the inequality (2.1) there exists a solution $x$ of problem (1.1) with

$$|y(t) - x(t)| \leq c\epsilon, \quad t \in \mathbb{N}_0^b.$$  

**Definition 2.10.** Problem (1.1) is generalized Ulam-Hyers stable if there exists $\zeta \in (\mathbb{N}_0^b, \mathbb{R}^+), \zeta(0) = 0$ such that for each solution $y$ of the inequality (2.1) there exists a solution $x$ of problem (1.1) with

$$|y(t) - x(t)| \leq \zeta(\epsilon), \quad t \in \mathbb{N}_0^b.$$  

**Definition 2.11.** Problem (1.1) is Hyers–Ulam–Rassias stable with respect to $\psi$, if there exists a real number $\xi > 0$ such that for each $\epsilon > 0$ and for each solution $y$ of the inequality (2.3), there exists a solution $x$ of problem (1.1) with

$$|y(t) - x(t)| \leq \xi\psi(t), \quad t \in \mathbb{N}_0^b.$$  

**Definition 2.12.** Problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\psi$, if there exists a real number $\xi > 0$ such that for each solution $y$ of the inequality (2.2), there exists a solution $x$ of problem (1.1) with

$$|y(t) - x(t)| \leq \xi\psi(t), \quad t \in \mathbb{N}_0^b.$$  

### 3. Main Results

Before proceeding to the main results, we start by the following theorems.

**Theorem 3.1.** The solution of problem (1.1) is given by

$$x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha - 1} \lambda(s) x(s) \nabla s$$

$$+ \frac{1}{\Gamma(\beta)} \frac{t^\alpha(b - t)}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha - 1} F(s, x(s)) \nabla s$$

$$- \frac{\mu t^\alpha(b - t)}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^\eta x(s) \nabla s$$

$$+ \frac{1}{\Gamma(\alpha + \beta)} \frac{t^\alpha(-b\alpha - b + 2t + 2\alpha)}{b^\alpha(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s$$

$$- \frac{1}{\Gamma(\alpha)} \frac{t^\alpha(-b\alpha - b + 2t + 2\alpha)}{b^\alpha(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha - 1} \lambda(s) x(s) \nabla s, \quad t \in \mathbb{N}_0^b.$$  

To prove the above statement, we give the following theorem.
Theorem 3.2. Consider a fractional difference Langevin equation with non-local boundary conditions

\begin{equation}
\begin{cases}
\frac{C^\beta}{C^\alpha} \frac{d}{db} x(t) + \lambda(t) x(t) = H(t), \quad t \in \mathbb{N}_0^b, \quad b \in \mathbb{N}_1 \\
x(0) = x(b), \quad \frac{C^\beta}{C^\alpha} x(0) + \frac{d}{db} x(b) = \mu \sum_{s=1}^{\eta} x(s),
\end{cases}
\end{equation}

where $0 < \alpha < 1$, $1 < \beta \leq 2$, $\mu \in \mathbb{R}$ and $0 < \eta < 1$. Then, its solution is given by

$$x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha+\beta-1} H(s) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s$$

$$+ \frac{1}{\Gamma(\beta) \Gamma(\alpha + 1)(\alpha b - 2\alpha - b)} \int_0^b (b - \rho(s))^{\beta-1} H(s) \nabla s$$

$$- \frac{\mu}{\Gamma(\alpha + 1)(\alpha b - 2\alpha - b)} \int_0^\eta x(s) \nabla s$$

$$+ \frac{1}{\Gamma(\alpha + \beta) \beta^b (\alpha b - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} H(s) \nabla s$$

$$- \frac{1}{\Gamma(\alpha) \beta^b (\alpha b - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s, \quad t \in \mathbb{N}_0^b.$$

Proof. Applying $\nabla_0^{-\beta}$ on both sides of Eq. (3.1), we have

$$\frac{C^\beta}{C^\alpha} \frac{d}{db} x(t) = \nabla_0^{-\beta} H(t) + a_0 + a_1 t - \lambda(t) x(t).$$

Now, applying $\nabla_0^{-\alpha}$ on both sides of Eq. (3.2), we get

$$x(t) - x(0) = \nabla_0^{-(\alpha + \beta)} H(t) + a_0 \nabla_0^{-\alpha}(1) + a_1 \nabla_0^{-\alpha} t - \nabla_0^{-\alpha} \lambda(t) x(t).$$

Using condition $x(0) = 0$, we get

$$x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha+\beta-1} H(s) \nabla s + \frac{a_0 t^\beta}{\Gamma(\alpha + 1)} + a_1 \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s.$$

Now, using condition $x(b) = 0$, we obtain

$$a_0 \frac{b^\beta}{\Gamma(\alpha + 1)} + a_1 \frac{b^{\alpha+1}}{\Gamma(\alpha + 2)} = - \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} H(s) \nabla s$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s.$$

Using Eq. (3.2) for third boundary condition, we get

$$2a_0 + a_1 b = - \frac{1}{\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta-1} H(s) \nabla s + \mu \int_0^\eta x(s) \nabla s.$$
Solving Eq. (3.4) and Eq.(3.5) for \(a_0\) and \(a_1\), we obtain

\[
a_1 = \frac{\alpha + 1}{\alpha b - 2\alpha - b} \left( -\frac{1}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\beta-1} \mathcal{H}(s) \nabla s + \mu \int_0^\eta x(s) \nabla s \right) + \frac{2\Gamma(\alpha + 2)}{b^\alpha (\alpha b - 2\alpha - b)} \left( \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \mathcal{H}(s) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s \right),
\]

and

\[
a_0 = \frac{-b(\alpha + 1)}{2(\alpha b - 2\alpha - b)} \left( -\frac{1}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\beta-1} \mathcal{H}(s) \nabla s + \mu \int_0^\eta x(s) \nabla s \right) - \frac{b \Gamma(\alpha + 2)}{b^\alpha (\alpha b - 2\alpha - b)} \left( \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \mathcal{H}(s) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s \right) - \frac{1}{2\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta-1} \mathcal{H}(s) \nabla s + \frac{\mu}{2} \int_0^\eta x(s) \nabla s.
\]

Putting the values of \(a_0\) and \(a_1\) in Eq. (3.3) and rearranging the terms, we get the desired solution as follows:

\[
x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha+\beta-1} \mathcal{H}(s) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s
+ \frac{1}{\Gamma(\beta)} \frac{t^\alpha (b - t)}{\Gamma(\alpha + 1)(\alpha b - 2\alpha - b)} \int_0^b (b - \rho(s))^{\beta-1} \mathcal{H}(s) \nabla s
\]

\[
- \frac{\mu}{\Gamma(\alpha + 1)(\alpha b - 2\alpha - b)} \int_0^\eta x(s) \nabla s
+ \frac{1}{\Gamma(\alpha + \beta)} \frac{t^\alpha (-b\alpha - b + 2t + 2\alpha)}{b^\alpha (\alpha b - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \mathcal{H}(s) \nabla s
\]

\[
- \frac{1}{\Gamma(\alpha)} \frac{t^\alpha (-b\alpha - b + 2t + 2\alpha)}{b^\alpha (\alpha b - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha-1} \lambda(s) x(s) \nabla s.
\]

The proof is complete. \(\Box\)

The following two lemmas prove helpful in the subsequent parts of the paper.

**Lemma 3.3.** Assume \(0 < \alpha < 1\) and \(b \in \mathbb{N}_1\). Then,

\[
\max_{t \in \mathbb{N}_0^b} \left[ t^\alpha (b - t) \right] = \zeta^\alpha (b - \zeta),
\]

where

\[
\zeta = \left[ \frac{1 + b\alpha}{1 + \alpha} \right].
\]

**Proof.** Denote by

\[
g(t) = t^\alpha (b - t), \quad t \in \mathbb{N}_0^b.
\]
Observe that $g(t) \geq 0$ for all $t \in \mathbb{N}_0$. Consider
\[
\nabla \left[ t^{\alpha} (b-t) \right] = (t-1)^{\alpha-1} + \alpha t^{\alpha-1} (b-t) \\
= -\frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} + \alpha \frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} (b-t) \\
= \frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} [-t(1) + \alpha (b-t)].
\]
Clearly,
\[
\frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} \geq 0, \quad t \in \mathbb{N}_0.
\]
Thus, $g$ is an increasing function of $t$ for $0 \leq t \leq \left\lfloor \frac{b+\alpha}{1+\alpha} \right\rfloor$ and decreasing function of $t$ for $\left\lceil \frac{b+\alpha}{1+\alpha} \right\rceil \leq t \leq b$. Thus,
\[
\max_{t \in \mathbb{N}_0} g(t) = g(\zeta) = \zeta^{\alpha} (b-\zeta).
\]

**Lemma 3.4.** Let $0 < \alpha < 1$ and $b \in \mathbb{N}_1$. Denote by
\[
h(t) = t^{\alpha} (-b\alpha - b + 2t + 2\alpha), \quad t \in \mathbb{N}_0.
\]
Then,
\[
\max_{t \in \mathbb{N}_0} |h(t)| = \max\{|h(\theta)|, h(b)\},
\]
where
\[
\theta = \begin{cases}
\left\lfloor \frac{\alpha + \frac{2}{b}}{2} \right\rfloor, & b \in \mathbb{N}_3, \\
1, & b = 2, \\
0, & b = 1.
\end{cases}
\]

**Proof.** We have $h(0) = 0$ and $h(b) = b^{\alpha} (-b\alpha + b + 2\alpha) > 0$. Consider
\[
\nabla h(t) = (t-1)^{\alpha-1} (2) + \alpha t^{\alpha-1} (-b\alpha - b + 2t + 2\alpha) \\
= 2 \frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} + \alpha \frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} (-b\alpha - b + 2t + 2\alpha) \\
= \frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} [2(t-1) + \alpha (-b\alpha - b + 2t + 2\alpha)].
\]
Clearly,
\[
\frac{\Gamma(t + \alpha - 1)}{\Gamma(t)} \geq 0, \quad t \in \mathbb{N}_0.
\]
Thus, $t^{\alpha} (-b\alpha - b + 2t + 2\alpha)$ is a decreasing function for $0 \leq t \leq \left\lfloor \frac{\alpha + \frac{2}{b}}{2} \right\rfloor$ and increasing function for $\left\lceil \frac{\alpha + \frac{2}{b}}{2} \right\rceil \leq t \leq b$. Then,
\[
\max_{t \in \mathbb{N}_0} |h(t)| = \max\{|h(\theta)|, h(b)\}.
\]
\[\blacksquare\]
Let $S$ be a Banach space of all continuous functions from $\mathbb{N}_0^b \to \mathbb{R}$ with norm defined as
\[
\|x\| = \sup_{t \in \mathbb{N}_0^b} |x(t)|.
\]
Let us have the following assumptions,

(A1) $\mathcal{F} : \mathbb{N}_0^b \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A2) $\mathcal{F}$ satisfies Lipschitz condition with $L$ being Lipschitz constant
\[
|\mathcal{F}(t, x) - \mathcal{F}(t, y)| \leq L|x - y|, \quad \forall \ t \in \mathbb{N}_0^b, x, y \in \mathbb{R}.
\]

(A3) There exists a nonnegative function $\phi : \mathbb{N}_0^b \to \mathbb{R}$ such that
\[
|\mathcal{F}(t, x)| \leq \phi(t), \quad \forall \ (t, x) \in (\mathbb{N}_0^b, \mathbb{R}).
\]

(A4) There exists two nonnegative functions $p, q$ such that
\[
|\mathcal{F}(t, x)| \leq p(t)|x| + q(t), \quad (t, x) \in \mathbb{N}_0^b \times \mathbb{R}.
\]

Let us denote by
\[
\mathcal{M}_1 = \sup_{t \in \mathbb{N}_0^b} \left[ \frac{t^\alpha (b - t)}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \right], \quad \mathcal{M}_2 = \sup_{t \in \mathbb{N}_0^b} \left[ \frac{t^\beta (-ab - b + 2t + 2\alpha)}{b^\beta (ab - 2\alpha - b)} \right],
\]
\[
\lambda^* = \max_{t \in \mathbb{N}_0^b} |\lambda(t)|.
\]

Suppose
\[
(3.6) \quad B = B_1 + LB_2,
\]
\[
(3.7) \quad C = \frac{LM_2b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\lambda^* M_2 b^\beta}{\Gamma(\alpha + 1)} + \mathcal{M}_1 \left( \eta|\mu| + \frac{Lb^\beta}{\Gamma(\beta + 1)} \right),
\]
\[
(3.8) \quad D = B_1 + \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} p(s) \nabla s
\]
\[
+ \frac{M_1}{\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta - 1} p(s) \nabla s,
\]
\[
(3.9) \quad E = \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} q(s) \nabla s + \frac{M_1}{\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta - 1} q(s) \nabla s,
\]
where
\[
(3.10) \quad B_1 = \frac{\lambda^*(1 + M_2)b^\alpha}{\Gamma(\alpha + 1)} + \eta|\mu| M_1,
\]
\[
(3.11) \quad B_2 = \frac{(1 + M_2)b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{M_1 b^\beta}{\Gamma(\beta + 1)}.
\]
We can write solution of problem (1.1) as mentioned in Theorem (3.1) as follows
\[
(3.12) \quad x = T(x) = T_1(x) + T_2(x),
\]
where operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined on Banach space $\mathcal{S}$ as follows:

\begin{equation}
\mathcal{T}_1 x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha - 1} \lambda(s)x(s) \nabla s,
\end{equation}

and

\begin{equation}
\mathcal{T}_2 x(t) = \frac{t^\eta (b-t)}{\Gamma(\beta) \Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\beta - 1} F(s, x(s)) \nabla s - \frac{\mu t^\eta (b-t)}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^\eta x(s) \nabla s + \frac{t^\eta (-\alpha b - b + 2t + 2\alpha)}{\Gamma(\alpha + \beta) b^\eta (ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s - \frac{t^\eta (-\alpha b - b + 2t + 2\alpha)}{\Gamma(\alpha) b^\eta (ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha - 1} \lambda(s)x(s) \nabla s.
\end{equation}

**Theorem 3.5.** Assume $(A_1)$ and $(A_2)$ hold. Then, problem (1.1) has a unique solution if $\mathcal{B} < 1$, where $\mathcal{B}$ is given by (3.6).

**Proof.** Define the closed ball $\mathcal{B}_r = \{ x \in \mathcal{S} : \| x \| \leq r \}$ with the radius $r \geq \frac{MB_2}{1 - \mathcal{B}}$. Let $\mathcal{M} = \sup_{t \in \mathbb{N}_0} |F(t, 0)|$. Then, for $x \in \mathcal{B}_r$, we have

\[ \| F(t, x(t)) \| = \sup_{t \in \mathbb{N}_0} |F(t, x(t)) - F(t, 0) + F(t, 0)| \leq \sup_{t \in \mathbb{N}_0} |F(t, x(t)) - F(t, 0)| + \sup_{t \in \mathbb{N}_0} |F(t, 0)| \leq \mathcal{L} \| x \| + \mathcal{M} \leq \mathcal{L} r + \mathcal{M}. \]

Using the above inequality and Lemmas 3.3 and 3.4, we obtain

\[ \|(T x)t\| = \sup_{t \in \mathbb{N}_0} \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha - 1} \lambda(s)x(s) \nabla s \right| + \frac{1}{\Gamma(\beta) \Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\beta - 1} F(s, x(s)) \nabla s - \frac{\mu}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^\eta x(s) \nabla s + \frac{1}{\Gamma(\alpha + \beta) b^\eta (ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s - \frac{1}{\Gamma(\alpha) b^\eta (ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha - 1} \lambda(s)x(s) \nabla s \]
Proof. Let the two operators \( T_1 \) and \( T_2 \) be defined in (3.13) and (3.14), respectively. Set \( \sup_{t \in \mathbb{N}_0} |\phi(t)| \leq \|\phi\| \). Consider closed ball \( B_r = \{ x \in \mathcal{S} : \|x\| \leq r \} \) be defined for

\[
\frac{Lr + M}{\Gamma(\alpha + \beta)} \sup_{t \in \mathbb{N}_0} \int_0^t (t - \rho(s))^\alpha \beta^{-1} \nabla s + \frac{\lambda^* r}{\Gamma(\alpha)} \sup_{t \in \mathbb{N}_0} \int_0^t (t - \rho(s))^\alpha \beta^{-1} \nabla s
\]

\[
+ \frac{M_1(Lr + M)}{\Gamma(\beta)} \int_0^b (b - \rho(s))^\beta \beta^{-1} \nabla s + \mu M_1 r \int_0^b \nabla s
\]

\[
+ \frac{M_2(Lr + M)}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^\alpha \beta^{-1} \nabla s + \frac{\lambda^* M_2 r}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^\alpha \beta^{-1} \nabla s
\]

\[
\leq \frac{(Lr + M)(1 + M_2)b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\lambda^* r(1 + M_2)b^{\alpha + \beta}}{\Gamma(\alpha + 1)} + M_1 \left( \eta |\mu| + \frac{(Lr + M)b^{\alpha + \beta}}{\Gamma(\beta + 1)} \right)
\]

\[
\leq r,
\]

implying that \( \|T x\| \leq r \). Let \( x, y \in \mathcal{S} \), and consider

\[
\| (T x)(t) - (T y)(t) \|
\]

\[
\leq \frac{1}{\Gamma(\alpha + \beta)} \sup_{t \in \mathbb{N}_0} \int_0^t (t - \rho(s))^\alpha \beta^{-1} |F(s, x(s)) - F(s, y(s))| \nabla s
\]

\[
+ \frac{\lambda^*}{\Gamma(\alpha)} \sup_{t \in \mathbb{N}_0} \int_0^t (t - \rho(s))^\alpha \beta^{-1} |x(s) - y(s)| \nabla s
\]

\[
+ \frac{1}{\Gamma(\beta)} \sup_{t \in \mathbb{N}_0} \frac{(t^{\alpha}(b - t))}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^\beta \beta^{-1} |F(s, x(s)) - F(s, y(s))| \nabla s
\]

\[
+ \mu \sup_{t \in \mathbb{N}_0} \frac{(t^{\alpha}(b - t))}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^\eta |x(s) - y(s)| \nabla s
\]

\[
+ \sup_{t \in \mathbb{N}_0} \frac{(t^{\alpha}(-b\alpha - b + 2t + 2\alpha))}{\Gamma(\alpha + \beta)b^{\beta}(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^\alpha \beta^{-1} |F(s, x(s)) - F(s, y(s))| \nabla s
\]

\[
+ \frac{\lambda^*}{\Gamma(\alpha)} \sup_{t \in \mathbb{N}_0} \frac{(t^{\alpha}(-b\alpha - b + 2t + 2\alpha))}{b^{\beta}(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^\alpha \beta^{-1} |x(s) - y(s)| \nabla s
\]

\[
\leq \|x - y\| \left\{ \frac{\mathcal{L}(1 + M_2)}{\Gamma(\alpha + \beta + 1)} b^{\alpha + \beta} + \frac{\lambda^* (1 + M_2)b^{\alpha + \beta}}{\Gamma(\alpha + 1)} + M_1 \left( \eta |\mu| + \frac{\mathcal{L}b^{\alpha + \beta}}{\Gamma(\beta + 1)} \right) \right\}
\]

\[
\leq \mathcal{B} \|x - y\|,
\]

implying that \( T \) is contraction mapping for \( \mathcal{B} < 1 \). Hence by Banach contraction theorem, problem (1.1) has a unique solution. \( \square \)

**Theorem 3.6.** Assume \((A_1), (A_2)\) and \((A_3)\) hold. Then, problem (1.1) has at least one solution if \( C < 1 \), where \( C \) is given by (3.7).

**Proof.** Let the two operators \( T_1 \) and \( T_2 \) are defined in (3.13) and (3.14), respectively. Set \( \sup_{t \in \mathbb{N}_0} |\phi(t)| \leq \|\phi\| \). Consider closed ball \( B_r = \{ x \in \mathcal{S} : \|x\| \leq r \} \) be defined for
\[ r \geq B_2 \| \phi \| (|1 - B_1|)^{-1}. \] Let \( x, y \in B_r \), and consider

\[
\| T_1 x(t) + T_2 y(t) \| \leq \frac{1}{\Gamma(\alpha + \beta)} \sup_{t \in [0, b]} \int_0^t (t - \rho(s))^{\alpha + \beta - 1} |F(s, x(s))| \nabla s \\
+ \frac{\lambda^*}{\Gamma(\alpha)} \sup_{t \in [0, b]} \int_0^t (t - \rho(s))^{\alpha - 1} |x(s)| \nabla s \\
+ \frac{M_1}{\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta - 1} |F(s, y(s))| \nabla s + |\mu|M_1 \int_0^\eta |y(s)| \nabla s \\
+ \frac{M_2}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} |F(s, y(s))| \nabla s \\
+ \frac{\lambda^* M_2}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\alpha - 1} |y(s)| \nabla s \\
\leq \frac{\| \phi \|}{\Gamma(\alpha + \beta + 1)} b^{\alpha + \beta} + \frac{\lambda^* r}{\Gamma(\alpha + 1)} b^\alpha + \frac{M_1 \| \phi \|}{\Gamma(\beta + 1)} b^\beta \\
+ |\mu| M_1 \eta r + \frac{M_2 \| \phi \|}{\Gamma(\alpha + \beta + 1)} b^{\alpha + \beta} + \frac{\lambda^* M_2 r}{\Gamma(\alpha + 1)} b^\alpha \\
\leq \left( \frac{(1 + M_2)b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{M_1 b^\beta}{\Gamma(\beta + 1)} \right) |\phi| + \left( \frac{\lambda^*(1 + M_2)b^\alpha}{\Gamma(\alpha + 1)} + \eta|\mu|M_1 \right) r \\
\leq B_2 \| \phi \| + rB_1 \leq r.
\]

This shows that \( T_1 x + T_2 y \in B_r \). Now using assumption \((A_2)\), we will show that \( T_2 \) is a contraction mapping if \( C < 1 \).

\[
\| T_2 x(t) - T_2 y(t) \| \\
\leq \frac{M_1}{\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta - 1} |F(s, x(s)) - F(s, y(s))| \nabla s \\
+ M_1 |\mu| \int_0^\eta |x(s) - y(s)| \nabla s \\
+ \frac{M_2}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} |F(s, x(s)) - F(s, y(s))| \nabla s \\
+ \frac{\lambda^* M_2}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\alpha - 1} |x(s) - y(s)| \nabla s \\
\leq \left( \frac{M_1 L b^\beta}{\Gamma(\beta + 1)} + M_1 |\mu| + \frac{M_2 L b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\lambda^* M_2 b^\alpha}{\Gamma(\alpha + 1)} \right) |x - y| \\
= C \| x - y \|.
\]
Hence $\mathcal{T}_2$ is contraction if $C < 1$. Since function $F$ is continuous, so $\mathcal{T}_1$ is continuous. Now for $x \in B_r$, we have
\[
\|\mathcal{T}_1 x(t)\| = \sup_{t \in \mathbb{N}_0} \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s \right|
\]
\[
- \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha - 1} \lambda(s) x(s) \nabla s \right| \leq \frac{\|\phi\| b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\lambda r b^\pi}{\Gamma(\alpha + \beta + 1)}.
\]
Hence $\mathcal{T}_1$ is uniformly bounded. Now let $0 \leq t_1 < t_2 \leq b$, then for $x \in B_r$, we have
\[
\|\mathcal{T}_1 x(t_2) - \mathcal{T}_1 x(t_1)\|
\]
\[
= \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_1} (t_2 - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - \rho(s))^{\alpha - 1} \lambda(s) x(s) \nabla s
\]
\[
- \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_1} (t_1 - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - \rho(s))^{\alpha - 1} \lambda(s) x(s) \nabla s
\]
\[
\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_1} [(t_2 - \rho(s))^{\alpha + \beta - 1} - (t_1 - \rho(s))^{\alpha + \beta - 1}] F(s, x(s)) \nabla s
\]
\[
+ \frac{1}{\Gamma(\alpha + \beta)} \int_{t_1}^{t_2} (t_2 - \rho(s))^{\alpha + \beta - 1} F(s, x(s)) \nabla s
\]
\[
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - \rho(s))^{\alpha - 1} - (t_2 - \rho(s))^{\alpha - 1}] \lambda(s) x(s) \nabla s
\]
\[
- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - \rho(s))^{\alpha - 1} \lambda(s) x(s) \nabla s
\]
\[
\leq \frac{\|\phi\|}{\Gamma(\alpha + \beta + 1)} (t_2^{\alpha + \beta} - t_1^{\alpha + \beta}) + \frac{\lambda r}{\Gamma(\alpha + \beta)} (t_2^\pi - t_1^\pi + 2(t_2 - t_1)^\pi),
\]
which is independent of $x$ and approaches to zero when we let $t_2 \to t_1$. This implies that $\mathcal{T}_1$ is relatively compact on $B_r$. Hence by Arzela-Ascoli Theorem, the operator $\mathcal{T}_1$ is completely continuous on $B_r$. Therefore, according to the Krasnoselskii Theorem, problem (1.1) has at least one solution on $B_r$. The proof is complete. $\square$

**Theorem 3.7.** Assume $(A_1)$ and $(A_4)$ hold. Then, problem (1.1) has at least one solution if $\mathcal{D} < 1$, where $\mathcal{D}$ is given by (3.8).

**Proof.** Consider an open subset of a Banach space $S$ as $\mathcal{G} = \{x \in S : \|x\| < l\}$ with $l = \mathcal{E}(1 - \mathcal{D})^{-1}$ where $\mathcal{E}$ is given by (3.9). We can see easily that operator $\mathcal{T} : \overline{\mathcal{G}} \to S$ given by (3.12) is completely continuous. Now let us suppose $x \in \partial \mathcal{G}$ such that...
\[ \delta x = T(x) \text{ for } \delta > 1. \text{ Then, we have} \]

\[ \delta l = \delta \|x(t)\| = \|T x(t)\| = \sup_{t \in \mathbb{R}_0^b} |T x(t)| \]
\[ \leq \frac{1}{\Gamma(\alpha + \beta)} \sup_{t \in \mathbb{R}_0^b} \int_0^t (t - \rho(s))^{\alpha+\beta-1} |F(s, x(s))| \|s\| \]
\[ + \frac{\lambda^*}{\Gamma(\alpha)} \sup_{t \in \mathbb{R}_0^b} \int_0^t (t - \rho(s))^{\beta-1} |x(s)| \|s\| \]
\[ + \frac{\mathcal{M}_1}{\Gamma(\beta)} \int_0^b (b - \rho(s))^{\beta-1} |F(s, x(s))| \|s\| + |\mu| \mathcal{M}_1 \int_0^b |x(s)| \|s\| \]
\[ + \frac{\mathcal{M}_2}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} |F(s, x(s))| \|s\| + \frac{\lambda^* \mathcal{M}_2}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\beta-1} |x(s)| \|s\| \]
\[ \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} (p(s) l + q(s)) \|s\| \]
\[ + \frac{\lambda^*}{\Gamma(\alpha) b^\pi} \int_0^b (b - \rho(s))^{\beta-1} (p(s) l + q(s)) \|s\| \]
\[ + \frac{\mathcal{M}_1}{\Gamma(\beta) b^\pi} \int_0^b (b - \rho(s))^{\beta-1} (p(s) l + q(s)) \|s\| + |\mu| \mathcal{M}_1 \eta l \]
\[ + \frac{\mathcal{M}_2}{\Gamma(\alpha + \beta) b^\pi} \int_0^b (b - \rho(s))^{\alpha+\beta-1} (p(s) l + q(s)) \|s\| + \frac{\lambda^* \mathcal{M}_2}{\Gamma(\alpha + 1) b^\pi} l \]
\[ \leq \left( \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} p(s) \|s\| \right) \|s\| + \frac{\lambda^*}{\Gamma(\alpha + 1) b^\pi} \]
\[ + \frac{\mathcal{M}_1}{\Gamma(\beta) b^\pi} \int_0^b (b - \rho(s))^{\beta-1} p(s) \|s\| + |\mu| \mathcal{M}_1 \eta + \frac{\mathcal{M}_2}{\Gamma(\alpha + \beta) b^\pi} \int_0^b (b - \rho(s))^{\alpha+\beta-1} p(s) \|s\| \]
\[ + \frac{\lambda^* \mathcal{M}_2}{\Gamma(\alpha + 1) b^\pi} l + \left( \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} q(s) \|s\| \right) \|s\| \]
\[ + \frac{\mathcal{M}_1}{\Gamma(\beta) b^\pi} \int_0^b (b - \rho(s))^{\beta-1} q(s) \|s\| + \frac{\mathcal{M}_2}{\Gamma(\alpha + \beta) b^\pi} \int_0^b (b - \rho(s))^{\alpha+\beta-1} q(s) \|s\| \]
\[ \leq D \delta l + E. \]

From this last expression, we get \( \delta \leq 1 \), which is a contradiction to hypothesis \( \delta > 1 \).

Hence using Leray–Schauder theorem, problem (1.1) has at least one solution. \( \Box \)
4. Ulam–Hyers Stability

In this section, we present Ulam–Hyers stability for problem (1.1). For \( \epsilon > 0 \), consider problem (1.1) with the following inequality

\[
|^{c}\nabla_{0}^{\alpha}(^{c}\nabla_{0}^{\alpha} + \lambda(t))y(t) - F(t, y(t))| \leq \epsilon, \quad t \in \mathbb{N}_{0}^{b}.
\]

**Definition 4.1.** A function \( y : \mathbb{N}_{0}^{b} \times \mathbb{R} \to \mathbb{R} \) is a solution of the inequality (4.1) if and only if there exists a function \( g : \mathbb{N}_{0}^{b} \times \mathbb{R} \to \mathbb{R} \) (which depends on \( y \)) such that

(i) \( |g(t)| \leq \epsilon, \quad t \in \mathbb{N}_{0}^{b} \),
(ii) \( ^{c}\nabla_{0}^{\alpha}(^{c}\nabla_{0}^{\alpha} + \lambda(t))y(t) = F(t, y(t)) + g(t), \quad t \in \mathbb{N}_{0}^{b} \).

**Lemma 4.2.** Let \( y \) be solution of inequality (4.1). Then \( y \) is solution of following integral inequality

\[
\left| y(t) - \left( \frac{1}{\Gamma(\alpha + \beta)} \int_{0}^{t}(t - \rho(s))^{\alpha+\beta-1}F(s, y(s))\nabla s - \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t - \rho(s))^{\alpha-1}\lambda(s)y(s)\nabla s \right) \right| \leq \epsilon c.
\]

**Proof.** As we know

\( ^{c}\nabla_{0}^{\alpha}(^{c}\nabla_{0}^{\alpha} + \lambda(t))y(t) = F(t, y(t)) + g(t) \),

its solution is given by

\[
y(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_{0}^{t}(t - \rho(s))^{\alpha+\beta-1}(F(s, y(s)) + g(s))\nabla s - \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t - \rho(s))^{\alpha-1}\lambda(s)y(s)\nabla s
\]

\[
\quad + \frac{1}{\Gamma(\beta)} \frac{t^{\alpha}(b - t)}{(ab - 2\alpha - b)} \int_{0}^{b}(b - \rho(s))^{\beta-1}F(s, y(s))\nabla s - \mu \frac{t^{\alpha}(b - t)}{\Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_{0}^{\eta}y(s)\nabla s
\]

\[
\quad + \frac{1}{\Gamma(\alpha + \beta)} \frac{t^{\alpha}(-b\alpha - b + 2t + 2\alpha)}{b^{\alpha}(ab - 2\alpha - b)} \int_{0}^{b}(b - \rho(s))^{\alpha+\beta-1}F(s, y(s))\nabla s
\]

\[
\quad - \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha}(-b\alpha - b + 2t + 2\alpha)}{b^{\alpha}(ab - 2\alpha - b)} \int_{0}^{b}(b - \rho(s))^{\alpha-1}\lambda(s)y(s)\nabla s.
\]
Now, we have
\[
\begin{align*}
|y(t) - \left( & \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha+\beta-1} F(s, y(s)) \nabla s \right. \\
& + \frac{1}{\Gamma(\beta)} \Gamma(\alpha + 1)(ab - 2\alpha - b) \int_0^b (b - \rho(s))^{\beta-1} F(s, y(s)) \nabla s \\
& + \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} F(s, y(s)) \nabla s \\
& - \frac{1}{\Gamma(\alpha)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \lambda(s) y(s) \nabla s | \\
& + \frac{1}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \lambda(s) y(s) \nabla s |
\end{align*}
\]
\[
\leq \left( \frac{b^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \frac{M_1 b^\beta}{\Gamma(\beta + 1)} + \frac{M_2 b^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} \right) \epsilon \leq c\epsilon.
\]

\[\square\]

**Theorem 4.3.** Let us suppose assumptions \((A_1)\) and \((A_2)\) holds. Then problem (1.1) is Ulam-Hyers stable provided \(Lc + c_1 < 1\).

**Proof.** Let \(y\) be a solution of inequality (4.1) and \(x\) be a solution of following problem

\[
\begin{align*}
x'(t) + \lambda(t)x(t) &= F(t, x(t)), \quad t \in \mathbb{N}_0, \quad b \in \mathbb{N}_1, \\
x(0) &= y(0), \quad x(b) = y(b), \quad c_0^\alpha \nabla_0^\alpha x(0) + c_0^\alpha \nabla_0^\alpha x(b) = \mu \sum_{s=1}^\eta y(s).
\end{align*}
\]

Its solution is given by
\[
x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha+\beta-1} F(s, x(s)) \nabla s \\
+ \frac{1}{\Gamma(\beta)} \Gamma(\alpha + 1)(ab - 2\alpha - b) \int_0^b (b - \rho(s))^{\beta-1} F(s, x(s)) \nabla s \\
- \frac{1}{\Gamma(\alpha + \beta)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \lambda(s) x(s) \nabla s \\
+ \frac{1}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + 1)(ab - 2\alpha - b)} \int_0^b (b - \rho(s))^{\alpha+\beta-1} \lambda(s) y(s) \nabla s.
\]
Now using Lemma 4.2, we get

\[
|y(t) - x(t)| = \left| y(t) - \left( \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha \beta - 1} F(s, y(s)) \nabla s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha - 1} \lambda(s) y(s) \nabla s 
\right.
\]
\[
+ \frac{1}{\Gamma(\beta)} \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^b (b - \rho(s))^{\beta - 1} F(s, y(s)) \nabla s
\]
\[
- \mu \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^\eta y(s) \nabla s
\]
\[
+ \frac{1}{\Gamma(\alpha + \beta)} \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} F(s, y(s)) \nabla s
\]
\[
- \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^b (b - \rho(s))^{\alpha - 1} \lambda(s) y(s) \nabla s
\]
\[
+ \left( \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - \rho(s))^{\alpha + \beta - 1} (F(s, y(s)) - F(s, x(s))) \nabla s
\]
\[
- \frac{1}{\Gamma(\alpha)} \int_0^t (t - \rho(s))^{\alpha - 1} \lambda(s) (y(s) - x(s)) \nabla s
\]
\[
+ \frac{1}{\Gamma(\beta)} \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^b (b - \rho(s))^{\beta - 1} (F(s, y(s)) - F(s, x(s))) \nabla s
\]
\[
+ \frac{1}{\Gamma(\alpha + \beta)} \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^b (b - \rho(s))^{\alpha + \beta - 1} (F(s, y(s)) - F(s, x(s))) \nabla s
\]
\[
- \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)(\alpha - 2\alpha - b)}{\Gamma(\alpha + 1)} \int_0^b (b - \rho(s))^{\alpha - 1} \lambda(s) (y(s) - x(s)) \nabla s \right|
\]
\[
\leq \epsilon c + \left\{ \mathcal{L} \left( \frac{b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\mathcal{M}_1 b^{\beta}}{\Gamma(\beta + 1)} + \frac{\mathcal{M}_2 b^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} \right) + \left( \frac{\lambda^* b^{\alpha}}{\Gamma(\alpha + 1)} + \frac{\lambda^* \mathcal{M}_2 b^{\beta}}{\Gamma(\alpha + 1)} \right) \right\}
\times \|y - x\|
\]
\[
\leq \epsilon c + (\mathcal{L} c + c_1) \|y - x\|.
\]

Hence we get \( 1 - (\mathcal{L} c + c_1) \) \( \|y - x\| \leq \epsilon c \). This implies

\[
\|y - x\| \leq \frac{\epsilon c}{1 - (\mathcal{L} c + c_1)}.
\]

Hence problem (1.1) is Ulam–Hyers stable.

5. Example

Consider the following problem

\[
\begin{cases}
\begin{aligned}
\left\langle c \nabla_0^\alpha \left( c \nabla_0^\beta + \frac{t^2}{8} \right) \right\rangle x(t) &= \frac{1}{2} \left( 1 + t \sin(tx) \right), & t \in \mathbb{N}_0^1, \\
x(0) = 0 = x(1), & c \nabla_0^\alpha x(0) + c \nabla_0^\alpha x(1) = 2 \sum_{s=1}^{\frac{1}{\mathbb{N}}} x(s).
\end{aligned}
\end{cases}
\]
Clearly assumptions $A_1$ and $A_2$ hold, that is, $F(t, x(t)) = \frac{1}{2}(1 + t \sin(tx))$ is continuous function on $N_0^1$ and satisfies Lipschitz condition as follows:

$$|F(t, x(t)) - F(t, y(t))| = \frac{1}{2}|t \sin(tx) - t \sin(ty)| \leq \frac{1}{2}t^2 \int_y^x |\cos(ts)|\, ds \leq \frac{1}{2}|x - y|.$$  

Moreover, for $\lambda(t) = \frac{t^2}{8}$, $\alpha = \frac{2}{5}$, $\beta = \frac{3}{2}$, $b = 1$, we get $M_1 = 0$, $M_2 = 0$, $\lambda^* = \frac{1}{8}$. Further calculations show that $B = 0.625 < 1$, where $B$ is given by Eq. (3.6). Hence using Theorem 3.5 (Banach Contraction Principle), there exists a unique solutions of problem (5.1).

Now, let us consider $\phi(t) = 1 + t$, such that assumptions $A_1$, $A_2$ and $A_3$ are satisfied with $C = 0 < 1$, where $C$ is given by Eq. (3.7). Then using Theorem 3.6 (Krasnoselskiis fixed point theorem), there exists at least one solution of problem (5.1).

Substituting $M_1 = 0$, $M_2 = 0$, $\lambda^* = \frac{1}{8}$, $\alpha = \frac{2}{5}$, $\beta = \frac{3}{2}$, $b = 1$, we get

$$\mathcal{L}c + c_1 < 1,$$

where

$$\mathcal{L}c+c_1 = \left\{ \frac{1}{2} \left( \frac{b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{M_1 b^\beta}{\Gamma(\beta+1)} + \frac{M_2 b^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) + \left( \frac{\lambda^* b^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^* M_2 b^\beta}{\Gamma(\alpha+1)} \right) \right\}.$$  

Moreover, assumptions $A_1$ and $A_2$ are satisfied. Hence, using Theorem 4.3, problem (5.1) is Ulam–Hyers stable.

### 6. Conclusion

In this paper, we initiated the study of fractional difference Langevin equations. Before proceeding to the main results, we converted problem (1.1) into a sum equation which will facilitate dealing with fixed point theorems. Besides and to overcome some obstacles in the proofs of the main results, we proved two essential lemmas. We have established sufficient conditions for the existence and uniqueness of solutions of problem (1.1) via fixed point theorems. Moreover and under specific assumptions and conditions, Ulam–Hyers stability results are obtained for the solution of the said equation. The theoretical results are demonstrated by an example that illustrates consistency.

The results of this paper evidently provide a promising platform for interested researchers to go further and investigate various types of fractional difference Langevin equations such as coupled fractional difference Langevin equations, $q$–fractional difference Langevin equations, fractional difference Langevin equations associated with different types of boundary conditions, fractional difference Langevin equations involving integer order discrete operators or fractional difference Langevin equations...
associated with periodic or antiperiodic boundary conditions. We leave the investigation of these topics to interested researchers.

**Authors’ contributions**

All authors have equally and significantly contributed to the contents of the paper.

**Conflict of interest**

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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