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DISCRETE TIME QUEUEING INVENTORY MODELS WITH INVENTORY DEPENDENT CUSTOMER ARRIVAL UNDER (s, S) POLICY

BALAGOPAL N, DEEPTHY C P, JAYAPRASAD P N, AND VARGHESE JACOB

Department of Computer Science and Engineering, N.S.S. College of Engineering, Palaghat - 678008, India. *balagopalan89@gmail.com*
Department of Mathematics, S. D. College, Alappuzha, Kerala - 688003, India. *deepthicp@gmail.com*
Department of Mathematics, Government College, Kottayam-686 013, India. *jayaprasadpn@gmail.com*
Department of Mathematics, Government College, Kottayam - 686 013, India. *drvarghesejacob@gmail.com*

ABSTRACT. In this paper, we presents two discrete queueing inventory models with positive service time and lead time where customers arrive according to a Bernoulli process and service time follows a geometric distribution. In model 1, we assume that an arriving customer joins the system only if the number in the queue is less than the number of items in the inventory at that epoch. In model 2, it is assumed that if the inventory level is greater than reorder level, s at the time of arrival of a customer, then he necessarily joins. However, if it is less than or equal to s (but larger than zero) then he joins only if the number of customers present is less than the on hand inventory. We analyse this queueing system using the matrix geometric method and we derive an explicit expression for the stability condition of the model-2. We obtain the steady-state behaviour of these systems and several system performance measures. An average system cost function is constructed for the models and are investigated numerically. The influence of various parameters on the system performances are also discussed through numercal example.

Subject Classification:

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1. Introduction

There is a growing research interest in discrete time queues mainly motivated by their applications in computer and communication systems because the basic time

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unit in these systems is a binary code.(See [2], [3]). Also the discrete time system can be used to approximate the continuous system in practice. Recently, due to the fast progress of computer and telecommunication network technologies, the discrete time models have received more attention from queueing researchers. BISDN (Broadband Integrated Service Digital Network) has been of significant interest because it can provide a common interface for future communication needs including video, voice and data communication signals through high speed Local Area Network (LAN), on-demand video distribution and video telephony communications. The Asynchronous Transfer Mode(ATM) is a key technology for accommodating such a wide area of services. Applications of discrete time queues were discussed in the books (see [4], [5], [19]).

In a discrete time analysis, the system is observed only at specific points in time. e.g., a system in which observation is made only at points of event occurrences such as arrivals or departures at specified points which may be equal and numbered sequentially as $0, 1, 2, \ldots$.

The first work on discrete time queues is due to Meisling (1958). (see [14]). Since 1958, many researches have dedicated time to study of such systems. One of the most outstanding works of the queueing theory has been carried out by Yang and Li (1998) (see [15]) who extended the queues with repeated attempts to the discrete time systems.

Inventory models have a wide range of application in industries, hospitals, banks, agriculture, educational institutions etc. The ultimate objective of any inventory model is to answer two basic questions: how much to order and when to order. The answer to the first question is expressed in terms of what we call the order quantity and that of the second, the reorder level. Order quantity is the optimum amount that should be ordered every time an order is placed so as to minimize the total system running cost. Reorder level depends on the type of inventory model.

The objective of inventory control is often to balance conflicting goal of making available the required item at a time of need and minimizing the related costs. In inventory models, the availability of items has also to be taken into consideration along with features of queueing theory. In inventory models with negligible service times, queue of customers is formed only when the system is out of stock and unsatisfied customers are permitted to wait. On the contrary for the case of inventory with positive service time, queue is formed even when inventoried items are available because new customers can join while a service is going on. If either service time or lead time or both are taken to be positive, then also a queue is formed, depending on assumptions on backlogging of demands/on other factors.

The analysis of inventory problem was started by Harris [7] in 1915. The cost analysis of different inventory policies is given by Naddor [16]. The book by Hadley and Whitin [8] provides inventory theory and applications. A systematic approach to (s, S) inventory policy is provided by Arrow, Karlin and Scarf [1] using renewal theory. One of the recent contributions of significance to inventory with positive service time is due to Schwarz et alia (et al.) [18]. Krishnamoorthy and Viswanath ([10],[11]) analyzed production inventory system with service time. One of the works of the queueing theory has been carried out by Yang and Li [15] who extended the queues with repeated attempts to the discrete time systems. Lian and Liu [12] developed a discrete time inventory model with geometric inter demand times and constant life time. The (s, S) inventory system with positive lead time has been studied by several researchers. (See [9], [13]). Deepthi (see [6]) have studied many discrete time inventory models with/withot positive time in her thesis.

Discrete time queueing system has been found to be more appropriate in modelling computer systems and communication network.

In this paper, we analyze two discrete time (s, S) inventory models with positive service time and lead time. In model 1, we assume that an arriving customer joins the system only if the number in the queue is less than the number of items in the inventory at that epoch. In model 2, it is assumed that if the inventory level is greater than or equal to $s + 1$ at the time of arrival of a customer, then he necessarily joins. However, if it is less than or equal to s (but larger than zero) then he joins only if the number of customers present is less than the on hand inventory since this guaranties that he gets service without waiting for replenishment.

This paper is organized as follows :- In section 2, we present the mathematical formulation of the model-1 its long run bevavior and some key performance measures. Section 3 discuss mathematical formulation of the model-2 and its stability condition. We also analyze the computation of steady-state probabilities of the system state and derive some performance measures. In section 4, we obtain a cost function for the models. Finally some numerical results are given in section 5.

2. **Mathematical Formulation of Model 1**

Consider a single product (s, S) inventory system in which customers asking for the product arrive according to a Bernoulli process with parameter p and no demand with probability $\overline{p} = 1 - p$. Each demand is for exactly one unit. Here we assume that an arriving customer joins the system only if the number in the queue is less than the number of items in the inventory at that epoch. The service time follows geometric distribution with parameter q. Denote $\overline{q} = 1 - q$. Whenever the inventory level depletes to s due to demand, an order is placed for replenishment up to S. Lead time for replenishment of the inventory has a geometric distribution with parameter r. Denote $\overline{r} = 1 - r$. Each time a replenishment is done, the on hand inventory is raised up to the maximum level S.

At time m^+ the system can be described by $\mathfrak{X}_1 = = \{(N_m, I_m) : m = 0, 1, \dots S\}$ where N_m is the number of customers in the system and I_m is the inventory level. It can be seen that $\mathfrak{X}_1$ is a finite Markov chain with state space

$$\mathfrak{E}_1 = \{(i, j) : 0 \leq i < j;\ 0 \leq j \leq S\}.$$

The corresponding one-step transition probability matrix $\mathcal{P}_1$ is given by

$$\mathcal{P}_1 = \begin{bmatrix} B_{0,0} & B_{0,1} & & & & & \\ B_{1,0} & A_{1,1} & A_{1,2} & & & & \\ & A_{2,1} & A_{2,2} & A_{2,3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_{S-2,S-3} & A_{S-2,S-2} & A_{S-2,S-1} & \\ & & & & A_{S-1,S-2} & A_{S-1,S-1} & A_{S-1,S} \\ & & & & & A_{S,S-1} & A_{S,S} \end{bmatrix}$$

where
$[B_{0,0}]$ is of dimension $(S+1) \times (S+1)$ and is given by

$$[B_{0,0}]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \dots, s \\ \overline{p}, & j = i, \quad i = s+1, s+2, \dots, S \\ \overline{p}r, & j = S, \quad i = 0, 1, \dots, s \\ 0, & \text{otherwise} \end{cases}$$

$[B_{0,1}]$ is of dimension $(S+1) \times S$ and is given by

$$[B_{0,1}]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \dots, s \\ p\,r, & j = S, \quad i = 0, 1, \dots, s \\ p, & j = i, \quad i = s+1, s+2, \dots, S \\ 0, & \text{otherwise} \end{cases}$$

$[B_{1,0}]$ is of dimension $S \times (S+1)$ and is given by

$$[B_{1,0}]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 1, 2, \dots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \dots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \dots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{1,1}]$ is of dimension $S \times S$ and is given by

$$[A_{1,1}]_{ij} = \begin{cases} \overline{q}\,\overline{r}, & j = i, \quad i = 1 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = 2, 3, \dots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \dots, S \\ p\;\; q\,r, & j = S-1, \quad i = 1, 2, \dots, s \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \dots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{1,2}]$ is of dimension $S \times (S-1)$ and is given by

$$[A_{1,2}]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i, \qquad i = 2, 3, \ldots, s \\ p, & j = i, \qquad i = s+1, s+2, \ldots, S \\ p\overline{q}r, & j = S, \qquad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{2,1}]$ is of dimension $(S-1) \times S$ and is given by

$$[A_{2,1}]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{2,2}]$ is of dimension $(S-1) \times (S-1)$ and is given by

$$[A_{2,2}]_{ij} = \begin{cases} \overline{q}\,\overline{r}, & j = i, \qquad i = s-1 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \qquad i = s \\ \overline{p}\,\overline{q}, & j = i, \qquad i = s+1, s+2, \ldots, S \\ \overline{p}\,\overline{q}r, & j = S, \qquad i = 2, \ldots, s \\ pqr, & j = S-1, \quad i = 2, 3, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{2,3}]$ is of dimension $(S-1) \times (S-2)$ and is given by

$$[A_{2,3}]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i, \qquad i = s \\ p, & j = i, \qquad i = s+1, s+2, \ldots, S \\ p\overline{q}r, & j = S, \qquad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$\vdots$

$$A_{S-1,S} = \begin{bmatrix} 0 \\ p \end{bmatrix};$$

$$A_{S,S-1} = \begin{bmatrix} q & 0 \end{bmatrix};$$

$$A_{S,S} = \overline{q}.$$

2.1. **Long run behaviour of the system.** Assuming $p, q, r \in (0,1)$, the Markov chain $\mathfrak{X}_1$ is seen to be irreducible and positive recurrent. An irreducible Markov Chain on finite state space is always stable. So there exists a unique steady-state probability vector $\boldsymbol{x}$.

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{S-1}, x_S)$ be the steady-state vector of $\mathfrak{X}_1$.
Then

$$\boldsymbol{x}\mathcal{P}_1 = \boldsymbol{x}; \qquad \boldsymbol{x}\mathbf{e} = 1 \tag{2.1}$$

gives

$$\boldsymbol{x}_0 = \boldsymbol{x}_1 D_0 \text{ where } D_0 = B_{1,0}(I - B_{0,0})^{-1};$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_2 A_{2,1}(I - A_{1,1})^{-1} D_1 \text{ where } D_1 = [I - D_0 B_{0,1}(I - A_{1,1})^{-1}]^{-1};$$

$$\boldsymbol{x}_2 = \boldsymbol{x}_3 A_{3,2}(I - A_{2,2})^{-1} D_2 \text{ where } D_2 = [I - A_{2,1}(I - A_{1,1})^{-1} D_1 A_{1,2}(I - A_{2,2})^{-1}]^{-1};$$

$$\boldsymbol{x}_3 = \boldsymbol{x}_4 A_{4,3}(I - A_{3,3})^{-1} D_3 \text{ where } D_3 = [I - A_{3,2}(I - A_{2,2})^{-1} D_2 A_{2,3}(I - A_{3,3})^{-1}]^{-1};$$

$$\vdots$$

$$\boldsymbol{x}_{S-1} = x_S A_{S,S-1}(I - A_{S-1,S-1})^{-1} D_{S-1}$$

$$\text{where } D_{S-1} = [I - A_{S-1,S-2}(I - A_{S-2,S-2})^{-1} D_{S-2} A_{S-2,S-1}(I - A_{S-1,S-1})^{-1}]^{-1}$$

x_S can be found using the normalizing condition

$$\boldsymbol{x}_0 \mathbf{e} + \boldsymbol{x}_1 \mathbf{e} + \ldots + \boldsymbol{x}_{S-1} \mathbf{e} + x_S = 1.$$

2.2. **System Performance Measures Model- 1.** Let the steady-state probability vector $\boldsymbol{x}$ be partitioned as $\boldsymbol{x}_0 = (x_{0,0}, x_{0,1}, \ldots, x_{0,S})$; $\boldsymbol{x}_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,S})$; $\boldsymbol{x}_2 = (x_{2,2}, x_{2,3}, \ldots, x_{2,S})$; $\ldots$ $\boldsymbol{x}_{S-1} = (x_{S-1,S-1}, x_{S-1,S})$; $x_S = x_{S,S}$.

We have obtained the following measures for evaluating performance of the system.

1. Expected number of customers in the system is, EC $= \sum_{i=0}^{S} i\ \boldsymbol{x}_i e$.
2. Expected inventory level is, EI $= \sum_{j=1}^{S} \sum_{i=0}^{j} j\ x_{i,j}$.
3. Expected reorder rate is, ER $= q \sum_{i=1}^{s} x_{i,s+1}$.
4. Expected replenishment rate is, ERR $= r \sum_{j=1}^{s} \sum_{i=0}^{j} x_{i,j}$.
5. Probability that the inventory level is zero is, $\sum_{i=0}^{S} x_{i,0}$.
6. Expected loss rate of fresh arrivals is, EL $= p \sum_{i=1}^{S} x_{i,i}$.
7. Expected rate of departure after completing service is, ED $= q \sum_{j=1}^{S} \sum_{i=1}^{j} x_{i,j}$.

3. Mathematical Formulation of Model 2

In this model, we assume that at the time of arrival of a customer, if the inventory level is $\geq s+1$, then he joins. However, if it is $\leq s$ (but larger than zero) then he joins only if the number of customers present is less than the on hand inventory.

At time m^+ the system can be described by $\mathfrak{X}_2 == \{(N_m, I_m) : m \in \mathcal{N}\}$ where N_m is the number of customers in the system and I_m is the inventory level at epoch after the occurence of probable events. It can be seen that $\mathfrak{X}_2$ is a Discrete Time Markov Chain (DTMC) with state space $\mathfrak{E}_2 = \{(i,j) : i \geq 0; 0 \leq j \leq S\}$.

The corresponding one step transition probability matrix $\mathcal{P}_2$ is

$$\mathcal{P}_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & s-1 & s & s+1 & s+2 & \cdots \end{array} \\ \begin{pmatrix} E_0 & C_0 & & & & & & & \\ B_1 & E_1 & C_1 & & & & & & \\ & B_2 & E_2 & C_2 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & & B_s & E_s & C_s & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

where each sub-matrix is of order $(S+1) \times (S+1)$. They are given by

$$[E_0]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \overline{p}r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[E_k]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0, \quad k = 1, 2, \ldots, s \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1, \ldots, k \quad k = 1, \ldots, s \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = k+1, \ldots, s, \quad k = 1, \ldots, s-1 \\ pq\overline{r}, & j = i-1, \quad i = k+1, \ldots, s, \quad k = 1, \ldots, s-1 \\ pq, & j = i-1, \quad i = s+1, s+2, \ldots, S \quad k = 1, 2, \ldots, s \\ \overline{p}r, & j = S, \quad i = 0, \quad k = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \quad k = 1, 2, \ldots, s \\ p\ \ qr, & j = S-1, \quad i = 1, 2, \ldots, s \quad k = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \quad k = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[C_0]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \dots, s \\ p\,r, & j = S, \quad i = 0, 1, \dots, s \\ p, & j = i, \quad i = s+1, s+2, \dots, S \\ 0, & \text{otherwise} \end{cases}$$

$$[C_k]_{ij} = \begin{cases} pr, & j = S, \quad i = 0 \quad k = 1, 2, \dots, s \\ p\overline{q}r, & j = S, \quad i = 1, 2, \dots, s \quad k = 1, 2, \dots, s \\ p\overline{q}\,\overline{r}, & j = i, \quad i = k+1, \dots, s, \quad k = 1, \dots, s-1 \\ p\overline{q} & j = i, \quad i = s+1, s+2, \dots, S \quad k = 1, 2, \dots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[B_k]_{ij} = \begin{cases} q\overline{r}, & j = i-1, \quad i = 1, \dots, k \quad k = 1, \dots, s \\ \overline{p}q\overline{r}, & j = i-1, \quad i = k+1, \dots, s \quad k = 1, \dots, s-1 \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \dots, S \quad k = 1, 2, \dots, s \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \dots, s \quad k = 1, 2, \dots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_1]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}r, & j = S, \quad i = 0 \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \dots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \dots, S \\ pqr, & j = S-1, \quad i = 1, 2, \dots, s \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \dots, s \\ pq, & j = i-1, \quad i = s+2, s+3, \dots, S \\ 0, & \text{otherwise} \end{cases}$$

$$[A_2]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 1, 2, \dots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \dots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \dots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} pr, & j = S, \quad i = 0 \\ p\overline{q}r, & j = S, \quad i = 1, 2, \dots, s \\ p, & j = i, \quad i = s+1, s+2, \dots, S \\ p\overline{q}, & j = i, \quad i = s+1, s+2, \dots, S \\ 0, & \text{otherwise} \end{cases}$$

3.1. **Stability condition.** In this section, we perform the steady-state analysis of the queueing model under study. For determining the stability condition for the system, we consider the transition probability matrix $A = A_0 + A_1 + A_2$, which is obtained as

$$[A]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ r, & j = S, \quad i = 0 \\ q\overline{r}, & j = i-1, \quad i = 1,2,\ldots,s \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1,2,\ldots,s \\ qr, & j = S-1, \quad i = 1,2,\ldots,s \\ \overline{q}r, & j = S, \quad i = 1,2,\ldots,s \\ \overline{p}q, & j = i-1, \quad i = s+1 \\ p+\overline{p}\,\overline{q}, & j = i, \quad i = s+1 \\ q, & j = i-1, \quad i = s+2, s+3, \ldots, S \\ \overline{q}, & j = i, \quad i = s+2, s+3, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \pi_{s+1}, \ldots, \pi_S)$ be the stationary probability vector of A. Then

$$\boldsymbol{\pi} A = \boldsymbol{\pi}; \quad \boldsymbol{\pi}\boldsymbol{e} = 1, \tag{3.1}$$

where $\boldsymbol{e}$ is a column vector of 1's of appropriate order.

From equation 3.1

$$\pi_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{(q\,\overline{r})^j}\pi_0, & j = 1,2,\ldots,s; \\ \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{\overline{p}\,q(q\,\overline{r})^{j-1}}\pi_0, & j = s+1; \\ \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-2}}{q(q\,\overline{r})^{j-2}}\pi_0, & j = s+2, s+3, \ldots, S-1. \end{cases}$$

$$\pi_S = \frac{(1-\overline{r})[q(q\,\overline{r})^s + \overline{q}(1-\overline{q}\,\overline{r})^s]}{q(q\,\overline{r})^s}\pi_0.$$

Further $\boldsymbol{\pi}\boldsymbol{e} = 1$ gives,

$$\pi_0 = \frac{q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[\overline{p}\,q + (S-s-1)r + r\,\overline{q}] + rq(q\,\overline{r})^s}$$

Now,

$$\boldsymbol{\pi} A_0 e = \frac{(1-\overline{q}\,\overline{r})^s[pr + (S-s-1)p\overline{p}\,\overline{q}r] + p\overline{p}qr(q\overline{r})^s}{\overline{p}q(q\,\overline{r})^s}\pi_0.$$

and

$$\boldsymbol{\pi} A_2 e = \frac{(1-\overline{q}\,\overline{r})^s[q\overline{r} + r + (S-s-1)\overline{p}r] - (q\,\overline{r})^{s+1}}{(q\,\overline{r})^s}\pi_0.$$

Thus we obtained the stationary probability vector, $\boldsymbol{\pi}$, explicitly in terms of the parameters of the model, and hence, we have the stability condition in the following theorem [see, [17]].

Theorem 3.1. *The Markov chain under study is stable if and only if*

$$\frac{(q\,\overline{r})^s(\overline{p}q^2\overline{r}+p\overline{p}qr)}{(1-\overline{q}\,\overline{r})^s\left[\overline{p}q^2\overline{r}+\overline{p}qr-pr+(S-s-1)\overline{p}r(\overline{p}-\overline{q})\right]}<1 \tag{3.2}$$

3.2. **Steady-state analysis.** We define the steady-state distribution of $\{N_m = n, I_m = i : m \in \mathcal{N}\}$ as follows :

$$\boldsymbol{x}_{n,i} = \lim_{m\to\infty} P(N_m = n, I_m = i); \quad (n,i) \in \mathfrak{E}_2.$$

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s, \boldsymbol{x}_{s+1}, \boldsymbol{x}_{s+2}, \ldots)$ be the steady-state probability vector of $\mathfrak{X}_2$, where $\boldsymbol{x}_n = (x_{n,0}, x_{n,1}, \ldots, x_{n,S})$ for $n \geq 0$.
Then

$$\boldsymbol{x}\mathcal{P}_2 = \boldsymbol{x}, \quad \boldsymbol{x}\mathbf{e} = 1. \tag{3.3}$$

Using Matrix-geometric method [see [17]], under the stability condition (3.2), the steady-state probability vector $\boldsymbol{x}$ is obtained as

$$\boldsymbol{x}_n = \boldsymbol{x}_{s+1}R^{n-(s+1)}; \; n \geq s+2 \tag{3.4}$$

where R is the minimal non-negative solution to the matrix equation

$$R^2A_2 + RA_1 + A_0 = R \tag{3.5}$$

The vectors $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s, \; \boldsymbol{x}_{s+1}$ are obtained from the equations

$$\boldsymbol{x}_0E_0 + \boldsymbol{x}_1B_1 = \boldsymbol{x}_0 \tag{3.6}$$

$$\boldsymbol{x}_{n-1}C_{n-1} + \boldsymbol{x}_nE_n + \boldsymbol{x}_{n+1}B_{n+1} = \boldsymbol{x}_n, \quad 1 \leq n \leq s-1 \tag{3.7}$$

$$\boldsymbol{x}_{s-1}C_{s-1} + \boldsymbol{x}_sE_s + \boldsymbol{x}_{s+1}A_2 = \boldsymbol{x}_s \tag{3.8}$$

$$\boldsymbol{x}_sC_s + \boldsymbol{x}_{s+1}A_1 + \boldsymbol{x}_{s+2}A_2 = \boldsymbol{x}_{s+1} \tag{3.9}$$

From (3.9), we get,

$$\boldsymbol{x}_sC_s + \boldsymbol{x}_{s+1}(A_1 - I - RA_2) = 0.$$

Thus

$$\boldsymbol{x}_{s+1} = \boldsymbol{x}_sR_s, \text{ where } R_s = C_s(I - A_1 - RA_2)^{-1}.$$

From (3.8), we have

$$\boldsymbol{x}_{s-1}C_{s-1} + \boldsymbol{x}_s(E_s - I - R_sA_2) = 0.$$

Thus

$$\boldsymbol{x}_s = \mathrm{x}_{\,s-1}R_{s-1}, \text{ where } R_{s-1} = C_{s-1}(I - E_s - R_sA_2)^{-1}.$$

From (3.7) we have,

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} R_{n-1}; 1 \leq n \leq s-1, \text{ where } R_{n-1} = C_{n-1}(I - E_n - R_{s-1}B_{n+1})^{-1}.$$

Finally $\boldsymbol{x}_0$ can be found from the normalizing condition

$$\boldsymbol{x}_0 \mathbf{e} + \boldsymbol{x}_1 \mathbf{e} + \ldots + \boldsymbol{x}_s \mathbf{e} + \boldsymbol{x}_{s+1}(I - R)^{-1}\mathbf{e} = 1. \tag{3.10}$$

That is, $\boldsymbol{x}_0 \left(I + \sum_{i=0}^{s-1} \prod_{j=0}^{i} R_j + \prod_{j=0}^{s} R_j (I - R)^{-1} \right) \mathbf{e} = 1.$

3.3. **System Performance Measures : Model - 2.**

1. Expected number of customers in the system is,
 $\text{EC} = \sum_{i=1}^{\infty} i\ \boldsymbol{x}_i \mathbf{e} = \sum_{i=1}^{s} i\ \boldsymbol{x}_i \mathbf{e} + \boldsymbol{x}_{s+1} R(I-R)^{-2}\mathbf{e} + (s+1)\boldsymbol{x}_{s+1}(I-R)^{-1}\mathbf{e}.$
2. Expected inventory level is, EI $= \sum_{i=0}^{\infty} \sum_{j=1}^{S} j\ x_{i,j}.$
3. Expected reorder rate is, ER $= q \sum_{i=1}^{\infty} x_{i,s+1}.$
4. Expected replenishment rate is, ERR $= r \sum_{i=0}^{\infty} \sum_{j=0}^{s} x_{i,j}.$
5. Expected loss rate of fresh arrivals is, EL $= p \sum_{i=0}^{\infty} x_{i,0}.$
6. Expected rate of departure after completing service is, ED $= q \sum_{i=0}^{\infty} \sum_{j=1}^{S} x_{i,j}.$

4. **Cost Analysis of the models**

In this section we study the Cost Analysis of the models and discuss it through figure 1 in section 5. To construct an objective cost function with the following.
Let

- c_0 denote the fixed ordering cost.
- c_1 - procurement cost/unit.
- c_2 - holding cost of inventory /unit/unit time.
- c_3 - holding cost of customers/unit/unit time.
- c_4 - cost due to the loss of customers /unit/unit time.
- c_5 - cost due to decay of items per unit per unit time.

Then for Model 1 and Model 2, the Expected Total Cost

$$\textbf{ETC} = [c_0 + (S - s)c_1]\, ER + c_2 EI + c_3 EC + c_4 EL + c_5 ED. \tag{4.1}$$

5. **Numerical Illustrations**

By fixing the parameters, $(p, q, r, s) = (0.3, 0.7, 0.4, 5)$, a look at the Table 1 shows the following observations.

- System idle probability, P_{idle} decreases as S increases and the rate of increase is almost negligible in Model-2, this is because of increase in the maximum inventory level results in more customers can join the system, so traffic intensity of the system increases.
- The measures expected number of customers, EC as well as expected inventory level, EI are non-decreasing function of S as expected.
- The measure expected loss of primary arrival, EL is a non-increasing function of S. This due to the fact that in model-1, customer arrival depends on the inventory level. So customer loss will be higher in model-1 and negligible in model-2.

TABLE 1. Effect of S on Model-1 when $(p, q, r, s) = (0.3, 0.7, 0.4, 5)$.

S	P_{idle}	EC	EI	ER	EL	ED
Model - 1						
20	0.41041	1.35927	12.51734	0.02506	0.00251	0.41272
25	0.40548	1.40085	15.04177	0.01894	0.00208	0.41616
30	0.40230	1.42963	17.56195	0.01523	0.00175	0.41839
35	0.40007	1.45077	20.07878	0.01273	0.00151	0.41995
40	0.39843	1.46692	22.59291	0.01095	0.00132	0.42110
45	0.39716	1.47962	25.10484	0.00960	0.00118	0.42199
Model - 2						
20	0.5716	0.52443	12.45552	0.01941	0.00004	0.01361
25	0.5715	0.52477	14.95997	0.01467	0.00003	0.01028
30	0.5715	0.52485	17.46261	0.01179	0.00003	0.00826
35	0.5715	0.52491	19.96405	0.00985	0.00002	0.00690
40	0.5715	0.52495	22.46473	0.00847	0.00002	0.00593
45	0.5714	0.52498	24.96490	0.00742	0.00002	0.00519

- Table 2 shows that as the arrival rate p increases, expected number of customers as well as expected inventory level increases. Expected reorder rate decreases which is completely against our expectation. This may be due to the increase in the number of customers above the reorder level. Expected rate of departure after completion of service increases.

- From table 3 we notice that the expected number of customers increases and the expected inventory level decreases as p increases which is as expected. Here expected reorder rate and ED also increase with increasing value of p.

TABLE 2. Effect of p on Model-1 when $(q, s, S) = (0.7, 5, 20)$

p	P_{idle}	EC	EI	ER	EL	ED
$r = 0.3$						
0.400	0.18429	3.10560	12.30936	0.02388	0.02734	0.57100
0.425	0.13985	3.69822	12.33383	0.02117	0.04100	0.60210
0.450	0.10388	4.30922	12.38669	0.01779	0.05797	0.62728
0.475	0.07627	4.91289	12.46673	0.01416	0.07776	0.64661
0.500	0.05602	5.48963	12.56962	0.01068	0.09967	0.66078
0.525	0.04170	6.02814	12.68933	0.00764	0.12302	0.67081
0.550	0.03181	6.52450	12.81963	0.00517	0.14723	0.67773
$r = 0.4$						
0.400	0.17118	3.24947	12.44302	0.02443	0.02636	0.58017
0.425	0.12573	3.88555	12.47622	0.02156	0.04004	0.61199
0.450	0.08945	4.53842	12.53896	0.01800	0.05715	0.63738
0.475	0.06219	5.17810	12.62933	0.01422	0.07718	0.65647
0.500	0.04272	5.78265	12.74208	0.01063	0.09938	0.67009
0.525	0.02939	6.34053	12.87043	0.00752	0.12302	0.67943
0.550	0.02052	6.84898	13.00772	0.00504	0.14751	0.68564
$r = 0.5$						
0.400	0.16281	3.35611	12.52091	0.02476	0.02569	0.58604
0.425	0.11690	4.02446	12.56128	0.02176	0.03937	0.61817
0.450	0.08072	4.70769	12.63205	0.01807	0.05657	0.64349
0.475	0.05401	5.37252	12.73057	0.01417	0.07673	0.66220
0.500	0.03536	5.99544	12.85078	0.01050	0.09909	0.67525
0.525	0.02293	6.56511	12.98532	0.00737	0.12288	0.68395
0.550	0.01490	7.07996	13.12730	0.0049	0.14750	0.68957

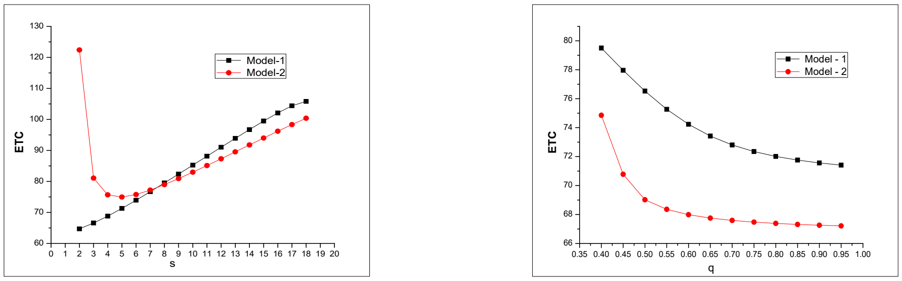
From figure 1, we give the optimum value of the expected total cost per unit time by varying the parameter one at a time and keeping others fixed. Here, we fixed maximum inventory level as 20 unit. In figure 1 (a) we can observe that for model-2, the cost function has the minimum value 74.9575 and optimum reorder level is $s = 5$. In figure1(b) it can be seen that the expected total cost **ETC** is getting reduce as the service rate q increases.

Table 3. Effect of p on Model-2 when $(q, s, S) = (0.7, 5, 20)$

p	P_{idle}	EC	EI	ER	EL	ED
$r = 0.3$						
0.400	0.4312	0.78998	12.09634	0.02503	0.00087	0.01534
0.425	0.3958	0.87849	12.03603	0.02655	0.00115	0.0156
0.450	0.3610	0.97460	11.98145	0.02803	0.00147	0.01583
0.475	0.3266	1.08523	11.92748	0.02953	0.00185	0.01600
0.500	0.2925	1.21591	11.87408	0.03109	0.0023	0.01612
0.525	0.2588	1.37615	11.82097	0.03278	0.00281	0.01620
0.550	0.2255	1.58437	11.76729	0.03473	0.00337	0.01622
$r = 0.4$						
0.400	0.4275	0.81025	12.25618	0.02583	0.00020	0.01535
0.425	0.3938	0.88429	12.22904	0.02720	0.00026	0.01572
0.450	0.3586	0.98327	12.18449	0.02876	0.00034	0.01592
0.475	0.3234	1.09829	12.14019	0.03036	0.00044	0.01605
0.500	0.2885	1.23601	12.09601	0.03203	0.00057	0.01611
0.525	0.2537	1.40836	12.05153	0.03385	0.00071	0.01612
0.550	0.2191	1.63981	12.00552	0.03601	0.00088	0.01626
$r = 0.5$						
0.400	0.4268	0.81367	12.36443	0.02621	0.00004	0.01545
0.425	0.3933	0.88673	12.34523	0.02759	0.00006	0.01587
0.450	0.3578	0.98719	12.30706	0.02920	0.00008	0.01607
0.475	0.3223	1.10471	12.26897	0.03086	0.00010	0.01618
0.500	0.2870	1.24682	12.23077	0.03260	0.00013	0.01623
0.525	0.2517	1.42754	12.19189	0.03452	0.00017	0.01631
0.550	0.2164	1.67716	12.15079	0.03683	0.00021	0.01642

Coclusion

In this paper, we studied two $Geo/Geo/1$ discrete time queueing systems with positive service time and lead time. In model 1, we assumed that an arriving customer joins the system only if the number in the queue is less than the number of items in the inventory at that epoch. In model 2, it is assumed that if the inventory level is greater than or equal to $s + 1$ at the time of arrival of a customer, then he necessarily joins. However, if it is less than or equal to $s\,(s > 0)$ then he joins only if the number of customers present is less than the on hand inventory since this guaranties that he gets service without waiting for replenishment. The systems are exhaustively analyzed. Steady-state analysis of the model is carried out using Matrix-geometric method. Several performance measures are derived. The influence of various parameters on the



(A) s versus **ETC** with $q = 0.4$ (B) q versus **ETC** with $s = 8$

FIGURE 1. **ETC** of Model-1 and Model-2 when $(p, r, S, c_0, c_1, c_2, c_3, c_4, c_5) = (0.4, 0.4, 20, 20, 1, 5, 1, 2, 3)$

system performance are also investigated through numercal example. Cost analysis for the models are numerically investigated. We observed that the loss probability of the primary arrival will be high in model-1 in comparison with model-2.

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