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NUMERICAL RESULTS FOR NONLINEAR CAPUTO FRACTIONAL IMPULSIVE DIFFERENTIAL EQUATIONS VIA GENERALIZED MONOTONE METHOD

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ABSTRACT. Existence and uniqueness of the solution of Caputo fractional impulsive differential equations has been established recently by generalized monotone method. In this work, we will demonstrate the numerical application of the generalized monotone method for Caputo fractional impulsive differential equations which arises in population models, and blow up problems. The numerical code developed here is applicable in a variety of other physical application problems also. In this work, we will be discussing the numerical applications of Caputo fractional impulsive differential equations.

AMS (MOS) Subject Classification. 34A08, 34A37.

1. Introduction

Study of fractional differential has gained great importance due to its myriad applications in various branches of science and engineering. For example, see [1, 3, 7, 16, 24, 26, 28, 29] and the references there in. Among the different types of fractional derivative involved, the most common type of fractional derivative used are the Caputo fractional derivative and the Riemann-Lioville fractional derivative. See [7, 16, 20, 26, 28] for details on Caputo derivative, Riemann-Liouville derivative and other types of derivative that are discussed in literature. However, the Caputo derivative reduces to an integer derivative when the order of the fractional derivative tends to an integer. In fact, the integer derivative is used as a tool to define the Caputo fractional derivative. In addition, in the study of Caputo dynamic equations the initial and boundary conditions are the same as that of the integer dynamic equations. See [7, 8, 12, 16, 18, 20, 21, 26, 28, 36] for some of the details. However, the fractional dynamic equations are considered as equations with memory, mainly because the fractional derivative is global in nature, where as the integer derivative is local in nature. Thus from modeling point of view, we can use the order of the

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fractional derivative as a parameter to enhance the model. We will illustrate this in our work numerically.

It is known that many mathematical models of real world problems arising from medicine, biology, economics, and financial market exhibit impulsive behavior at regular interval of time. Although, there is a vast literature on this topic, see [13, 19] for a few references, on the analysis and applications of ordinary impulsive differential equations. See [2, 4, 14, 25, 38] for the existence and uniqueness of the solutions of Caputo fractional impulsive differential equations with initial or boundary conditions. Majority of the existence results for Caputo fractional impulsive differential equations has been obtained using some kind of fixed point theorem results. However, the fixed point theorem approach does not guarantee the interval of existence. Existence of solution for the nonlinear dynamic equations by the method of lower and upper solutions or the method of coupled lower and solutions combined with a monotone iterative technique is both theoretical as well as computational. In addition, the interval of existence is guaranteed. These methods are known as monotone method,generalized monotone method, quasilinearization method or generalized quasilinearization method. These method yields linear convergence or quadratic convergence depending on whether monotone method is used or quasilinearization method is used. See [17, 23] for integer order nonlinear ordinary and partial differential equations. See [6, 9, 10, 15, 30, 37, 34, 35] for Caputo and Riemann Liouville fractional dynamic equations. In a recent article [40], we have obtained a closed form solution for the linear Caputo fractional impulsive differential equation when the impulses are in the forcing function in the form of the characteristic function or the unit step function. We have achieved this by using the Laplace transform method since the Caputo derivative is in convolution integral form. See [7, 16, 26, 28, 38, 39, 40] for references where Laplace transform method has been used to solve linear fractional differential equations. In addition, in [40] we have developed a generalized comparison result for nonlinear Caputo fractional impulsive differential equation when the forcing function is the sum of an increasing and decreasing functions. Thus, we have all tools to develop generalized monotone method for the nonlinear Caputo fractional impulsive differential equation with initial condition.

In this work, we have developed the numerical application of the generalized monotone method for Caputo fractional impulsive differential equations. The examples discussed are blow up problems and population models. See [36] for the theoretical approach of blow problems in Caputo fractional dynamic equations with initial conditions without impulses. See [15] for the Caputo fractional logistic equation without impulses. See [11, 22, 27, 32, 33, 35, 41] for some of the numerical methods developed for Caputo fractional initial and boundary value problems. Majority of the numerical results developed for fractional dynamic equations are analogous to the well known numerical results of the integer dynamic equation, which is approximating the derivative. Thus the procedure to approximate the derivative increases the computational complexity of the fractional dynamic equation. However in [22, 32, 35] the integral representation has been used. In out present work also we have used the integral representation for each of the linear iterates (approximate solution of the nonlinear problem). This provides an opportunity to compare the solutions of the first order impulsive differential equation with the solution of the Caputo fractional impulsive differential equation with initial conditions of order q and $0.5 < q < 1$. Our solution tends to the integer order solution, as $q \to 1$. In fact we have also established numerically using monotone method starting from the lower solution that the solution of the Caputo fractional impulsive differential equation blows up on before the solution of the first order impulsive differential equation when the forcing function is u^2, together with a positive impulse. In this work, it is also established numerically a similar thing happens for a simple logistic equation. The examples discussed are basically to enhance the mathematical model from the available data by a proper choice of the vale q. Thus, we can choose the order of the fractional derivative to match with the available data whose forecasting will be more accurate.

2. **Preliminary Results**

In this section, we introduce some known definitions and results, which are needed in the main results.

Definition 2.1. The Caputo (left) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:

$$ {}^cD_t^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s) \mathrm{d}s. \tag{2.1}$$

Throughout this work, we assume that the value of q is such that $0 < q < 1$. In our main result, we further restrict q values such that $0.5 < q < 1$. Consider nonlinear Caputo fractional impulsive differential equation with initial condition of the form:

$$\begin{cases} {}^cD^q u(t) = \lambda u(t) + \sum\limits_{i=1}^{N} c_i \chi(t-t_i) s_i(t-t_i) u(t_i) \\ \qquad + \sum\limits_{i=1}^{N} b_i \chi(t-t_i) r_i(t-t_i) u(t_i) + f(t,u(t)) + g(t,u(t)) \\ u(o) = u_0. \end{cases} \tag{2.2}$$

where $t \in [0,T]$, and $0 < t_1 < t_2 < \cdots < t_N = T$. Also, $\chi(t-t_i)$ is the Heaviside unit step function which is left continuous,

$$\chi(t-t_i) = \begin{cases} 1 & if,\ t > t_i \\ 0 & if,\ t \le t_i. \end{cases} \tag{2.3}$$

Furthermore, we assume that $\lambda \neq 0$, and $c_i\chi(t-t_i)s_i(t-t_i) \geq 0$ and $b_i\chi(t-t_i)r_i(t-t_i) \leq 0$ for each $i = 0, 1, 2...N$. The function $f(t,u)$ is nondecreasing in u and $g(t,u)$ is non-increasing in u. In addition, $s_i(t-t_i)$ and $r_i(t-t_i)$ are continuous on each interval $[t_i, t_{i+1}]$ for $i = 1, \ldots, N-1$. Therefore, they are bounded on each interval.

We need the following definition to recall known theoretical results on generalized monotone method relative to equation (2.2). See [15] for other types of coupled lower and upper solutions.

Definition 2.2. We say v and w are coupled lower and upper solutions of type 1 if they satisfy the inequalities:

$$\begin{aligned} {}^cD^q v(t) &\leq \lambda v(t) + \sum_{i=1}^{N} a_i\chi(t-t_i)s_i(t-t_i)v(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)w(t_i) \\ &\quad + f(t,v) + g(t,w) \\ v(0) &\leq u_0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} {}^cD^q w(t) &\geq \lambda w(t) + \sum_{i=1}^{N} a_i\chi(t-t_i)s_i(t-t_i)w(t_i) + \sum_{i=1}^{N} b_i\chi(t-t_i)r_i(t-t_i)v(t_i) \\ &\quad + f(t,w) + g(t,v) \\ w(0) &\geq u_0. \end{aligned} \tag{2.5}$$

Next we merely state the generalized monotone method relative to equation (2.2) from [42].

Theorem 2.3. *Assume*
(A_1). v_0 and w_0 be coupled lower and upper solutions of type 1 of the equation (2.2), such that $v_0 \leq w_0$ on $[0,T]$;
(A_2). $f(t,u)$ and $g(t,u)$ be nondecreasing and non-increasing respectively on Ω;
Then the sequences of coupled lower and upper solutions $\{v_n\}$ and $\{w_n\}$ are well defined satisfy the following results:
(i). $\{v_n\}$ and $\{w_n\}$ satisfy the inequality,

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0, \qquad \forall t \in [0,T]. \tag{2.6}$$

(ii). If u is any solution of equation (2.2) such that $v_0 \leq u \leq w_0$, then the sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly and monotonically to the coupled minimal and maximal solutions $v(t)$ and $w(t)$ respectively such that $v(t) \leq u \leq w(t)$.
(iii). Furthermore, if $f(t,u)$ and $g(t,u)$ satisfies the one-sided Lipschitz condition of the form

$$f(t,u_1) - f(t,u_2) \leq L_1(u_1 - u_2), \qquad g(t,u_1) - g(t,u_2) \geq L_2(u_1 - u_2), \tag{2.7}$$

where $u_1 \geq u_2$, $L_1 \geq 0$ and $L_2 \geq 0$, $\forall t \in [0,T]$, then we have $v(t) = w(t) = u(t)$ the unique solution of (2.2) on $[0,T]$.

See [42] for proof and other details. Note that the results of 2.3 holds true for $\lambda = 0$. This is precisely what we use in our main result.

3. Main Results

In this section, we develop numerical applications of the generalized monotone method for the blow up type of nonlinear function and for the logistic type of nonlinear function with impulses in the nonhomogeneous terms. Our computation of the linear approximations do not involve the Mittag-Leffer functions since we have chosen $\lambda = 0$ in 2.3 for our examples. Our numerical iterates depends on the value of q. In addition as $q \to 1$ we can demonstrate that the solution tends to the solution of the integer impulsive differential equation. We have used MATLAB code for all our numerical results and graphs.

In our first example, we consider the Caputo fractional impulsive differential equations with initial conditions of the form:

$$\begin{cases} {}^cD^q u(t) = u^2 + \chi(t-1)(t-1)u(1) \\ u(0) = \frac{1}{2} \end{cases} \tag{3.1}$$

where $t \in [0, 1.6]$.

It is easy to see that $v_0(t) = \frac{1}{2}$ is the lower solution for equation (3.1). Then the linear iterates relative to equation (3.1) are given by:

$$ {}^cD^q v_n(t) = v_{n-1}^2 + \chi(t-1)(t-1)v_n(1) \tag{3.2}$$

To compute $v_1(t)$, we initially compute $v_1(t)$ on the interval $t \in [0, 1)$ and use it for the impulsive part. Then $v_1(t)$ is given by:

$$v_1(t) = \frac{1}{2} + \frac{t^q}{4\Gamma(q+1)}. \tag{3.3}$$

Thus we get

$$v_1(1) = \frac{1}{2} + \frac{1}{4\Gamma(q+1)}. \tag{3.4}$$

For $t \in [1, 1.6]$, $v_1(t)$ is the solution of

$$ {}^cD^q v_1(t) = v_0^2 + (t-1)v_1(1). \tag{3.5}$$

By using the Laplace transformation, we can get the solution on $[0, 1.6]$ as,

$$v_1(t) = \frac{1}{2} + \frac{t^q}{4\Gamma(q+1)} + v_1(1)\frac{(t-1)^q}{\Gamma(q+2)}. \tag{3.6}$$

Similarly, we compute $v_2(t)$ and $v_3(t)$ using equation (3.2).

In the following Figure 1, we present the graph of the iterates $v_1(t)$, $v_2(t)$ and $v_3(t)$ for values of $q = 0.7$, 0.8, 0.9, and 1 respectively, relative to equation (3.1).

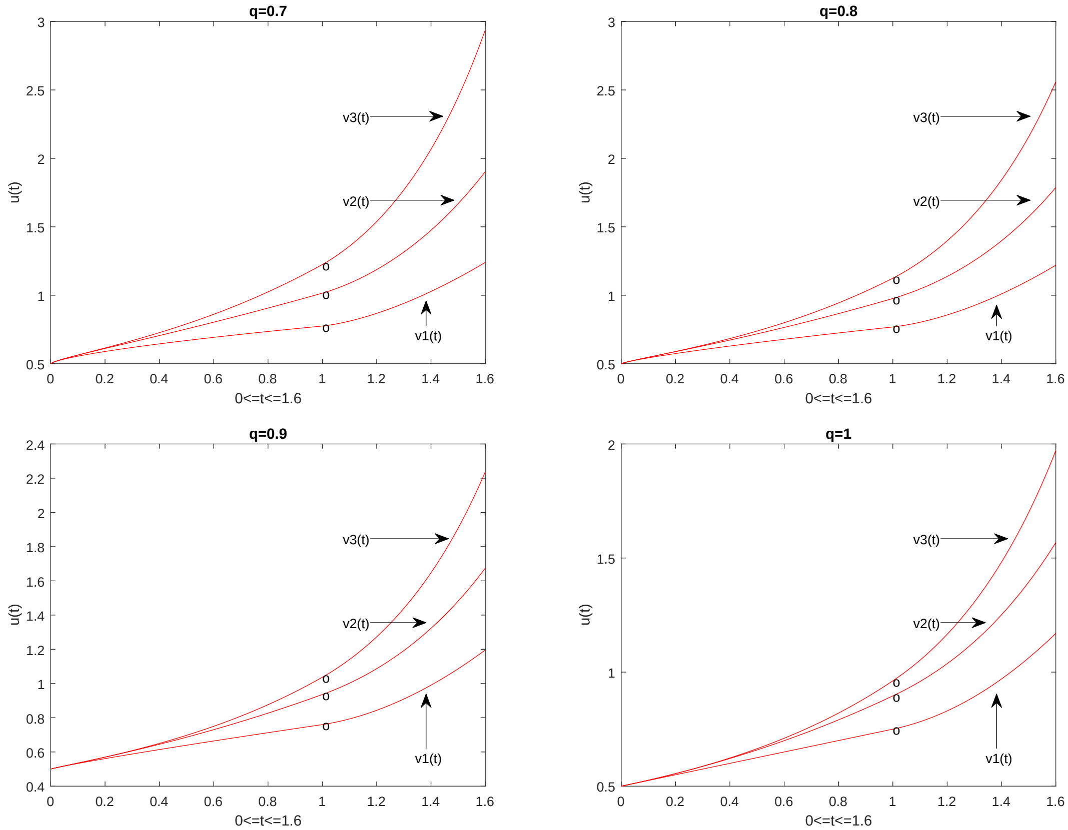

FIGURE 1. Lower solutions $v_1(t)$, $v_2(t)$ and $v_3(t)$ with different values of q.

In Figure 2-4, we have compared the iterates $v_1(t)$, $v_2(t)$ and $v_3(t)$ for values of $q = 0.7$, 0.8, 0.9, and 1. By labeling $v_{i,q}(t)$ as the i^{th} iterate with the fractional order of q. It is easy to observe that, in Figure 2-4,

$$v_{i,0.7}(t) \geq v_{i,0.8}(t) \geq v_{i,0.9}(t) \geq v_{i,1}(t) \qquad i = 1, 2, 3. \tag{3.7}$$

This has also been illustrated in the numerical Table 1 for $t = 0.5$ and $t = 1.2$, respectively.

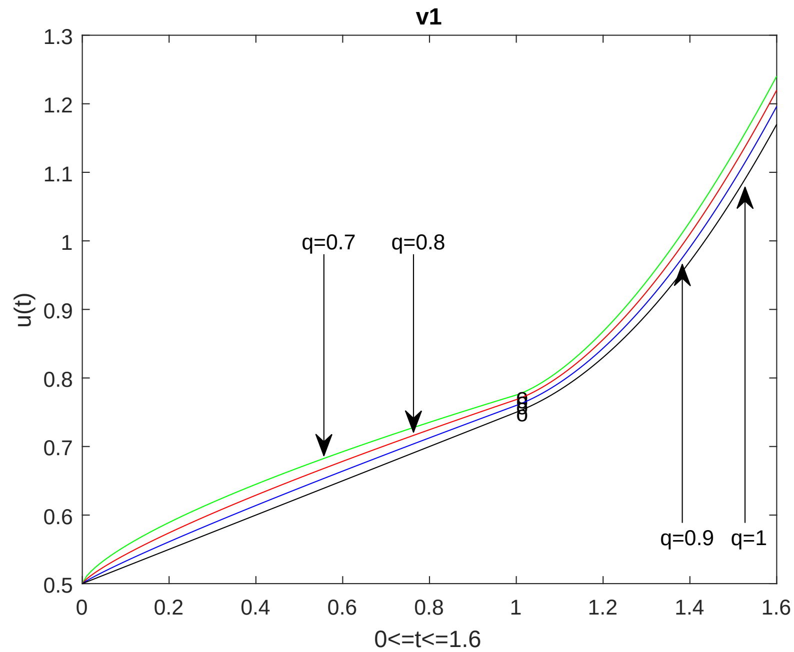

FIGURE 2. Lower solutions $v_1(t)$ with different values of q.

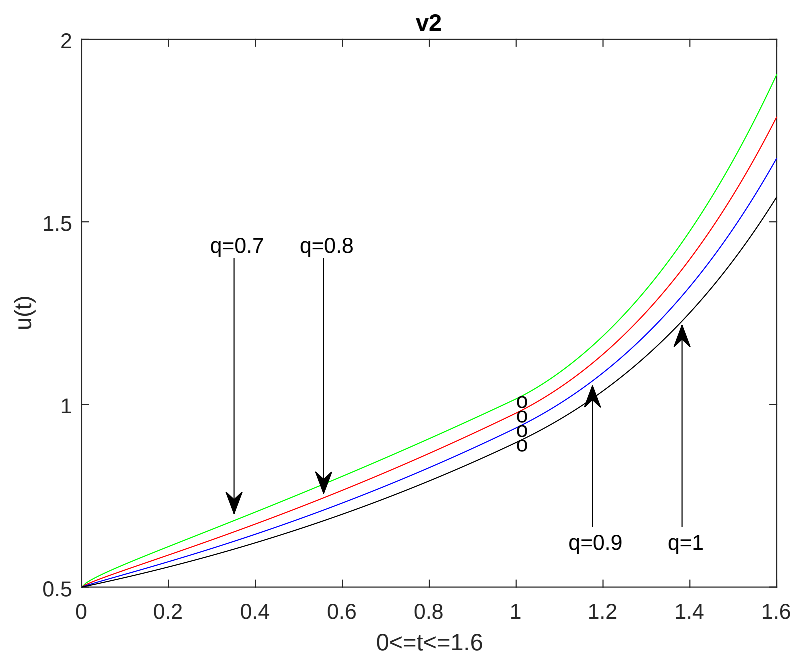

FIGURE 3. Lower solutions $v_2(t)$ with different values of q.

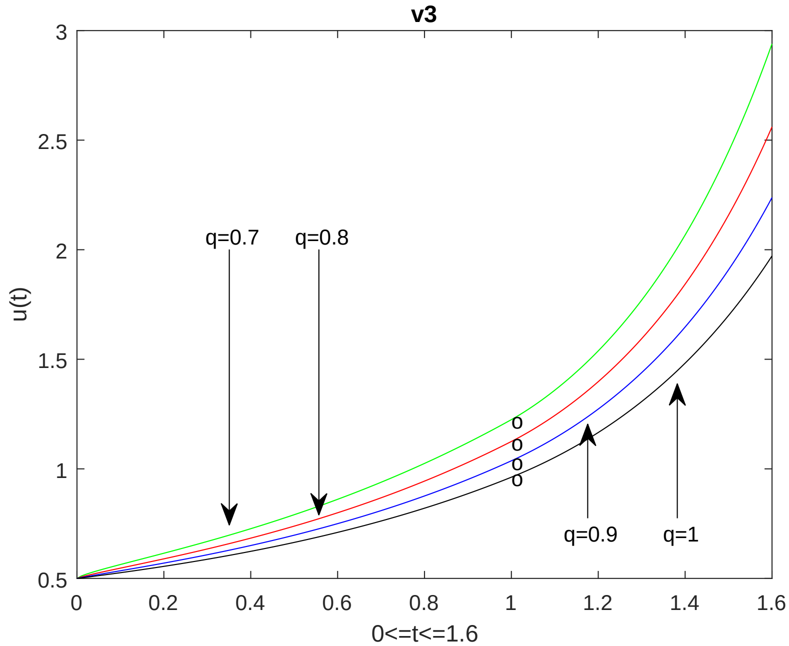

FIGURE 4. Lower solutions $v_3(t)$ with different values of q.

	$q = 0.7$	$q = 0.8$	$q = 0.9$	$q = 1$
$v_1(0.5)$	0.6694	0.6542	0.6393	0.6250
$v_2(0.5)$	0.7538	0.7178	0.6862	0.6589
$v_3(0.5)$	0.7907	0.7386	0.6972	0.6643
$v_1(1.2)$	0.8126	0.8106	0.8063	0.8000
$v_2(1.2)$	1.1875	1.1376	1.0868	1.0371
$v_3(1.2)$	1.5383	1.3980	1.2729	1.1659

TABLE 1. Numerical values of the iterates for values $q = 0.7$, 0.8, 0.9, and 1.

In our next example, we consider the Caputo fractional impulsive differential equations with initial conditions of the form:

$$\begin{cases} {}^cD^q u(t) = u^2 + \chi(t-0.4)(t-0.4)u(0.4) \\ u(0) = 1 \end{cases} \tag{3.8}$$

where $t \in [0, 0.8]$.

It is easy to see that $v_0(t) = 1$ is the lower solution for equation (3.8). Then the linear iterates relative to equation (3.8) are given by:

$$^cD^q v_n(t) = v_{n-1}^2 + \chi(t-0.4)(t-0.4)v_n(0.4) \tag{3.9}$$

To compute $v_1(t)$, we initially compute $v_1(t)$ on the interval $t \in [0, 0.4)$ and use it for the impulsive part. Then $v_1(t)$ is given by:

$$v_1(t) = 1 + \frac{t^q}{\Gamma(q+1)}. \tag{3.10}$$

Thus we get

$$v_1(0.4) = 1 + \frac{0.4^q}{\Gamma(q+1)}. \tag{3.11}$$

For $t \in [0.4, 0.8]$, $v_1(t)$ is the solution of

$${}^cD^q v_1(t) = v_0^2 + (t-1)v_1(0.4). \tag{3.12}$$

By using the Laplace transformation, we can get the solution on $[0, 0.8]$ as:

$$v_1(t) = \frac{1}{2} + \frac{t^q}{\Gamma(q+1)} + v_1(0.4)\frac{(t-1)^q}{\Gamma(q+2)}. \tag{3.13}$$

Similarly, we compute $v_2(t)$ and $v_3(t)$ using equation (3.9).

In the following Figure 5, we present the graph of the iterates $v_1(t)$, $v_2(t)$ and $v_3(t)$ for values of $q = 0.7$, 0.8, 0.9, and 1 relative to equation (3.8)

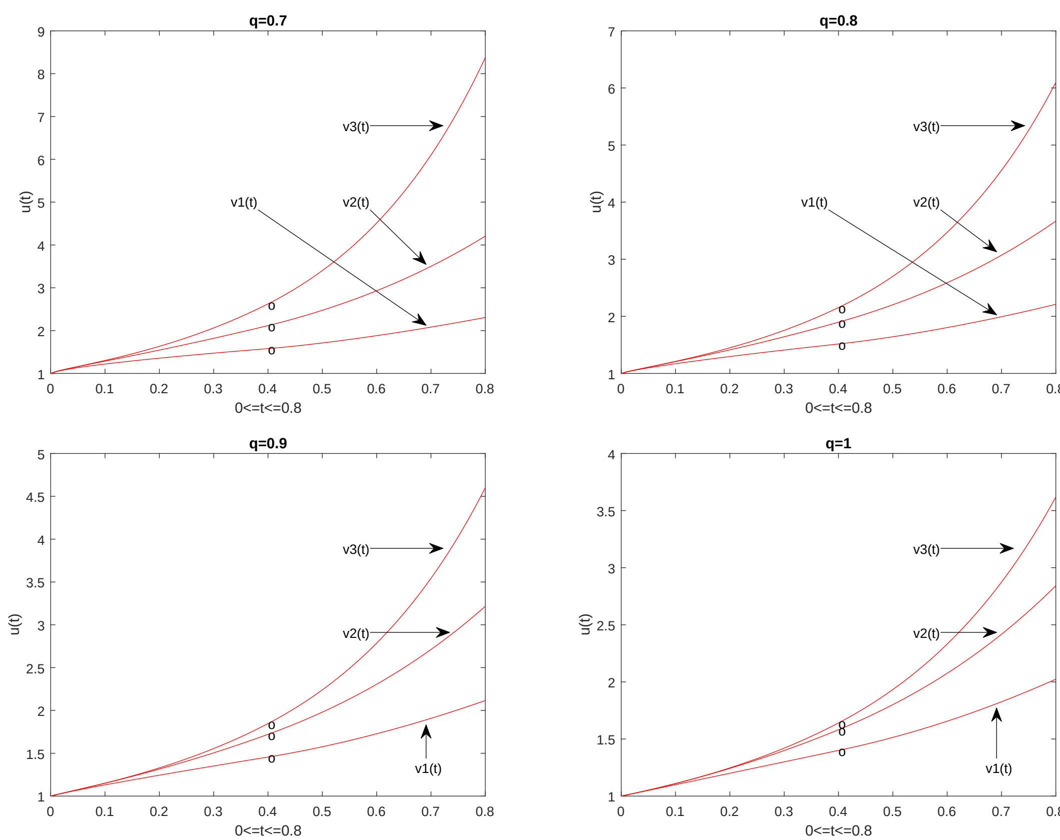

FIGURE 5. Lower solutions $v_1(t)$, $v_2(t)$ and $v_3(t)$ with different values of q.

In Figure 6-8, we have compared the iterates $v_1(t)$, $v_2(t)$ and $v_3(t)$ for values of $q = 0.7$, 0.8, 0.9, and 1. By labeling $v_{i,q}(t)$ as the i^{th} iterate with the fractional order of q. It is easy to observe that, in Figure 2-4,

$$v_{i,0.7}(t) \geq v_{i,0.8}(t) \geq v_{i,0.9}(t) \geq v_{i,1}(t) \qquad i = 1, 2, 3. \tag{3.14}$$

This has also been illustrated numerical Table 2 for $t = 0.3$ and $t = 0.7$, respectively.

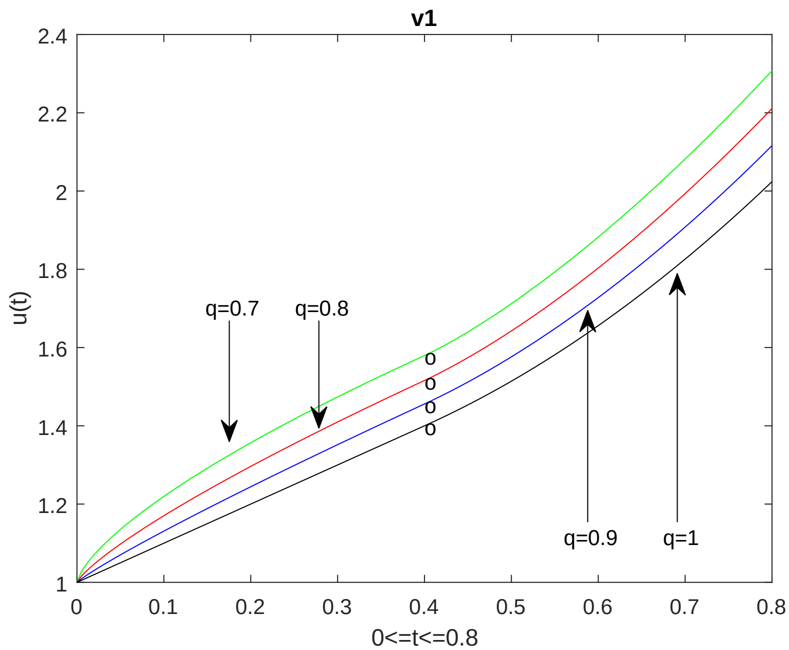

FIGURE 6. Lower solutions $v_1(t)$ with different values of q.

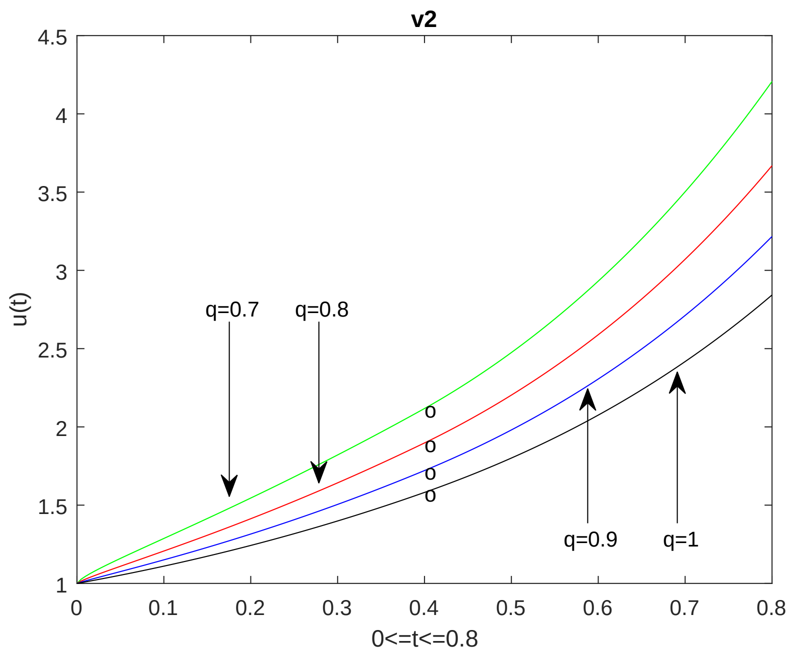

FIGURE 7. Lower solutions $v_2(t)$ with different values of q.

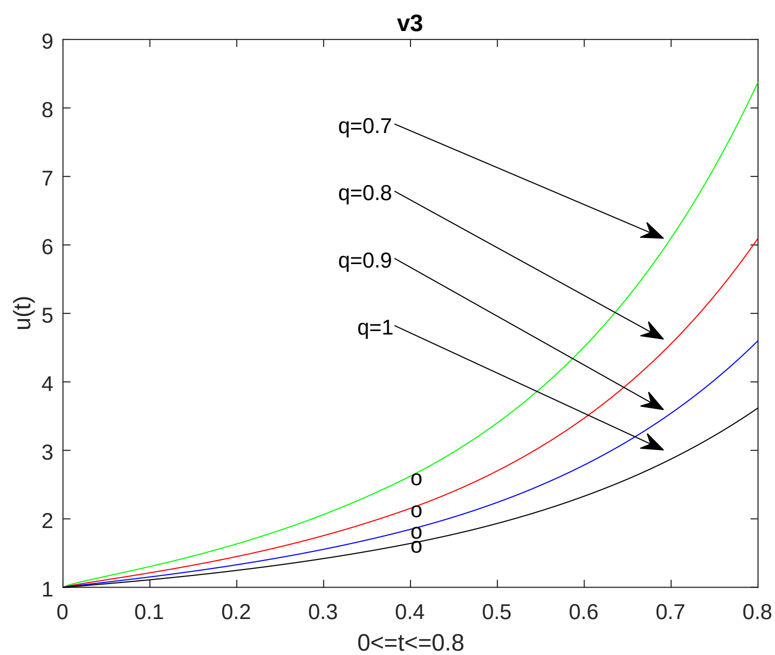

FIGURE 8. Lower solutions $v_3(t)$ with different values of q.

	$q=0.7$	$q=0.8$	$q=0.9$	$q=1$
$v_1(0.3)$	1.4738	1.1098	1.3518	1.3000
$v_2(0.3)$	1.8208	1.6425	1.5049	1.3990
$v_3(0.3)$	2.0631	1.7553	1.5553	1.4203
$v_1(0.7)$	2.0819	1.9935	1.9079	1.8260
$v_2(0.7)$	3.5021	3.0711	2.7120	2.4174
$v_3(0.7)$	6.1008	4.5574	3.5434	2.8738

TABLE 2. Numerical values of the iterates for of $q = 0.7$, 0.8, 0.9, and 1.

Note: From our examples 1 and 2, the solutions of the integer case without the impulses blows up in a finite time $t = 2$ and $t = 1$ respectively. With a positive impulses it definitely it blows up on or before $t = 2$ and $t = 1$ respectively. In [36], we have established that the the integer solution with t replaced by $\frac{t^q}{\gamma q+1}$, will be a lower solution for (3.1) and (3.8)having a similar initial conditions. This proves theoretically, the solution of the Caputo fractional impulsive differential equations of (3.1) and (3.8)blows up on or before $t = 2$, and $t = 1$, respectively.In this work, this has been illustrated numerically in Figure 2-4 and Figure 6-8, for (3.1) and (3.8), respectively.

In example three, we consider the nonlinear Caputo fractional impulsive differential equations with initial conditions of the form:

$$\begin{cases} {}^cD^q u(t) = u - u^2 - \chi(t-0.3)(t-0.3)u(0.3) \\ u(0) = \frac{1}{2} \end{cases} \tag{3.15}$$

where $t \in [0, 0.6]$.

In this case, it is not easy to compute coupled lower and upper solutions of Type 1 for (3.15). Therefore, we use monotone method with coupled lower and upper solutions of natural type initially, to use it as a tool to compute coupled lower and upper solutions of Type 1. It is easy to observe that $v_0(t) = 0, \quad w_0(t) = 1$ are natural lower and upper solutions of equation (3.15) since the impulsive effect is negative. Then, we will compute the iterates as in generalized monotone method (ie Theorem 2.3 with $\lambda = 0$.) as the solution of

$$\begin{cases} {}^cD^q v_1(t) = v_0 - w_0^2 - \chi(t-0.3)(t-0.3)w_1(0.3) \\ {}^cD^q w_1(t) = w_0 - v_0^2 - \chi(t-0.3)(t-0.3)v_1(0.3). \\ v_1(0) = w_1(0) = \frac{1}{2} \end{cases} \tag{3.16}$$

The solutions $v_1(t)$ and $w_1(t)$ on $[0, 0.3]$ will be solutions of

$$
\begin{aligned}
{}^cD^q v_1(t) &= -1^2 \\
{}^cD^q w_1(t) &= 1,
\end{aligned} \tag{3.17}
$$

respectively. Thus, we get

$$
\begin{aligned}
v_1(t) &= \frac{1}{2} - \frac{t^q}{\Gamma(q+1)} \\
w_1(t) &= \frac{1}{2} + \frac{t^q}{\Gamma(q+1)},
\end{aligned} \tag{3.18}
$$

on $[0, 0.3]$. Now using this, we can compute $v_1(t)$ and $w_1(t)$ on $[0.3, 0.6]$, which is given by

$$
\begin{aligned}
v_1(t) &= \frac{1}{2} - \frac{t^q}{\Gamma(q+1)} - w_1(t_1)\frac{(t-t_1)^q}{\Gamma(q+2)} \\
w_1(t) &= \frac{1}{2} + \frac{t^q}{\Gamma(q+1)} - v_1(t_1)\frac{(t-t_1)^q}{\Gamma(q+2)}.
\end{aligned} \tag{3.19}
$$

In the following Figure 9, we present the graph of the iterates $v_1(t)$, $v_2(t)$, $v_3(t)$ and $w_1(t)$, $w_2(t)$, $w_3(t)$ for values of $q = 0.7$, 0.8, 0.9, and 1 relative to equation (3.15) **Note:** Suppose $v_1(t)$ and $w_1(t)$ meets $v_0(t)$ and $w_0(t)$ at say t_1 and t^1. Then we will

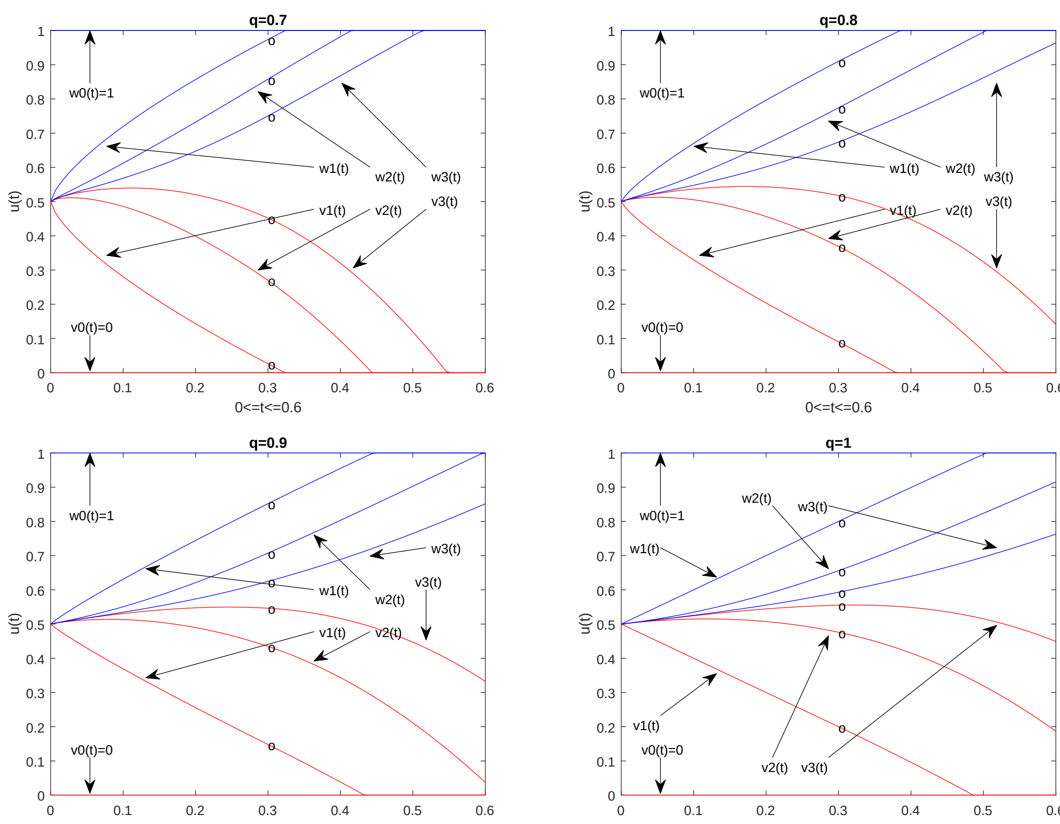

FIGURE 9. Iterations of coupled lower and upper solutions of type 1 with different values of q.

redefine $v_1(t)$ and $w_1(t)$ as computed above on $[0, t_1]$ and $[0, t^1]$ respectively. Further $v_1(t) = v_0(t)$ on $[t_1, T]$ and $w_1(t) = w_0(t)$ on $[t^1, T]$ respectively.We continue this process for $v_i(t)$ and $w_i(t)$. See ([34, 35]) for fractional differential equations without impulses.

The linear approximation $v_n(t)$ and $w_n(t)$, $n = 1, 2, 3.$ of the solution of ((3.15)are computed as below.

$$\begin{cases} {}^cD^q v_n(t) = v_{n-1} - w_{n-1}^2 - \chi(t-0.3)(t-0.3)w_n(0.3) \\ {}^cD^q w_n(t) = w_{n-1} - v_{n-1}^2 - \chi(t-0.3)(t-0.3)v_n(0.3). \\ v_n(0) = w_n(0) = \frac{1}{2} \end{cases} \tag{3.20}$$

where $n = 1, 2, 3$. Since the explicit formulas of v_n and w_n cannot be obtained, we use numerical method to get an approximation.

In Figure 10-12, we have compared the iterates $v_1(t)$, $v_2(t)$, $v_3(t)$ and $w_1(t)$, $w_2(t)$, $w_3(t)$ for values of $q = 0.7$, 0.8, 0.9, and 1. By labeling $v_{i,q}(t)$ and $w_{i,q}(t)$ as the i^{th} iterate with the fractional order of q, respectively. It is easy to observe that, in Figure 10-12,

$$\begin{aligned} v_{i,0.7}(t) \le v_{i,0.8}(t) \le v_{i,0.9}(t) \le v_{i,1}(t) \qquad i = 1, 2, 3 \\ w_{i,0.7}(t) \ge w_{i,0.8}(t) \ge w_{i,0.9}(t) \ge w_{i,1}(t) \qquad i = 1, 2, 3. \end{aligned} \tag{3.21}$$

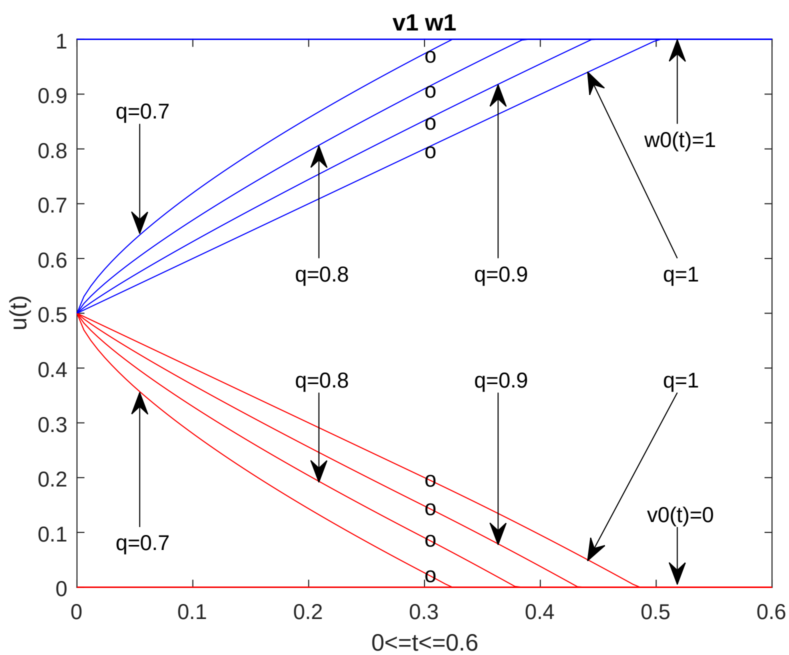

FIGURE 10. Coupled Lower and upper solutions $v_1(t)$ and $w_1(t)$ with different values of q.

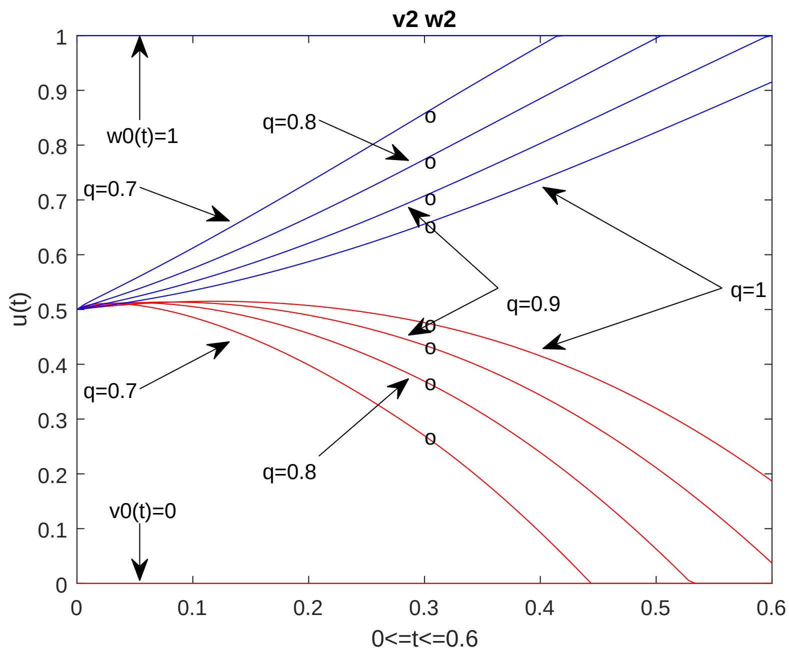

FIGURE 11. Coupled Lower and upper solutions $v_2(t)$ and $w_2(t)$ with different values of q.

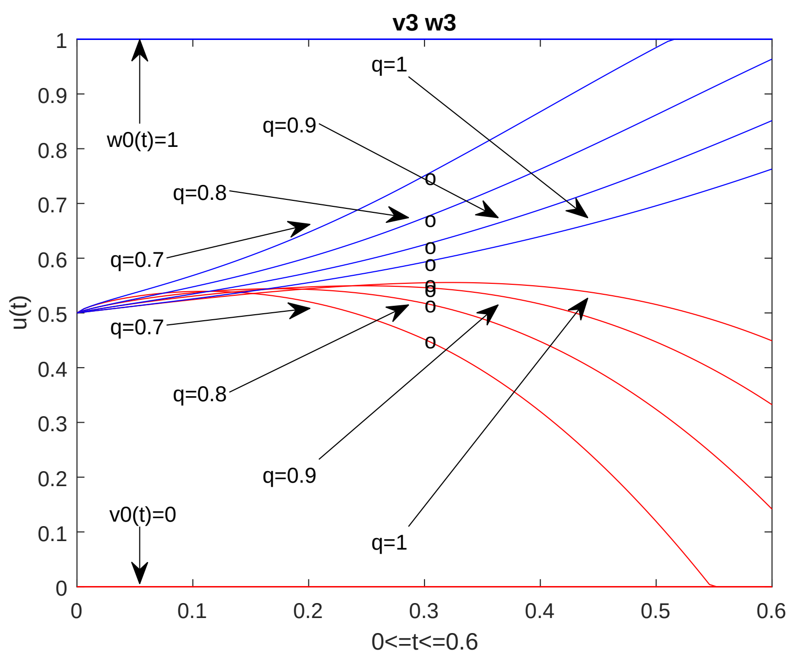

FIGURE 12. Coupled Lower and upper solutions $v_3(t)$ and $w_3(t)$ with different values of q.

In this part, we can see it is very clear that $v_{n-1}(t) \leq v_n(t)$ and $w_n(t) \leq w_{n-1}(t)$, which satisfies the generalized monotone method result. In addition, the numerical results show that $v_n(t)$ and $w_n(t)$ is sandwiching the exact solutions of equation (3.15). The real solution $u(t)$ is sandwiched between $v_n(t)$ and $w_n(t)$. The error in computing the solution can be minimized by computing enough number of iterates. Also, the value of q can be chosen as a parameter to improve the model to suit the available data.

4. **Conclusion**

In this work, we have developed numerical results for application problems such as blow up in finite time and the logistic equation by using generalized monotone method for Caputo fractional impulsive differential equations when the impulses introduced in the forcing function are through unit step functions. See [36] for the theoretical justification of blow up results for Caputo fractional differential equation without impulses. Generalized monotone method for Caputo fractional equations are applicable to more general nonlinear terms including a linear term which is either increasing or decreasing. In this case we will have $\lambda \neq 0$, in Theorem 2.3. When $\lambda \neq 0$, the computation of each iterates will involve the computation of Mittag-Leffler function. Although a lot of literature is available about Mittag-Leffler function, computation of series involving Mittag-Leffler functions, product of Mittag-Leffler functions etc are nontrivial. See [5, 31] for some known results. We plan to take up this in our future work to include a variety of other physical application problems.

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