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ON RANDOM POLYNOMIALS-I: A SURVEY

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ABSTRACT. Herein, we summarise a set of fundamental results of various random polynomials including algebraic polynomials, binomial polynomials, trigonometric polynomials, Weyl polynomials, hyperbolic polynomials, etc. This article contains a survey of selected results in random polynomials on the real zeros and distribution of zeros of random polynomials. A few applications are also presented.

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1 INTRODUCTION

An exciting topic in statistical physics is the study of properties of the zeros of random polynomials ([9], [10]). In fact, the positions of quantized vortices in 2D rotating ideal Bose gas can be mapped to the zeros of the random polynomial depicting the atomic state. Further the zeros of random polynomials are popping up in many areas of physics as peak points of signals; vacua in compactifications of M-Theory on Calati-Yau manifolds with flux; extremal black holes, peak points of galaxy distributions etc. Algebraic Geometers are interested in zeros of holomorphic sections of any positive holomorphic line bundle over any Kähler manifold. We try to provide a life jacket to swim in the theory of random polynomials.

Definition 1.1. Let I be an interval of the real axis. Define smooth functions $f_0, f_1, \ldots, f_n : I \to \mathbb{R}$; and consider the random variables X_k, with mean μ_k and variance $\sigma_k^2 > 0$, $k = 0, 1, \ldots, n$, defined on the same probability space $(\Omega, \mathscr{F}, P)$. We form a linear

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combination

$$F(t) \equiv F_n(\omega, t) = \sum_{k=0}^{n} X_k(\omega) f_k(t) \tag{1.1}$$

This is called a random function. The value of this random function at each point $t \in I$ is a random variable. When $f_k(t) = t^k, k = 0, 1, \ldots, n$, and $f_k(t) = \sin(kt)$ or $\cos(kt)$ or $A_k \sin(kt) + B_k \cos(kt)$, then F is known as a random algebraic polynomial(RAP) and a random trigonometric polynomial(RTP) respectively.

In particular, F is called Gaussian random polynomial when X_k's are normal random variables with mean μ_k and variance $\sigma_k^2 > 0$. It is known that the real Gaussian random polynomials, whose coefficients are independent identically distributed (IID) standard normal random variables $\mathcal{N}(0; 1)$, have zeros concentrated on -1 and +1. The zeros of complex Gaussian random polynomials really tend to concentrate on unit circle in the complex domain. Further if we orthonormalize the polynomials on the boundary of any simply connected bounded domain in the complex plane, the zeros of the associated random polynomials concentrate on the boundary. This situation triggers a question whether there are random polynomials whose zeros are uniform on the complex plane. Surprisingly the answer is positive with suitable choice of inner product and metric. This is a classical topic of research in probabilistic analysis. These observations motivate us to study, summarise, and explore the results as well as new avenues for further investigation in the theory of random polynomials. Some times it may become a hard nut to crack.

Let us denote the number of real zeros of random polynomial F with degree n as $\mathcal{N}_n(F, \mathbb{R})$, its expectation as $\mathbb{E}[\mathcal{N}_n(F, \mathbb{R})]$, and the variance as $\mathbb{V}[\mathcal{N}_n(F, \mathbb{R})]$.

The study on random polynomials has been broadly focused on the following topics in this survey.

- Bounds for the number of real zeros
- Average number of real zeros with moments conditions on the random coefficients
- Maxima and minima of number of real roots of random algebraic curves
- Variance of the number of real zeros with moments conditions on the coefficients
- Central Limit Theorem and Large Deviation Principle on the number of real zeros
- Average number of points of inflection
- Other random polynomials and applications in pure mathematics, mathematical physics, social sciences, computer science etc.

Our purpose is to give a bird's eye view of the above topics. We may not cover all contributions. In literature, one may find many interesting contributions by many passionate researchers.

Block and Polya [8] have initiated the study of the expectation of the number of real zeros of a random polynomial in the thirties. Further investigations have been made by

Littlewood and Offord [55]. However, the first sharp result is by Kac (see [50], [51]), who gives the asymptotic average value

$$\mathbb{E}[\mathcal{N}_n(F,\mathbb{R})] \sim \frac{2}{\pi}\log n \text{ as } n \to +\infty$$

when the coefficients of univariate polynomial $F = \sum_{k=0}^{n-1} X_k t^k$ of degree $n-1$ are Gaussian centered independent random variables $\mathcal{N}(0;1)$. One may consult the books by Bharucha-Reid and Sambandham [1] for historical developments and Farahmand [31] for innovative devices to capture the behavior of random polynomials. This is the first time that these methods appear in book form. In [31], he has obtained the number of *sharp crossings, maxima below a level* and the *exceedance measure* to compensate for the absence of a graphical image.

At the first instance, consider a random quadratic equation(QE) $ax^2+bx+c=0$ where a, b, and c are IID uniform random variables over $[0,1]$. The probability of real roots of this equation is an exercise in Karlin [[53], p.36 Problem 1]. It is cleverly evaluated using shadow method as

$$\begin{aligned}
P(\text{Real roots of QE}) &= P(R) \\
&= P(b^2 > 4ac) \\
&= 1 - P(b^2 < 4ac) \\
&= 1 - \iiint_{b^2<4ac} \mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a \\
&= 1 - \left\{ \int_{\frac{1}{4}}^{1}\int_{0}^{1}\int_{\frac{b^2}{4a}}^{1} \mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a + \int_{0}^{\frac{1}{4}}\int_{0}^{2\sqrt{a}}\int_{\frac{b^2}{4a}}^{1} \mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a \right\} \\
&= \frac{5+3\log 4}{36}.
\end{aligned}$$

Also when a, b, and c are IID uniform random variable over $[-1,1]$,

$$\begin{aligned}
P(\text{Real roots of QE}) &= \frac{1}{2} + \frac{1}{2}\left\{ \int_{0}^{\frac{1}{4}}\int_{0}^{1}\int_{2\sqrt{ac}}^{1} \mathrm{d}b\,\mathrm{d}a\,\mathrm{d}c + \int_{\frac{1}{4}}^{1}\int_{0}^{\frac{1}{4c}}\int_{2\sqrt{ac}}^{1} \mathrm{d}b\,\mathrm{d}a\,\mathrm{d}c \right\} \\
&= \frac{1}{2} + \frac{1}{2}\frac{5+3\log 4}{36} \\
&= \frac{41}{72} + \frac{\log 2}{12}.
\end{aligned}$$

Here we could get the exact value for the probability of real roots of a quadratic equation. Further Hamblen [41] in his Ph.D. Dissertation has derived the probability of real roots and complex roots of quadratic equation $x^2+\xi_1 x+\xi_2=0$ when (ξ_1,ξ_2) follow general bivariate normal distribution with means μ_1,μ_2 standard deviation σ_1,σ_2 and constant correlation ρ

and gamma type density $\exp\{-x-y\}, x \geq 0, y \geq 0$. The probabilities are expressed in terms of computable integrals.

As a generalisation, consider the cubic equation(CE) $ax^3 + bx^2 + cx + d = 0$ where a, b, c, and d are IID uniform over $[-1, +1]$. In view of homogeneity among the random coefficients a, b, c, and d, the probability of real roots of this equation is given by

$$P(\text{Real roots of CE}) = \frac{1}{16} \int\int\int\int_{[-1,+1]^4} I(\Delta > 0) \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d.$$

where $\Delta = -27a^2d^2 + 18abcd - 4ac^3 - 4b^3d + b^2c^2$. But it is open to evaluate this integral. For random cubic polynomial, Soileau [91] has discussed a cubic equation of the type $x^3 + \xi_1 x + \xi_2 = 0$. A well-known result in the theory of equations gives the following information about the roots of $x^3 + \xi_1 x + \xi_2 = 0$: if

$$\frac{\xi_2^2}{4} + \frac{\xi_1^3}{27} \begin{cases} > 0, \text{then there are exactly one real root and two conjugate imaginary roots;} \\ = 0, \text{then there are exactly one real root and two conjugate imaginary roots;} \\ < 0, \text{then there are three distinct real roots.} \end{cases}$$

These three conditions define a disjoint partition of Ω into three events, $\mathcal{D}, \mathcal{S}$, and $\mathcal{K}$ respectively. Since $\mathcal{S}$ is a zero-probability event, it is omitted and considered the events $\mathcal{D}$ and $\mathcal{K}$, both of which are assumed to have nonzero probabilities. Then the densities $h(u, v|\mathcal{D})$ and $h(u, v, |\mathcal{K},)$ have been calculated in [91] .

The degree of difficulty increases when the polynomial of degree increases. When the degree is n, we have to use calculus carefully as enunciated by Kac [50], and, Edelman and Kostlan [27]. We make a modest attempt to review recent different developments.

2 HOW TO COUNT THE NUMBER OF REAL ZEROS?

Now our journey starts on a rosy garden. First let us state the Kac's counting formula. For this purpose, we state the known definition.

Definition 2.1. Let $F : [a, b] \to \mathbb{R}$ be a C^1−function. Then we say that F is convenient if the following conditions are satisfied.

- $F(a) \cdot F(b) \neq 0$,
- if $F(t) = 0$, then $F'(t) \neq 0$.

In 1637, Descartes recorded his famous rule of counting real roots of algebraic equations on p.373 of 'gé0métrie'. Next, in 1829 Sturm gave the most precise result on counting the number of roots of an equation. The introduction of random character in the equations has created a challenge. At first, Waring in 1782 and Sylvester in 1864 used probabilistic method in random polynomials. During 1932, Bloch and Pólya, Littlewood and Offord in 1938 started the game of calculating the number of roots of random polynomials in a very tricky way. But Kac is the first person to give elegant and beautiful formula for finding average number of real zeros of random algebraic polynomial with centered Gaussian coefficients.

2.1 Kac's Counting Method-1 Let $F : [a,b] \to \mathbb{R}$ be a convenient C^1-function. Then

$$\mathcal{N}_n(F,[a,b]) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_a^b \mathbb{1}_{\{|F(t)|<\varepsilon\}}|F'(t)|\mathrm{d}t.$$

As in Kac[50], let us start with Dirac delta function

$$\int_{\mathbb{R}}\delta_0(x)\mathrm{d}x = 1, \quad \text{where } \delta_0(x) = \infty \text{ when } x = 0;\ 0 \text{ when} x \neq 0.$$

and its approximation

$$\eta_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad x \to \frac{1}{2\varepsilon}\mathbb{1}_{\{|x|<\varepsilon\}}$$

which is by the change of variables $x \mapsto F(t)$, gives

$$\int_{\mathbb{R}}\delta_0(F(t))|F'(t)|\mathrm{d}t = \int_{I_k}\delta_0(F(t))|F'(t)|\mathrm{d}t = 1,$$

for $k = 1, 2, \ldots, n$. We take $\cup_{k=1}^n I_k \subset [a,b]$. Now summing over k on both sides of the last equality, and using the fact that F crosses level $u = 0$ in each of the subinterval I_k at $t = s_k$, it follows

$$\int_{[a,b]}\delta_0(F(t))|F'(t)|\mathrm{d}t = n$$

which leads to Kac's counting formula. The expectation of number of real zeros of polynomials with IID random coefficients is given by

$$\mathbb{E}[\mathcal{N}_n(F,[a,b])] = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_a^b \mathbb{E}\left[\mathbb{1}_{\{|F(t)|<\varepsilon\}}|F'(t)|\right]\mathrm{d}t.$$

Let $G(x,y) = \mathbb{1}_{\{|x|<\varepsilon\}}|y|$. Then the above formula can be written as

$$\mathbb{E}[\mathcal{N}_n(F,[a,b])] = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_a^b \mathbb{E}[G(F(t),F'(t))]\mathrm{d}t.$$

Pemantle in [65] has remarked that Kac has said, roughly (this doesn't require Gaussian assumption): F has a zero in $[t-\varepsilon, t+\varepsilon] \Leftrightarrow |F(t)| \leq \varepsilon|F'(t)|$: Letting $\varepsilon \to 0$ and multiplying by ε^{-1}, the expectation is given by

$$(\text{density of } F(t) \text{ at } 0)\cdot\mathbb{E}\{|F'(t)|/F(t) = 0\}.$$

For any Gaussian pair (X,Y) with covariances $\begin{bmatrix} A & B \\ B & C\end{bmatrix}$ the density of X at zero is $1/\sqrt{A}$ and

$$\mathbb{E}[|Y|/X = 0] = \sqrt{\Delta/A}$$

where $|\Delta|$ is the determinant $AC - B^2$. Thus

$$(\text{density of } X \text{ at } 0) \cdot \mathbb{E}[|Y|/X = 0] = \frac{\sqrt{\Delta}}{A}.$$

The vector $(F(t), F'(t))$ has covariance structure (i.e. by taking $K(s,t) = \mathbb{E}[F(s)F(t)]$)

$$\begin{bmatrix} K(t,t) & K_s(s,t)|_{s=t} \\ K_s(s,t)|_{s=t} & K_{st}(s,t)|_{s=t} \end{bmatrix}$$

which leads to

$$\frac{\sqrt{\Delta}}{A} = \sqrt{\partial^2_{st} \log K(s,t)}.$$

Here

$$\begin{aligned} A(t) = K(x,y)|_{x=y=t} &= \sum_{k=0}^{n-1} t^{2k}, \\ C(t) = K_{xy}(x,y)|_{x=y=t} &= \sum_{k=1}^{n-1} k^2 t^{2k-2}, \quad \text{and} \\ B(t) = K_x(x,y)|_{x=y=t} &= \sum_{k=1}^{n-1} k t^{2k-1} \end{aligned}$$

Theorem 2.2. *(Kac-Rice Formula) Let $\mathbb{E}[\mathcal{N}_n(F, I)]$ denote the expected number of zeros of F in the real interval I. Then $\mathbb{E}$ is a measure with density*

$$\rho(t) = \frac{1}{\pi} \left. \sqrt{\partial^2_{xy} \log K(x,y)} \right|_{x=y=t}.$$

2.2 Kac's Counting Method-2 We know from the basic calculus course that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin(\xi\alpha)}{\xi} \mathrm{d}\xi &= \pi \operatorname{sgn}(\alpha) \\ \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \cos \eta y) \eta^{-2} \mathrm{d}\eta &= |y| \end{aligned}$$

Using these two identities, Kac [52] has derived a formula in an analytically elegant way for the number of real zeros of a random polynomial.

$$\mathcal{N}_n(F, [a,b]) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_a^b \cos(\xi F(t)) |F'(t)| \mathrm{d}t.$$

2.3 Edelman's Geometric Method-3 Now we present some basic geometric arguments and show their relationship with real roots of certain deterministic smooth functions. Let S_n be the surface of the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Edelman and Kostlan count zeros for Gaussian random polynomials in a geometric way via the Crofton formula which expresses the arc length of the curve γ in terms of an integral over the space of all oriented lines.

Definition 2.3. Let P be a point on the sphere S^n, the corresponding equator $P_\perp$ is the set of points of S^n which lie on the plane through origin that is perpendicular to the line passing through the origin and the point P.

Definition 2.4. Let $\gamma(t)$ be a rectifiable curve on the sphere S^n parametrized by $t \in \mathbb{R}$, then $\gamma_\perp := \{P_\perp | P \in \gamma\}$ is the set of equators of the curve γ.

Let us explain how random polynomial can be viewed in this approach. Consider the curve Γ in $\mathbb{R}^{n+1}$ defined by

$$\Gamma = \{x_0 = 1, x_1 = t, \ldots, x_n = t^n\}, \quad t \in \mathbb{R},$$

Projecting on $\mathbb{S}^n$, one obtains

$$\gamma = \left\{ x_0 = \frac{1}{\left(\sum\limits_{j=0}^{n} t^{2j}\right)}, \ldots, x_n = \frac{t^n}{\left(\sum\limits_{j=0}^{n} t^{2j}\right)} \right\}.$$

Intersecting γ by a random great circle is equivalent to counting the number of real zeros of

$$\sum_{j=0}^{n} X_j t^j.$$

We define $|\gamma_\perp|$ as the area swept out by γ counting multiplicities. If γ is a rectifiable curve then

$$\frac{|\gamma_\perp|}{\text{area of} S^n} = \frac{|\gamma|}{\pi}.$$

This observation leads to

$$\mathbb{E}[\mathcal{N}_n(F, \mathbb{R})] = \frac{|\gamma|}{\pi}.$$

Using calculus to obtain the integral formula for the length of γ and hence the expected number of zeros of a random polynomial is derived as in Theorem 2.2.

3 BOUNDS FOR THE NUMBER OF REAL ZEROS

Littlewood and Offord[55] have laid a strong foundation in the field of random polynomials. They have considered the random coefficients with probability distributions of the types given below.

(i) Standard Gaussian distribution $\mathscr{N}(0, 1)$
(ii) Uniform over $[-1, +1]$
(iii) Symmetric Bernoulli $P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$ with $X_0 = 1$ a.s.

Theorem 3.1. *For the above cases, the lower and upper bounds of the number of real zeros are respectively, for sufficiently large n,*

$$\mathcal{N}_n < 25(\log n)^2 \quad \textit{except for a set of measure at most} \quad 12(\log n)/n$$

and

$$\mathcal{N}_n > \frac{\alpha \log n}{\log\log\log n} \text{ except for a set of measure at most } \frac{A}{\log n}.$$

Notice that the measure of the exceptional set depends on the degree of the polynomial. If it is independent of the degree of the polynomial, it is called **strong result.**

Littlewood and Offord [56], Samal[72], Evans[29], and, Samal and Mishra [73], [74], [75] have assumed that the coefficients X_i are independent and identically distributed random variables. For dependent coefficients, Sambandham [81] has considered the **upper bound** for $\mathcal{N}_n$ in the case when the $X_i, i = 0, 1, \dots, n$, are normally distributed with mean zero and joint density function

$$|M|^{1/2}(2\pi)^{-(n+1)/2}\exp\{-(1/2)\boldsymbol{a}'M\boldsymbol{a}\}, \tag{3.1}$$

where M^{-1} is the moment matrix with $\sigma_i = 1, \rho_{ij} = \rho, 0 < \rho < 1, (i \neq j), i, j = 0, 1, \dots, n$ and $\boldsymbol{a}'$ is the transpose of the column vector $\boldsymbol{a}$. Also, Uno and Negishi [97] obtained the same result as Sambandham in the case of the moment matrix with $\sigma_i = 1, \rho_{ij} = \rho_{|i-j|}$, $(i \neq j), i, j = 0, 1, \dots, n$, where ρ_j is a nonnegative decreasing sequence satisfying $\rho_1 < 1/2$ and $\sum_{j=1}^{\infty} \rho_j < \infty$ in (3.1).

The **lower bound** for $\mathcal{N}_n$ in the case of dependent normally distributed coefficients was estimated by Renganathan and Sambandham [69], and, Nayak and Mohanty[61] under the same condition of Sambandham[81]. Uno [97] has corrected results of the above papers and obtained the result for the lower bound. Additionally, Uno [98] has estimated the strong result for this particular problem in the sense of Evans[29]. Samal and Mishra [73] have considered the random algebraic equation $\sum_{i=0}^{n} X_i x^i$ where the X_i's are independent random variables with a common characteristic function

$$\phi(t) = \exp(-C|t|^{\alpha}), \alpha > 1, \text{ and } C, \text{a positive constant.}$$

Then for $n > n_0$,

$$\mathcal{N}_n > (\mu \log n)/(\log\log n)$$

outside a set of measure at most $\mu' / \{\log[(\log n_0)/(\log\log n_0)]\}^{\alpha-1}$.

Their result is true for all $\alpha > 1$, but its importance lies in the range $1 < \alpha < 2$ when the variance is infinite. However, when $\alpha = 2$ one may get the corresponding result of Evans [29] as a special case, although their exceptional set is larger than Evans[29].

4 POLYNOMIALS WITH SYMMETRIC BERNOULLI COEFFICIENTS

4.1 Zeros Counting Method-4 Erdös and Offord have used the following approximation $\mathcal{N}_n^*(F; [a, b])$ to calculate $\mathcal{N}_n(F; [a, b])$ where

$$\mathcal{N}_n^*(F; [a, b]) = \begin{cases} 1 & \text{if } F(a)F(b) < 0 \\ \frac{1}{2} & \text{if } F(a)F(b) = 0 \\ 0 & \text{if } F(a)F(b) > 0. \end{cases}$$

Theorem 4.1. *(Erdös and Offord* [28]*) When X_k's in* (1.1) *assume +1 and -1 with equal probabilities, the number of real roots of most of the random equations* $\sum_{k=0}^{n-1} X_k t^k = 0$ *is*

$$\frac{2}{\pi}\log n + o\{(\log n)^{2/3}\log\log n\}$$

and the exceptional set does not exceed a proportion $o\{(\log\log n)^{-1/2}\}$ *of the total number of equations.*

The analysis used in this paper is very tricky and noteworthy. But in his doctoral dissertation, Stevens [92] has remarked as a footnote regarding a computational error in Lemma 14 in [28].

5 POLYNOMIALS WITH BINOMIAL COEFFICIENTS

Nezakatt and Farahmand [63] have derived asymptotic estimate for $\mathbb{E}(\mathcal{N}_n(\mathbb{R})$ of random algebraic polynomial F. Let n be separated into two multipliers such that $n = k \cdot m$, where $k = f(n)$ is an integer and increasing function of n, such that $f(n) = O(\log n)^2$. The random variables X_j, $j = 0, 1, 2, \ldots, n-1$ are normally distributed with means zero and $\mathrm{var}(X_j) = \binom{k-1}{j-ik}$, $j = ik, ik+1, \ldots, (i+1)k-1, i = 0, 1, \ldots, m-1$. Then the expected number of real zeros of F is

$$\mathbb{E}[\mathcal{N}_n(\mathbb{R}] \sim \sqrt{k-1} \text{ as } n \to \infty.$$

Farahmand and Sambandham [35] have studied the behavior of F when the mean and variance of coefficients are

$$\mathbb{E}[X_j] = \binom{n}{j}\mu^{j+1} \quad\text{and}\quad \mathbb{V}[X_j] = \binom{n}{j}\sigma^{2j}.$$

They have obtained the following estimates for average number of real zeros.

$$\begin{aligned}
\text{Case-(i)}\ \ \mathbb{E}[\mathcal{N}_n(F;(-\infty,0)] &= \mathbb{E}[\mathcal{N}_n(F;(0,\infty)] \\
&= \frac{\sqrt{n}}{2}\ \text{ when } \mu = 0 \text{ and } \sigma^2 > 0 \\
\text{Case-(ii)}\ \ \mathbb{E}[\mathcal{N}_n(F;(0,\infty)] &\ \begin{cases} = O(1), & \text{if } \mu = \sigma^2 > 1, \\ \sim \left(\dfrac{\sqrt{n}}{2}\right)\left\{1 - \arctan\left(\dfrac{2\sqrt{\mu}}{1-\mu}\right)\right\}, & \text{if } 0 < \mu = \sigma^2 < 1. \end{cases} \\
\text{Case-(iii)}\ \ \mathbb{E}[\mathcal{N}_n(F;(-\infty,0)] &\sim \frac{\sqrt{n}}{2}\ \text{ if } x \text{ is negative and for every } \mu = \sigma^2
\end{aligned}$$

6 POLYNOMIALS WITH GAUSSIAN AND NON-GAUSSIAN COEFFICIENTS

We present here some note-worthy results when the coefficients are Gaussian and non-Gaussian with different moment assumptions in both RAP and RTP.

Kac's polynomial: A random algebraic polynomial of the form $F(t) = \sum_{n=0}^{n-1} X_k t^k$ where the coefficients X_k are IID Gaussian random variables of mean zero and variance one is called a **Kac Polynomial**.

Sambandham Algebraic Polynomial(SAP): A random algebraic polynomial of the form $F(t) = \sum_{n=0}^{n-1} X_k t^k$ where the coefficients X_k are (i) identically distributed dependent Gaussian random variables of mean zero, variance one, and correlation between any two random variables ρ and (ii) identically distributed dependent Gaussian random variables of mean zero, variance one, and correlation between X_i and X_J $\rho^{|i-j|}$ is called a Sambandham algebraic polynomial.

Sambandham Trigonometric Polynomial(STP): A random trigonometric polynomial of the form $T(t) = \sum_{n=1}^{n} X_k \cos(k\theta)$ where the coefficients X_k are (i) identically distributed dependent Gaussian random variables of mean zero, variance one, and correlation between any two random variables ρ and (ii) identically distributed dependent Gaussian random variables of mean zero, variance one, and correlation between X_i and X_J $\rho^{|i-j|}$ is called a Sambandham trigonometric polynomial.

After computation of Kac's counting formula gives the asymptotic value when the coefficients are standard normal random variables as

$$\mathbb{E}[\mathcal{N}_n(F,\mathbb{R})] \sim \frac{2}{\pi}\log n \ \text{ as } n \to \infty.$$

Kac has also obtained the same result when the coefficients are uniformly distributed.

Recently Edelman and Kostlan [27] have established a nice formula to estimate the average number of real zeros of RAP with standard Gaussian coefficients.

$$\mathbb{E}[\mathcal{N}_n(F,\mathbb{R})] \sim \frac{2}{\pi}\log n + C + \frac{2}{n\pi} + O(\frac{1}{n^2}) \ \text{ as } n \to \infty \ \text{ where } \ C = 0.6257358072\ldots.$$

On the other hand Ibragimov and Maslova([47],[48]) have improved Erdös and Offord method for variables with mean zero (and non-zero mean) and belong to the domain of attraction of normal law and established the asymptotics for the mean (same as Kac's result in the zero mean case) and variance of real zeros of RAP.

They have shown that

$$\mathbb{V}(\mathcal{N}_n(F,\mathbb{R})) \sim \frac{4}{\pi}(1-\frac{2}{\pi})\log n \text{ as } n \to \infty.$$

Apart of this, they have also established an CLT for the number of real roots of Kac polynomials.

In 1988, Wilkins[102] has established another interesting result which paves a way for further investigation in other polynomials.

Theorem 6.1. *Let $\mathbb{E}(\mathcal{N}_n)$ be the expected number of real zeros of a polynomial of degree n whose coefficients are independent random variables, normally distributed with mean 0* *and variance 1. Then an asymptotic expansion for* $\mathbb{E}(\mathcal{N}_n)$ *is of the form*

$$\mathbb{E}(\mathcal{N}_n) = \frac{2}{\pi}\log(n+1) + \sum_{p=0}^{\infty} A_p(n+1)^{-p},$$

in which $A_0 = 0.625735818, A_i = 0, A_2 = -0.24261274, A_3 = 0, A_4 = |0.08794067, A_5 = 0$. *The numerical values of* $\mathbb{E}(\mathcal{N}_n)$ *calculated from this expansion, using only the first four, or six, coefficients, agree with previously tabulated seven decimal place values* $(1 < n < 100)$ *with an error of at most* 10^{-7} *when* $n \geq 30$, *or* $n \geq 8$.

Theorem 6.2. *(Ibragimov and Maslova*[48]*) Let F be an random algebraic polynomial whose coefficients belonging to the domain of attraction of Gaussian law with* $\mathbb{E}(X_j) = a \neq 0$. *Then*

$$\begin{aligned} \mathbb{E}[\mathcal{N}_n(F,(-\infty,0))] &\sim \frac{1}{\pi}\log n, \quad as \;\; n \to \infty, \\ \mathbb{E}[\mathcal{N}_n(F,(0,\infty))] &\sim o(\log n), \quad as \;\; n \to \infty. \end{aligned}$$

Stevens [92] in 1965 for the first time has obtained an upper bound for the variance of the number of real zeros of a random algebraic polynomial with IID real-valued standard Gaussian coefficients. The upper bound is

$$\mathbb{V}[\mathcal{N}_n(\mathbb{R})] < 32\mathbb{E}[\mathcal{N}_n(\mathbb{R})] + 2.5 + (\log n)^2/\sqrt{n}, \quad \text{for } n \geq 32.$$

Fairly [30] in 1968 has computed the exact variances in this case and in the case with the coefficients of the random algebraic polynomial take the values ± 1 with equal probabilities for polynomials of degree up to 11. In 1974, Maslova [59] has considered the case when the random algebraic polynomial has IID real-valued coefficients $\{X_i\}$ such that $P[X_i = 0] = 0, \mathbb{E}[X_i] = 0$, and $\mathbb{E}[|X_i|^{2+s}| < \infty$ for some $s > 0$. For this case she has obtained the asymptotic variance as

$$\mathbb{V}[\mathcal{N}_n(\mathbb{R})] \sim \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right)\log n, \text{as } n \to \infty \;\; \text{and} \;\; \frac{\mathcal{N}_n(\mathbb{R}) - \mathbb{E}[\mathcal{N}_n(\mathbb{R})]}{\sqrt{\mathbb{V}[\mathcal{N}_n(\mathbb{R})]}} \xrightarrow{d} \mathscr{N}(0,1), \quad \text{as} \;\; n \to \infty,$$

where d denotes convergence in distribution. We note that Nguyen and Vu [64] have recently generalized Maslova's results to hold under the assumption that the distribution of coefficients are independent (but not necessarily identically distributed) and have moderate growth.

Qualls [67] has investigated the only known variance of the number of real roots of a random trigonometric polynomial of the type

$$\mathcal{T}_n(\theta) = \sum_{k=0}^{n}(A_i\cos(i\theta) + B_i\sin(i\theta))$$

which is a stationary stochastic process and for which a special theorem has been developed by Cramer and Leadbetter [15]. Farahmand [33] has derived a nice result of the variance of the number of real zeros of $\mathcal{T}_n(\theta)$.

Theorem 6.3. *Let $X_1, X_2, \ldots, X_n$ be the independent random variables following a Gaussian distribution with mean zero. Then the variance of the number of real roots of $\mathcal{T}_n(\theta) = \sum_{k=1}^{n} X_k \cos(k\theta)$ is*

$$\mathbb{V}[\mathcal{N}([0, 2\pi])] = O[n^{24/13}(\log n)^{16/13}].$$

In general terms, random functions of particular interest are random trigonometric polynomials of the form

$$T_n = \sum_{k=1}^{n} X_k \cos(k\theta) \text{ or } \sum_{k=1}^{n} A_k \cos(k\theta) + Y_k \sin(k\theta),$$

where $\{A_k\}, k = 1, 2, 3, \ldots$ and $\{B_k\}, k = 1, 2, 3, \ldots$, since the distribution of the zeros of such polynomials occurs in a wide range of problems in science and engineering.

The asymptotics of the mean number of real zeros of random trigonometric polynomials with independent standard and centered Gaussian coefficients was first obtained by Dunnage in [22], where it is shown that this number is asymptotically proportional to the degree n of the considered polynomial. Since then, the researchers have engaged in estimating the level sets of random trigonometric polynomials in various directions.

In literature, the only known results in the case of dependent Gaussian random variables as in STP have only been derived in Sambandham[82] , Renganathan and Sambandham [70] which focus on the two particular and somehow “extreme” cases of a constant correlation $\mathbb{E}(X_i X_j) = \rho, 0 < \rho < 1$ and a geometric correlation $\mathbb{E}(X_i X_j) = \rho^{|i-j|}$. In both cases, it is shown that the expected number of real roots lying in $[0, 2\pi]$ has the same value as

$$\lim_{n\to\infty} \frac{\mathbb{E}[\mathcal{N}_n(F, [0, 2\pi])]}{n} = \frac{2}{\sqrt{3}}.$$

Further it is interesting to note the contribution in Sambandham [76] for many more such results. Notably, when the coefficients are dependent Gaussian random variables in SAP with correlation ρ,

$$\mathbb{E}[\mathcal{N}(F, \mathbb{R})] \sim \frac{1}{\pi} \log n \text{ as } n \to \infty$$

and when the correlation between X_i and X_j is $\rho^{|i-j|}, 0 < \rho < 1/2$

$$\mathbb{E}[\mathcal{N}(F, \mathbb{R})] \sim \frac{2}{\pi} \log n \text{ as } n \to \infty.$$

Sambandham et al. [83] have obtained

$$\mathbb{V}(\mathcal{N}(F_n, \mathbb{R})) \sim \begin{cases} \dfrac{2}{\pi}\left(1 - \dfrac{2}{\pi}\right) \log n & \text{when correlation is } \rho, \\ \dfrac{4}{\pi}\left(1 - \dfrac{2}{\pi}\right) \log n & \text{when correlation is } \rho^{|i-j|}, \text{as } n \to \infty. \end{cases}$$

Thangaraj and Renganathan [95] have shown that the bound the random polynomial considered by Maslova[59] is $2\pi^{-1}(1 - \pi^{-1})$ for sufficiently large n.

7 WEIGHTED RANDOM POLYNOMIALS

Further Sambandham [77] has considered the following weighted random algebraic polynomial. As a by-product, we get the maxima of random algebraic curves.

Theorem 7.1. *Let*

$$\sum_{k=0}^{n} k^p X_k t^k \tag{7.1}$$

where $\{X_k, k = 0, 1, \ldots\}$ *is a sequence of dependent normal random variables with mean zero, variance one and the correlation between any two random variables is* $\rho, (0 < \rho < 1)$. *Then the average number of real zeros is aymptotic to*

$$(2\pi)^{-1}[1 + (2p + 1)^{1/2}] \log n, \ \ \textit{when}\ \ 0 \leq p < \infty\ \ \textit{for large}\ \ n. \tag{7.2}$$

This result subsumes many of the known results. When $p = 0$, that is, for the polynomial $\sum_{k=0}^{n} X_k t^k$ the average number of real zeros is estimated in Sambandham [78] and this asymptotic average is $\pi^{-1} \log n$. Since the maxima or minima of $\sum_{k=0}^{n} X_k t^k$ is only half of the average number of real zeros of $\sum\limits_{k=0}^{n} kX_k t^k$ by giving $p = 1$ in Theorem 7.1, we get the average number of maxima of $\sum_{k=0}^{n} X_k t^k$. This average has been already estimated in Sambandham and Bhatt [79] and its value is $(4\pi)^{-1}[1 + \sqrt{3}] \log n$.

When the random variables are independent and normally distributed Das [19] has estimated the average number of real zeros of (7.1) and the asymptotic average is $\pi^{-1}[1 + (2p + 1)^{1/2}] \log n$. Under the same condition the average number of maxima of $\sum_{k=0}^{n} X^k t^k$ is $(2\pi)^{-1}[1 + \sqrt{3}] \log n$ and the average number of real zeros of $\sum_{k=0}^{n} X_k t^k$ is $(2/\pi) \log n$. These two results are respectively in Das [17] and Kac [50]. We note that when the random variables are independent the average number of zeros and the average number of maxima is twice that of the case when the random variables are dependent normal with a constant correlation.

It is also pertinent to note that Maslova[59] has also obtained the **maxima and minima** of the number of real zeros of random algebraic polynomial with IID coefficients.

Theorem 7.2. *(Maslova*[59]*) Let* $\{X_j\}$ *be an IID sequence of random variables with* $P(X_j = 0) = 0, \mathbb{E}(X_j) = 0,$ *and* $\mathbb{E}(X_j^2) < \infty$. *Let* $\mathbb{E}(M_n)$ *and* $\mathbb{E}(m_n)$ *be the average number of maxima and minima of* $F(t) = \sum_{j=0}^{n} X_j t^j$. *Then*

$$\mathbb{E}[M_n] \sim \mathbb{E}[m_n] \sim (2\pi)^{-1}\{\sqrt{3} + 1\} \log n, \ \ \textit{as}\ \ n \to \infty.$$

Sambandham and Maruthachalam [80] have considered a trigonometric polynomial of the type with constantly correlated Gaussian random variables X_k,

$$T_n(\theta, \omega) = \sum_{k=1}^{n} k^p X_k(\omega) \cos k\theta. \tag{7.3}$$

and obtained the following asymptotic estimate.

Theorem 7.3. *(Sambandham and Maruthachalam*[80]*) The probable number of real zeros in the interval* $0 \le \theta \le 2\pi$ *all except an exceptional set of functions* (7.3) *with a probability measure in* Ω *does not exceed* $n^{-2\varepsilon_1}$, *where* $0 < \varepsilon_1 < \frac{1}{13}$ *is asymptotically equal to*

$$2n\left(\frac{2p+1}{2p+3}\right)^{1/2} + O(n^{(11/13)+\varepsilon_1})$$

when n is large.

When $p = 0$, one gets Sambandham's result for dependent case in [82] and independent case by Dunnage's classical paper, and obtains the following asymptotic estimate for the probable number of real zeros.

Theorem 7.4. *(Dunnage*[22]*) The probable number of real zeros in the interval* $0 \le \theta \le 2\pi$ *all except an exceptional set of functions* (7.3)*(*$p = 0$*) with a measure does not exceed* $(\log n)^{-1}$, *is asymptotically equal to*

$$\frac{2n}{\sqrt{3}} + O(n^{11/13}(\log n)^{3/13})$$

when n is large.

Das [20] has discussed random trigonometric polynomials

$$\sum_{k=1}^{n} k^p (X_{2k-1}(\omega)\cos k\theta + X_{2k}\sin k\theta), p > -\frac{1}{2}$$

and X_k's are independent standard Gaussian random variables and Qualls [67] has discussed similar polynomials.

In a series of papers Wilkins [99],[100], [101], and [102] have established a new way of thinking in the derivation of average number of real zeros of random polynomials. He has obtained a convergent series representation for the average number of real zeros for algebraic, trigonometric, hyperbolic polynomials with weights. We just state some of his results to think beyond the box.

Theorem 7.5. *Suppose that* $X_i (i = 1, 2, \ldots, n)$ *are independent, normally distributed random variables with mean 0 and variance 1, and that* $\mathcal{N}_{np}$ *is the mean value of the number of zeros on the interval* $(0, 2\pi)$ *of the random trigonometric polynomial*

$$T_n(x) = \sum_{i=1}^{n} i^p X_i \cos(ix)$$

in which p is a nonnegative real number. If p is a nonnegative integer, there exist constants $D_{0p} = 1, D_{1p}, D_{2p}$, *and* D_{3p} *such that*

(7.4)

$$\mathbb{E}[\mathcal{N}_{np}] = (2n+1)\mu_p \sum_{r=0}^{3} (2n+1)^{-r} D_{rp} + O((2n+1)^{-3}) \text{ where } \mu_p = (2p+1)/(2p+3)^{1/2}.$$

Das [16] has shown that, for large n,

$$\mathbb{E}[\mathcal{N}_{np}] = 2\mu_p n + O(n^{1/2}), \text{ where } \mu_p = \{(2p+1)/(2p+3)\}^{1/2}.$$

When $p = 0$, Wilkins[99] has shown that the error term $O(n^{1/2})$ is actually $O(1)$. Moreover, the error term is also $O(1)$ when p is a positive integer [100]. Wilkins and Souter [101] have also derived a relation of the form (7.4) when $p = 1/2$. He has shown that (7.4) remains valid when $p = (2s+1)/2$, in which s is a positive integer. In combination with the earlier results, this implies that a relation of the form (7.4) is valid when $2p$ is any nonnegative integer, although he could not construct a unified derivation that covers the various cases in [99],[100],[101] including the above result. He could not extend his own techniques to the case in which $2p$ is not a nonnegative integer.

Recently the work of Flasche and Kabluchko [38], [39] are worth mentioning. In [37], the following result has been proved.

Theorem 7.6. *(Flasche* [37]*) The asymptotic estimate of the expected number of real zeros of random trigonometric polynomial*

$$G_n(t) = u + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (A_k \cos(kt) + B_k \sin(kt)), t \in [0, 2\pi], u \in \mathbb{R}$$

whose coefficients A_k, B_k, $k \in \mathbb{N}$ are IID random variables with mean zero and unit variance. Then

$$\lim_{n\to\infty} \frac{\mathbb{E}[\mathcal{N}_n[a,b]]}{n} = \frac{b-a}{\pi\sqrt{3}} \exp\left(-\frac{u^2}{2}\right), \text{ where } [a,b] \subseteq [0, 2\pi].$$

The references therein give more information about the latest developments.

Brania et al.[11] have proved the following result.

Theorem 7.7. *(Brania et al.* [11]*) The asymptotic average number of real zeros of a class of trigonometric polynomials of the form*

$$T(\theta) = \sum_{k=1}^{n} b_k X_k \cos(k\theta)$$

where the X_k's are independent standard normally distributed random variables and the b_k's are binomial coefficients $\binom{n}{k}^{1/2}$ is n for large n.

For the trigonometric random polynomials, i.e. $T_n(x) = \sum_{i=0}^{n} \eta_i \cos(ix)$, we note that asymptotics for the variance of the number of real zeros in $[0, 2\pi]$ has been well studied (cf. Bogomolny, Bohigas, Leboeuf [9], Farahmand[33], Grandville and Wigman [46], and, Su and Shao [93]). Similarly we mention the works of Forrester and Honner[40], Hannay[42], Shiffman and Zeldtich [90], Bleher and Di[2], that respective asymptotics for variance of the number of zeros for weighted random polynomials, i.e. random polynomials of the form $P_n = \sum_{i=0}^{n} \eta_i c_i z^i$ where either $c_i = \sqrt{\binom{n}{i}}$, or $c_i = 1/i!$.

In 2019, Yeager[105] has studied mean and the variance of the number of zeros in $\Omega \subset \mathbb{C}$ for

$$P_n(z) = \sum_{i=0}^{n} \eta_i \phi_i(z)$$

where $\{\eta_k\}$ are complex-valued random variables, and $\{\phi_i\}$ are orthogonal polynomials on the unit circle (OPUC).

Important Observation: The phenomenon of obtaining half of the existing results is noticed by Sambandham [77] because when the random variables are dependent with a constant correlation ρ, most of the random variables have a tendency to be of the same sign as they are interdependent. As the most of the random variables preserve the same sign $\sum_{k=0}^{n} X_k k^p t^k$ has a tendency of behaving like $\pm \sum_{k=0}^{n} X_k k^p t^k$. Under this condition when $t > 0$, the consecutive terms have a tendency to cancel each other and when $t < 0$ the cancellation does not become possible. This fact reduces the average number of real zeros for $t > 0$ to $o(\log n)$.

Remark 7.8. The conditions on moments have some influence in the average and variance of the number of real zeros of random algebraic polynomials. For instance, consult the works of Ibragimov and Maslova [48] for non-zero mean, Sambandham, Samal, Farahmand and their co-workers for non-zero mean cases as well as constant correlation cases, and Logan and Shepp [57][58] for Cauchy and infinite variance cases to infer how the moment assumption has the influence on the asymptotic value.

Shenker Polynomials This is an another branch of activity initiated in the theory of random polynomials. Shenker [87] has obtained an asymptotic value for average number of real zeros of random algebraic polynomial. We now call such polynomials **Shenker Polynomials.**

Theorem 7.9. *(Shenker*[87]*) Let $\{X_j\}, j = 0, 1, \ldots$ be a Gaussian stationary sequence of random variables with mean zero and variance one. For any n, the distribution of $\{X_0, X_1, \ldots, X_n\}$ is non-singular. Let $\rho_j = \mathbb{E}(X_0 X_j)$ and $\mathcal{N}_n(a, b)$ be the number of real zeros of $\sum_{j=0}^{n} X_j z^k$ in the interval (a, b). If $\sum_{j=0}^{\infty} \rho_j < 1/2$ then*

$$\mathbb{E}[\mathcal{N}_n(0, 1)] \sim \frac{1}{2\pi} \log n$$

and

$$\mathbb{E}[\mathcal{N}_n(-\infty, +\infty)] \sim \frac{2}{\pi} \log n$$

as $n \to \infty$.

Let $\mathcal{N}_n$ be the number of real zeros of $F_n(t) = \sum_{i=0}^{n} X_i t^i, -\infty < t < \infty$. Logan and Shepp[57] have shown that if the coefficients X_i are independent random variables with a common Cauchy distribution with characteristic function $\exp\{-|z|\}$, then, as $n \to \infty$,

$$\mathbb{E}[\mathcal{N}_n] \sim c \log n, \quad c = \frac{8}{\pi^2} \int_0^{\infty} \frac{x e^x}{x - 1 + 2e^{-x}} \mathrm{d}x. \tag{7.5}$$

Stevens[92] has extended the work of Kac and others by showing that if the coefficients are independent with mean zero and variance one and satisfy some additional general conditions, in place of (7.5) he has obtained $\mathbb{E}\mathcal{N}_n \sim \dfrac{2}{\pi}\log n$. We notice that $c \approx 0.7413$ and $2/\pi \approx 0.6366$. But the order of growth is same and we infer that there are more real zeros in the Cauchy case. Further Logan and Shepp [58] have considered the case where the coefficients are independent random variables with common characteristic function $\exp(-|z|^{\alpha}), 0 < \alpha \leq 2$ and obtained

$$\mathbb{E}[\mathcal{N}_n] \sim c\log n,$$

where the constant c depends on α and is given by

$$c = c(\alpha) = \frac{4}{\pi^2\alpha^2}\int_{-\infty}^{+\infty} \mathrm{d}x \log \int_0^{\infty} [|x-y|^{\alpha}/|x-1|^{\alpha}] \exp\{-y\}\mathrm{d}y.$$

Hence $c(2) = 2/\pi$ and $c(0+) = 1$. Note that $c(\alpha)$ decreases in α which leads to the tight bounds $(2/\pi) \leq c(\alpha) < 1$.

8 CORRELATION FUNCTION STUDY OF THE ZEROS OF RANDOM POLYNOMIALS

The study of correlation function of real zeros has opened a new chapter in the Theory of Random Polynomials. To start with, consider the random polynomial $F \equiv F_n(t) = \sum\limits_{k=0}^{n} X_i t^k$ where X_k are independent random variables. We make a legitimate assumption that all zeros of F are simple with probability one. Denote by $\mathcal{N}$ the empirical measure counting the real zeros of F.

$$\mathcal{N} = \sum_{\{t:F(t)=0\}} \delta_t$$

where δ_t is the unit point mass at t. The distribution of $\mathcal{N}$ can be described by its *correlation function*. We know from Hough et. al. [43] that the correlation functions of $\mathcal{N}$ are function $\rho_k : \mathbb{R}^k \to \mathbb{R}^+$ for $k = 1, 2, \ldots, n$ such that for any family of disjoint Borel subsets $B_1, B_2, \ldots, B_k \subset \mathbb{R}$,

$$\mathbb{E}\left[\prod_{i=1}^{k} \mathcal{N}(B_i)\right] = \int_{B_1}\int_{B_2}\cdots\int_{B_k} \rho_k(t_1, t_2, \ldots, t_k)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_k. \tag{8.1}$$

To evaluate this integral, we need the following extension of the Kac-Rice formula (cf. [6], [7]):

$$\rho_k(t_1, t_2, \ldots, t_k) = \int_{\mathbb{R}^k} |s_1 s_2 \cdots s_k| D_k(\mathbf{0}, \mathbf{s}, t_1, t_2, \ldots, t_k)\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k,$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ and $D_k(`.', t_1, t_2, \ldots, t_k)$ is the joint density function of the random vectors

$$(F(t_1), F(t_2)\ldots, F(t_k)) \text{ and } (F'(t_1), (F'(t_2)\ldots, F'(t_k)).$$

A key to evaluate this integral is *Coarea Formula* which is stated below.

Theorem 8.1. *(Coarea Formula) Let $B \subset \mathbb{R}^k$ be a region. Let $g : B \to \mathbb{R}^k$ be a Lipschitz function and $h : \mathbb{R}^k \to \mathbb{R}$ be an L^1–function. Then*

$$\int_{\mathbb{R}^k} \#\{\boldsymbol{x} \in B : g(\boldsymbol{x}) = \boldsymbol{y} d\boldsymbol{y} = \int_B |det\, J_g(\boldsymbol{x})| h(g(\boldsymbol{x}) d\boldsymbol{x}$$

where $J_g(\boldsymbol{x})$ is the Jacobian matrix of $g(\boldsymbol{x})$.

When the random coefficients are independent random variables, using the idea of Schur functions, Götze et al. [45] have evaluated the integral (8.1) and obtained average number of real zeros in the following cases: (1) Coefficients are uniformly distributed over $[-1, 1]$, (2) Gaussian random variables $\mathcal{N}(0, \sigma_i^2)$, and (3) Exponential random variables $e^{-t}, t > 0$.

9 FROM KAC'S MATRIX TO KAC'S POLYNOMIALS

Beresford Parlett has introduced Kac matrix to Edelman and Kostlan while going up on a staircase in Evans Hall at UC Berkeley. Without this fortuitous discussion, they would never have known that the matrix that they were studying in the context of Kac's polynomial also was named for Kac.

Definition 9.1. (Kac matrix) Kac matrix is defined as an $(n+1)\times(n+1)$ tridiagonal matrix.

$$K_{n+1} = \begin{pmatrix} 0 & n & \cdots & \cdots & \cdots \\ 1 & 0 & n-1 & \cdots & \cdots \\ \cdots & 2 & 0 & n-2 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & n-1 & 0 & 1 \\ \cdots & \cdots & \cdots & n & 0 \end{pmatrix}$$

It is also known as Clement matrix. It is the matrix that describes a random walk on a hypercube as well as the Ehrenfest urn model of diffusion. The first interesting and surprising result is its eigenvalues. The eigenvalues of K_{n+1} are integers! i.e. $\{2k - n : k = 0, 1, 2, \ldots, n\}$. Here, we note the link between the Kostlan's random polynomial

$$F(t) = \sum_{k=0}^{n} \binom{n}{k} X_k t^k, \text{where } X_k \sim \mathcal{N}(0,1) \text{ are IID random variables}$$

and the Kac matrix and Kostlan [54] has proved that the expected number of real roots of $F(t) = 0$ is exactly $\sqrt{n}$. For details of the proof, one may refer to [54].

10 OTHER RANDOM POLYNOMIALS

Flasche and kabluchko [38] have established a nice result.

Theorem 10.1. *(Flasche and Kabluchko* [38]*) Let* $\{X_k, k = 0, 1, \ldots\}$ *be IID random variables with zero mean and unit variance. Consider a random Taylor series of the form*

$$f(z) = \sum_{k=0}^{\infty} c_k X_k z^k,$$

where $\{c_k, k = 0, 1, \ldots\}$ *is a real sequence such that* c_n^2 *is regularly varying with index* $\gamma - 1$*, where* $\gamma > 0$*. Then*

$$\mathbb{E}[\mathcal{N}[0, 1-\varepsilon]] \sim \frac{\sqrt{\gamma}}{2\pi} |\log \varepsilon|, \text{ as } \varepsilon \downarrow 0,$$

where $\mathcal{N}[0, r]$ *denotes the number of real zeroes of* f *in the interval* $[0, r]$.

Flasche and Kabluchko [39] have considered the following four families of random analytic functions. Let

$$P_n(z) = \sum_{k=0}^{\infty} f_{n,k} X_k z^k, \tag{10.1}$$

where $z \in \mathbb{C}$ is a complex variable, $\{f_{n,k}\}_{(n=1,2,\ldots),(k=0,1,2,\ldots)}$ is a sequence of real deterministic coefficients to be specified below and X_k's IID real-valued random variables. Now the four cases are as given below.

$$f_{n,k} = \begin{cases} \sqrt{\binom{n}{k}} \mathbb{1}_{\{k \le n\}}, & \text{binomial, elliptic, or spherical(SP) or SU(2)} \\ \sqrt{\dfrac{n^k}{k!}}, & \text{flat random analytic function (FAF) or ISO(2)} \\ \sqrt{\binom{n+k+1}{n}}, & \text{hyperbolic random analytic function(HAF) or SU(1,1)} \\ \sqrt{\dfrac{n^k}{k!}} \mathbb{1}_{\{k \le n\}} & \text{Weyl polynomials (WP)} \end{cases} \tag{10.2}$$

Let $\{X_k, k = 0, 1, \ldots\}$ have zero mean and unit variance, and $v_n(z)$ denote the variance of P_n. Let $\mathcal{D}$ be a open and connected subset of $\mathbb{C}$. Then the variances of four cases are taken as

$$v_n(z) = \begin{cases} (1+z^2)^n, & \text{SP case,} z \in \mathbb{C} \\ \exp(nz^2), & \text{FAF case,} z \in \mathbb{C} \\ (1-z^2)^{-n}, & \text{HAF case,} |z| < 1 \\ \sum\limits_{k=0}^{n} \dfrac{(nz^2)^k}{k!}, & \text{WP case,} z \in \mathbb{C}. \end{cases} \tag{10.3}$$

All four families of random polynomials fulfill a condition that is sufficient for proving almost everything what follows. Namely, there exists an open, connected set $\mathcal{D} \subseteq \mathbb{C}$ and an analytic function $p : \mathcal{D} \to \mathbb{C}$ such that

$$\lim_{n \to \infty} \frac{v_n(z)}{e^{np(z)}} = 1.$$

So we have to choose $p(z)$ properly. Flasche and Kabluchko [39] have proved the following very interesting result.

Theorem 10.2. *Let $\{X_k\}$ be IID. real-valued random variables with zero mean and unit variance. Let P_n be one of the four random analytic functions defined in* (10.1) *and* (10.2)*, and choose proper $p : D \to \mathbb{C}$. If $\mathcal{N}[a,b]$ denotes the number of real zeroes of P_n in the interval $[a,b] \subseteq \mathcal{D}$ $(\mathbb{R}\ \{0\})$, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}[\mathcal{N}[a,b]]}{\sqrt{n}} = \frac{1}{2\pi}\int_a^b \sqrt{\frac{p'(t)}{t} + p''(t)}\mathrm{d}t \tag{10.4}$$

For proof and other delicate ideas, one may refer to [39]

What happens when the random trigonometric polynomials have symmetric long-tailed coefficients. This is answered by Shepp and Farahmand [88] in the following result.

Theorem 10.3. *The expected number of real zeros of the $n-$th degree polynomial with real independent identically distributed coefficients with common characteristic function $\phi(z) = e^{-A(\log|1/z|)^{-a}}$ for $0 < |z| < 1$ and $\phi(0) = 1, \phi(z) \equiv 0$ for $1 \le |z| < \infty$, with $1 < a$ and $A \ge a(a-1)$, is $(a-1)/(a-(1/2))\log(n)$ asymptotically as $n \to \infty$.*

The important contributions on random hyperbolic polynomials and orthogonal polynomials by Das [18], Sambandham [76], Faramand [32], Farahmand and Girigorash [34], Pritsker and Xie [66], and others are worth mentioning. Farahmand et al. [36] and Sambandham et al.[84] have introduced a new field activity in the theory of random polynomials viz. the study of number of *points of inflection* of random polynomials. The literature shows that the upto third moment assumption, the studies have been carried over. The question of less moment assumption has been investigated by Logan and Shepp [57], [58] in the case of random algebraic polynomials and Jamron [49] in the case of random hyperbolic polynomials (proof is not available).

One knows that Das [18] first calculated the expected number of real zeros of $Q_n(t) = \sum_{k=1}^{n} X_k \cosh(kt)$ where $\{X_k\}, k = 1, 2, \ldots$ is a centered Gaussian random coefficients and he has obtained the asymptotic value

$$\mathbb{E}[\mathcal{N}_n] \sim \frac{1}{\pi}\log n \text{ as } n \to \infty.$$

Wilkins[104] has determined the asymptotic value of average number of real zeros when zero mean and variance of X_k as k^p.

Theorem 10.4. *Let n and p be integers such that $n \ge 2$ and $p \ge 0$. We suppose that $X_k(k = 1, 2, ..., n)$ are independent, normally distributed random variables, each with mean 0 and variance 1, and define the random hyperbolic polynomial*

$$Q_n(x) = \sum_{k=1}^{n} k^p X_k \cosh(kx).$$

If $\mathbb{E}[\mathcal{N}_{np}]$ *denotes the mean number of real zeros of* $Q_n(x)$, *then*

$$\mathbb{E}[\mathcal{N}_{np}] = \frac{1}{\pi}\log n + O(\log n)^{1/2}), \quad as \quad n \to \infty.$$

This weighted polynomial makes the difference with weighted algebraic polynomial as the principal part is independent of p.

Edelman and Kostlan [27] have found out that the expected number of real zeros of random algebraic polynomials increases significantly if the variance of the coefficients changes from unity to $\sqrt{\binom{n}{k}}$. Therefore, it forces to examine what effect this new assumption on variance of the coefficients has on the number of oscillations of $Q_n(x)$ with different weights.

11 UNIVERSALITY OF ZEROS OF RANDOM POLYNOMIALS

Tao and Vu [94] have established some local universality results concerning the correlation functions of the zeroes of random polynomials with independent coefficients. Their analysis relies on a general replacement principle, motivated by some recent work in random matrix theory. This principle enables one to compare the correlation functions of two random functions F and $\tilde{F}$ if their log magnitudes $\log|F|$, $\log|\tilde{F}|$ are close in distribution, and if some non-concentration bounds are obeyed.

Universality phenomenon: As per the arguments of Tao and Vu [94], in the case when the distribution X_i is a real or complex Gaussian, the correlation functions $\rho(k,1)$ (in the real case) or $\rho(k)$ (in the complex case) can be computed explicitly using tools such as the Kac-Rice formula; see Hough et al [43]. When the distribution is not Gaussian, the Kac-Rice formula is still available, but is considerably less tractable. Nevertheless, it has been widely believed that the asymptotic behavior of the correlation functions in the non-Gaussian case should match that of the Gaussian case once one has performed appropriate normalizations, at least if the distribution X_i is sufficiently short-tailed. This type of meta-conjecture is commonly referred to as the universality phenomenon.

They have established the local universality phenomenon in three cases

(1) Flat polynomials : $\sum_{i=0}^{n} \frac{1}{\sqrt{i!}} X_i z^i$,

(2) Elliptic polynomials : $\sum_{i=0}^{n} \sqrt{\binom{n}{i}} X_i z^i$, and

(3) Kac polynomials : $\sum_{i=0}^{n} X_i z^i$.

For results and proofs, one may consult Tao and Vu [94].

12 LARGE DEVIATION PRINCIPLE (LDP) AND RANDOM POLYNOMIALS

Consider a sequence $\{X_i, i = 0, 1, \ldots\}$ of random variables with mean μ and variance σ^2. Take $\{X_i\}$ as IID r.v's with mean μ and variance 1. Then the law of large numbers says that

$$\frac{S_n}{n} \xrightarrow[a.s.]{p} \mu \text{ as } n \to \infty \text{ if } E|X_i| < \infty \ (\xrightarrow[SLLN]{WLLN} resp.).$$

Next, take $\{X_i\}$ as IID r.v's. with mean μ and variance 1. As $n \to \infty$,

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sqrt{n}}\right) = \Phi(x).$$

In the case of Law of Large Numbers, S_n deviates from μ in the order of n (LLN for typical events) whereas in the Central Limit Theorem, S_n deviates from μ in the order of $\sqrt{n}$ (CLT for typical events). We want to probe the situation where S_n deviates from $a > \mu$ in the order of n beyond $\sqrt{n}$ (LDP for rare events). This is well explained by the Large Deviation Principle.

Definition 12.1. A sequence of random variables $\{X_i, i = 1, 2, \ldots\}$ with values in a metric space is said to satisfy a **large deviation principle** with

- speed $a_n \to \infty$ and
- rate function I,

if, for all Borel sets $A \subset \mathscr{M}$(Metric space),

$$\begin{aligned} \limsup_{a_n\to\infty} \frac{1}{a_n} \log P(X_n \in A) \quad &\leq \quad -\inf_{x\in clA} I(x) \\ \liminf_{a_n\to\infty} \frac{1}{a_n} \log P(X_n \in A) \quad &\geq \quad -\inf_{x\in intA} I(x) \end{aligned}$$

Let us walk on the known territory. Start with a coin tossing experiment.

$$X_i = \begin{cases} 1 & \text{w.p.} \quad \frac{1}{2} \\ 0 & \text{w.p.} \quad \frac{1}{2}. \end{cases}$$

All X_i's are IID with $E(X_i) = \dfrac{1}{2}$. Fix $a > E(X_i) = \dfrac{1}{2}$. Now we want to study the behaviour of $P\left(\dfrac{S_n}{n} \geq a\right)$ large n. To proceed, we first look at

$$P(S_n \geq na) = \sum_{k\geq na} \left(\frac{1}{2}\right)^n.$$

After simplification, it leads to

$$\log P(S_n \geq na) = -[\log 2 + a \log a + (1-a) \log(1-a)] + \text{lower order}.$$

$$\frac{1}{n} \log P(S_n \geq na) = -I(a) \text{ where } I(a) = \log 2 + a \log a + (1-a) \log(1-a).$$

This function $I(a)$ has unique root at $a = \frac{1}{2}$. It is indeed a good catch for the law of large numbers!

Now, let us take $X_i \sim \mathscr{N}(0, 1)$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{y^2}{2}} \mathrm{d}y, \quad x \in \mathbb{R}.$

$$P\left(\left|\frac{S_n}{n}\right| > a\right) = 2[1 - \Phi(a\sqrt{n})]. \tag{12.1}$$

For any $y > 0, (1 - \frac{3}{y^4})\phi(y) < \phi(y) < (1 + \frac{1}{y^2})\phi(y)$. Integrating over $[z, \infty), z > 0$, we get

$$\left(\frac{1}{z} - \frac{1}{z^3}\right)\phi(z) < [1 - \Phi(z)] < \frac{1}{z}\phi(z). \tag{12.2}$$

Equations (12.1) and (12.2) give

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\left|\frac{S_n}{n}\right| > a\right) = I(a) = -\frac{a^2}{2}.$$

Thus the 'rare event' $\left\{\left|\frac{S_n}{n}\right| > a\right\}$ has probability of order $e^{-\frac{na^2}{2}}$.
This is the LDP for Gaussian random variables with rate function $I(a) = -\frac{a^2}{2}$.

Now, let us study the set of zeros $\{t_1, t_2, \ldots, t_n\}$ of the random polynomial $F_n(t) = \sum_{k=0}^{n} X_k t^k$. where X_k's are IID random variables. Let us define an empirical measure

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n} \delta_{t_k}.$$

Let $\mathcal{M}_1(X)$ denote the space of probability measures on X, equipped with the topology of weak convergence which makes it into a Polish space. Let pol+ denote the collection of polynomials (over $\mathbb{C}$) with coefficients that are real positive. For $F \in$ pol+, let $\mu_F \in \mathcal{M}_1(\mathbb{C})$ denote the empirical measure of zeros of F. Note that μ_F depends on the set of zeros and not on a particular labeling of the zeros, that μ_F is symmetric with respect to the transformation $z \to z^*$, and that $\mu_F(\mathbb{R}^+) = 0$. (Here and in the sequel, we use $\mathbb{R}^+$ to denote the interval $(0, \infty)$.) Finally, for any space X and subset $\subset X$, we let A^c denote the complement of A in X.

LDP for zeros of Kac Polynomial: Let F_n denote a random polynomial, with IID standard complex Gaussian random coefficients $\{X_i\}$ and associated empirical measure of zeros L_n. Zeitouni and Zelditch [106] have proved that the sequence of empirical measures of zeros denoted by $\mu_n^{\mathbb{C}}$ for this model satisfies the large deviations principle (LDP) in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function $I_{\mathbb{C}}$ defined by

$$\begin{aligned} I_{\mathbb{C}}(\mu) \quad = \quad & \iint \left(\log(|z-w| - \frac{1}{2}\log(1+|z|^2) - \frac{1}{2}\log(1+|z|^2)\right) \mathrm{d}\mu(z)\mathrm{d}\mu(z) \\ & + \sup_{z\in S^1} \int \left(\log|z-w|^2 - \log(1+|w|^2)\right) \mathrm{d}\mu(w) \end{aligned}$$

When $\int \log(1+|z|^2)\mathrm{d}\mu(z)$ is finite, it simplicfies to

$$I_{\mathbb{C}}(\mu) - - \iint (\log(|z-w|\mathrm{d}\mu(z))\, \mathrm{d}\mu(w) + \sup_{z\in S^1} \int \left(\log|z-w|^2\right) \mathrm{d}\mu(w).$$

This has been extended by Butez [12] to the case of real-valued IID standard Gaussians random variables $\{X_i\}$. The empirical measure of zeros, denoted $\mu_n^{\mathbb{R}}$ for that model, satisfies the LDP in $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function $I_{\mathbb{R}_+}$ defined by

$$I_{\mathbb{R}_+}(\mu) = \begin{cases} \frac{1}{2} I_{\mathbb{C}}(\mu) & \text{if } \mu \in \mathcal{P}, \\ \infty & \text{otherwise.} \end{cases}$$

where $\mathcal{P}$ is the set of empirical measures of zeros of polynomials with positive coefficients and $\overline{\mathcal{P}}$ is its closure for the weak topology.

LDP for zeros of Polynomial with exponential random coefficients:

Let F_n denote a random polynomial, with IID exponential (of parameter 1) coefficients $\{X_i\}$ and associated empirical measure of zeros L_n. We introduce the closure of the collection of empirical measures of polynomials with positive coefficients

$$\mathcal{P} = \overline{\{\mu_F : F \in \text{pol}_+\}} \subset \mathcal{M}_1(\mathbb{C}).$$

Obviously, $L_n \in \mathcal{P}$. In order to record the result, we need the following definitions.

Definition 12.2. For any measure $\mu \in \mathcal{M}_1(\mathbb{C})$, define the logarithmic potential function to be

$$\mathbb{L}_F(z) = \int \log |(z - w)| \mathrm{d}\mu(w),$$

and the logarithmic energy to be

$$\sum(\mu) = \int \int \log(|z - w|)\mu(z)\mu(w).$$

Definition 12.3. Define the function $I : \mathcal{M}_1(\mathbb{C}) \to \mathbb{R}_+$ by

$$I(\mu) = \begin{cases} \int \log |(1 - z)| \mathrm{d}\mu(z) - \frac{1}{2} \int \int \log |(z - w)| \mathrm{d}\mu z \mathrm{d}\mu(w) & \text{if } \mu \in \mathcal{P}, \\ \infty & \text{if } \mu \notin \mathcal{P}. \end{cases}$$

The Large Deviation Principle for zeros of random polynomial with IID exponential coefficients is given below.

Theorem 12.4. *The random measures L_n satisfy a large deviation principle in the space $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function I. Explicitly, we have:*

(i) The function $I : \mathcal{M}_1(\mathbb{C}) \to [0, 1]$ has compact level sets, i.e. the sets $\{\mu : I(\mu) \le M\}$ are compact subsets of $\mathcal{M}_1(\mathbb{C}) \to [0, 1]$ for each $M \in \mathbb{R}$.

(ii) For each open set $O \in \mathcal{M}_1(\mathbb{C}) \to [0, 1]$, we have

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in O) \ge - \inf_{\mu \in O} I(\mu),$$

(iii) For each closed set $F \in \mathcal{M}_1(\mathbb{C})$, we have

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in F) \le - \inf_{\mu \in F} I(\mu)$$

Remark 12.5. We infer that in spite of the fact that we are dealing with zeros of random polynomials, the rate function is closer to a random matrix theory rate function than to the one appearing in the Gaussian case.

13 APPLICATIONS IN CRYPTOGRAPHY

Now we give some basic concepts to trace the connection between random polynomials and Cryptography. In Cryptography, a message is encrypted with a key. To give mathematical description of the key, we define the following.

Definition 13.1. Let m be a positive integer and $q = 2^m$. A Boolean function with m variables is a map from $\{0,1\}^m \to \{0,1]\}$ i.e. $V_m = \mathbb{F}_2^m \to \mathbb{F}$.

Definition 13.2. A Boolean function is linear if it is a linear form on the vector space $\mathbb{F}_2^m$.

Definition 13.3. It is affine if it is equal to a linear function up to a constant.

The key uses Boolean functions. In Cryptography, one develops cryptography algorithms to resist attacks for which one seeks functions that are with mixing as much as possible, and cannot be recovered by some spy. Hence one need the functions most distinct from linear functions. i.e. non-linear functions. In order to resist modern cryptographic attacks one uses highly nonlinear Boolean functions.

Definition 13.4. We define non-linearity of a boolean function $f : V_m \to \mathbb{F}_2$, the distance from f to the set of affine functions with m variables, as

$$\mathrm{nl}(f) = \min_{h \text{ affine}} d(f,h)$$

where d is the Hamming distance.

The non-linearity is equal to

$$\mathrm{nl}(f) = 2^{m-1} - \frac{1}{2}\|\widehat{f}\|_\infty$$

where

$$\|\widehat{f}\|_\infty = \sup_{v\in V_m}\left|\sum_{x\in V_m}(-1)^{f(x)+v\cdot x}\right|$$

and $\widehat{f}$ denotes the Fourier transform of $(-1)^f$ on V_m and $\|\widehat{f}\|_\infty$ is the spectral amplitude of the Boolean function f.

Now we will trace the analogy between Boolean functions f and random polynomials $F_n(t) = \sum\limits_{k=0}^{n} X_k t^k$.

$$\|\widehat{f}\|_\infty = \sup_{v\in V_m}\left|\sum_{x\in V_m}(-1)^{f(x)+v\cdot x}\right| \quad \text{with} \quad (-1)^{f(x)} = \pm 1$$

and

$$\|F_n(t)\|_\infty = \sup_{x\in\mathbb{R}/\mathbb{Z}}\left|\sum_{k=0}^{n} X_k e^{2\pi i k x}\right| \quad \text{with} \quad X_k = \pm 1.$$

Erdös (1957) has conjectured that there exists $\delta > 1$ such as for any integer n there exists a complex number z of modulus 1 such that $|F_n(t)| \geq \delta\sqrt{n+1}$. Littlewood (1966) has raised a question on the contrary if there were polynomials F_n with coefficients X_k such that

$$|F_n(t)| = \sqrt{n+1} = o(n).$$

for every $k \in T = \mathbb{R}/\mathbb{Z}$ and with $X_k = \pm 1$ or with $|X_k| = 1$. Kahane (1980) has solved the problem for complex coefficients X_k of modulus 1. For $|X_k| = 1$, he has established that

$$\lim_{m} \inf_{f\in V_m} \frac{\|F_n(t)\|_\infty}{\sqrt{n}} = 1.$$

but the initial problem $X_k = \pm 1$ case is still not solved.

In 1983, Patterson and Wiedemann[62] have shown that one can do better for $m \geq 15$. They have produced a boolean function f such that

$$\|\widehat{f}\|_\infty = \frac{27}{32}\sqrt{21} \text{ if } m \geq 15, \text{ odd}.$$

They have conjectured that

$$\inf_{f\in V_m} \|\widehat{f}\|_\infty \sim \sqrt{q}.$$

Theorem 13.5. *If f is a boolean function on V_m, we have*

$$\lim_{m\to\infty} \frac{\|\widehat{f}\|_\infty}{\sqrt{2q\log q}} = 1$$

with probability one.

Notice that

$$\sqrt{q} = \|\widehat{f}\|_2 \leq \|\widehat{f}\|_4 \leq \|\widehat{f}\|_\infty.$$

A weaker conjecture is stated as follows.

Conjecture: For f a Boolean function, it has been conjectured as

$$\lim_{m} \inf_{f\in V_m} \frac{\|\widehat{f}\|_4}{\sqrt{q}} = 1.$$

14 APPLICATIONS IN STRING AND M-THEORY

A polynomial of degree$n$N in one complex variable is

$$F(z) = \sum_{j=1}^{n} X_j z^j, X_j \in \mathbb{C}$$

is specified by its coefficients $\{X_j\}$. A **random** polynomial is short for a probability measure P on the coefficients. Let

$$\begin{aligned} \mathcal{P}_n^{(1)} &= \left\{ \sum_{j=0}^{n} X_j z^J,\ (X_1, X_2, \ldots, X_n) \in \mathbb{C}^n \right\} \\ &\simeq \mathbb{C}^n. \end{aligned}$$

Endow $\mathbb{C}^n$ with probability measure dP. We call $(\mathcal{P}_n^{(1)}, P)$ an *ensemble* of random polynomials.

In the case of random algebraic polynomials with real random coefficients, how are zeros or critical points distributed? Analogously, random complex geometry generalizes polynomials to holomorphic sections of line bundles.

Counting universes in string and M-theory refers to 'Universes'(i.e. 'vacua' of string and M-theory) which are critical points of 'superpotentials' on the moduli space of Calabi-Yau manifolds. This leads to investigate how many vacua are there and how are they distributed.

Complex zeros of random algebraic polynomials concentrate in small annuli around the unit circle S^1. In the limit as the degree $n \to \infty$, the zeros asymptotically concentrate exactly on S^1.

When the Gaussian random polynomials adapted to domains, we orthonormalize polynomials on the boundary $\delta\Omega$ of any simply connected, bounded domain $\Omega \subset \mathbb{C}$, the zeros of the associated random polynomials concentrate on $\partial\Omega$.

i.e. define the inner product on $\mathcal{P}_n^{(1)}$ by

$$\langle f, \overline{g} \rangle_{\partial\Omega} = \int\limits_{\partial\Omega} f(z)\overline{g(z)}|\mathrm{d}z|$$

Let $\gamma_{\partial\Omega}^n$ be the Gaussian measure induced by $\langle f, \overline{g} \rangle_{\partial\Omega}$ and say that the Gaussian measure is adapted to Ω. How do zeros of random polynomials adapted to Ω concentrate?

Denote the expectation relative to the ensemble $(\mathcal{P}_n^{(1)}, \gamma^n)$ by $\mathbb{E}_{\partial\Omega}^n$.

Theorem 14.1. $\mathbb{E}_{\partial\Omega}^n[Z_f^n] = \nu_\Omega + O(1/n)$

where ν_Ω is the equilibrium measure of Ω. The equilibrium measure of a compact set K is the unique probability measure $d\nu_K$ which minimizes the energy

$$\mathbb{E}[\mu] = -\int_K \int_K \log|z - w| \mathrm{d}\mu(z)\mathrm{d}\mu(w).$$

Thus, in the limit as the degree $n \to \infty$, random polynomials adapted to Ω act like electric charges in Ω.

Algebraic geometers are interested in zeros of holomorphic sections. Now we focus on critical points $\nabla F(z) = 0$, where ∇ is a metric connection.

Critical points of Gaussian random functions come up in many areas of physics:

- as peak points of signals (Rice [71]);
- as vacua in compactifications of string and M-theory on Calabi-Yau manifolds with flux (Giddings-Kachru-Polchinski, Gukov-Vafa-Witten);
- as extremal black holes (Strominger, Ferrara-Gibbons-Kallosh) , peak points of galaxy distributions (Szalay et al. , Zeldovich), etc.

The vacuum selection problem: Which X forms the 'small' or 'extra' dimensions of our universe? How to select the right vacuum?

The vacuum selection problem in string and M-theory has applications to string and M-theory. According to string and M-theory, our universe is 10- (or 11-) dimensional. In the simplest model, it has the form $M^{3,1} \times X$ where X is a complex 3-dimensional Calabi-Yau manifold.

Finally, counting candidate universes in string theory amounts to counting critical points of integral superpotentials, which form a lattice in the hyperbolic shell.

15 APPLICATIONS IN STATISTICAL PHYSICS

The zeros of complex Gaussian random polynomials, with coefficients such that the density in the underlying complex space is uniform, are known to have the same statistical properties as the zeros of the coherent state representation of one dimensional chaotic quantum systems. It is to be noted that these polynomials arise as the wave functions for quantum particles in a magnetic field constructed from a random superposition of states in the lowest Landau level. A study of the statistical properties of the zeros has been undertaken by Forrester and Honner [40] using exact formulae for the one and two point distribution functions. They have analysed the moments of the two-point correlation in the bulk, the variance of a linear statistic, and the asymptotic form of the two-point correlation at the boundary. They have made a comparative study with the same quantities for the eigenvalues of complex Gaussian random matrices.(see [40] for results and proofs.)

An important field of study in theoretical statistical physics concerns the properties of the roots of random polynomials (see Bogomolny et al. [10]). Of particular interest is the so-called Weyl polynomial for the complex variable where the X_k's are independent random complex numbers with the same Gaussian probability distribution. The roots of F in the complex plane can be mapped to a two-dimensional (2D) gas of particles with repulsive interactions. They are spatially antibunched, and have a uniform mean density for the large value of n. The statistical properties of the roots of the Weyl polynomial have been well studied theoretically without any physical system to observe them directly.

Castin[13] has shown that a 2D rotating ideal Bose gas is a well suited system for this observation. The positions of the vortices appearing in the gas are mapped to the zeroes of the random polynomial describing the atomic state.

16 FINGERPRINTING BY RANDOM POLYNOMIAL

Prime numbers are used in several contexts to yield efficient randomized algorithm for many problems. In these applications a randomly chosen prime p is used to *fingerprint* a long character-string by computing the residue of that string, viewed as a large integer modulo p. This method requires performing fixed -point arithmetic on $k-$bit integers, where $k = \lceil \log_2 p \rceil$, or atleast addition/subtraction on such integers. Rabin [68] has proposed a randomly chosen irreducible (prime) polynomial $p(t) = \mathbb{Z}_2[t]$ of an appropriate small degree k instead of the prime integer p. It turns out that it is very easy to effect a random choice of an irreducible polynomial. The implementation of $\mod p(t)$ arithmetic requires just length$-k$ shift registers and the exclusive-or operation on $k-$bit vectors. These operations are fast, involve simple circuit, and in VLSI require little chip area.

By randomizing the choice of the irreducible polynomial $p(t)$, he has obtained a provably highly dependable and efficient algorithm for every instance of the string matching problem to be solved and protect every file against any deliberate modification, etc. For further details one may refer to [68].

17 APPLICATIONS IN WIRELESS COMMUNICATIONS

We now focus on a problem concerning the GSM (Global System for Mobile Communications)/ EDGE (Enhanced Data Rates for GSM Evolution) standard for mobile phones. When designing digital receivers for such a system, the properties of the so-called discrete-time overall channel impulse response becomes important. Specifically, the location of the zeros of the $z-$transform of the discrete-time overall channel impulse response determines the receiver's performance. The randomness inherent in mobile communications results in such a $z-$transform being a random polynomial. For wireless communications in urban areas it is common for the coefficients of F to be mean zero complex Gaussians, with exponentially increasing or decreasing variances. Under these assumptions, Schober and Gerstacker [85] have derived explicit results for the location of the zeros when the coefficients are independent. This assumption of independence, however, was made to facilitate the computations. In practice, they state that the coefficients are approximately uncorrelated.

With that in mind, a study the behavior of the complex zeros when the coefficients are dependent mean zero complex Gaussians with exponentially increasing or decreasing variances has been initiated. Using a result from Hughes and Nikeghbali [44], it is shown that in the limit, the roots accumulate around a circle in the complex plane, uniformly in the angle, where the radius is determined by the coefficient variances. This behavior holds without any restrictions on the covariance function of the coefficients and corresponds with the behavior observed by Schober and Gerstacker [85] in the independent case. The drawback is that this result applies only to the limiting behavior, and it fails to give any detail as to how fast this occurs or how close to the circle the zeros accumulate. Thus, to get a more detailed analysis we need the techniques developed by Shepp and Vanderbei[89]. In order to apply these techniques when the coefficients are dependent, some concessions must be made. Namely, it will be necessary to assume that the covariance function of the coefficients is absolutely summable and that the spectral density does not vanish. Another way to interpret these conditions is that it is required for the covariance of the coefficients to decay fast.

We state a result from Hughes and Nikeghbali [44]. Let $F(z)$ be of the random polynomial, and let $\nu_n(\Omega)$ be the number of zeros of $F_n(z)$ in the set $\mathbb{C}$. Also, for $0 < r < 1$ define the annulus $a(r) = \{z \in \mathbb{C} : 1 - r \le |z| \le 1/(1 - r)\}$, and for $0 \le \theta_1 < \theta_2 \le 2\pi$, let $C(\theta_1, \theta_2)$ be the cone in the complex plane consisting of all points with arguments between θ_1 and θ_2.

Theorem 17.1. *(Hughes and Nikeghbali) Assume the coefficients of F are complex Gaussians with mean zero and unit variance. Then there exists a deterministic positive sequence (α_n), subject to $0 < \alpha_n \le n$ for all n and $\alpha_n = o(n)$ as $n \to \infty$, such that*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\nu_n\left(a\left(\frac{\alpha_n}{n}\right)\right) &= 1, \quad a.s.\\ \lim_{n\to\infty} \frac{1}{n}\nu_n\left(C(\theta_1, \theta_2)\right) &= \frac{\theta_2 - \theta_1}{2\pi}, \quad a.s.\end{aligned}$$

In other words, we infer from the above theorem that for mean zero complex Gaussian coefficients with unit variance, the zeros will accumulate around the unit circle in the limit, uniformly in the angle.

Matayoshi[60] has succeeded in showing that, for the Chebyshev polynomials of the first kind, the zeros of F converge to the equilibrium distribution. It leads to believe that similar results should hold for the Legendre polynomials, as well as the Jacobi polynomials.

18 APPLICATIONS IN GAME THEORY

Random polynomials are indispensable in the modelling and analysis of complex systems in which very limited information is available or where the environment changes so rapidly. The statistical science of equilibria in large random systems provides important insight into the understanding of the underlying physical, biological and social system such as the complexity-stability relationship in ecosystems, bio-diversity and maintenance of polymorphism in multi-player multi-strategy games, and the learning dynamics.

A key challenge in such study is due to the large (but finite) size of the population in an ecological system, the number of players and strategies in an evolutionary game and the number of nodes and connections in a social network. It is well-known that the behaviour of the system at finite size or characterizing its asymptotic behaviour when the size tends to infinity are of both theoretical and practical interest.

Consider the number of internal equilibria in $(n + 1)-$player two strategy random evolutionary games. We consider an infinitely large population that consists of individuals using two strategies, A and B. We denote by $y, 0 \le y \le 1$, the frequency of strategy A in the population. The frequency of strategy B is thus $(1 - y)$. The interaction of the individuals in the population is in randomly selected groups of $(n + 1)-$participants, that is, they interact and obtain their fitness from $(n + 1)-$player games. Consider symmetric games where the payoffs do not depend on the ordering of the players. Suppose that a_i (respectively, b_i) is the payoff that an $A-$strategist (respectively, B) achieves when interacting with a group of n other players consisting $i(0 \leq i \leq n)A$ strategists and $(n-i)B$ strategists.

The average payoffs (fitnesses) of strategies A and B are respectively given by

$$\pi_A = \sum_{i=0}^{n} a_i \binom{n}{i} y^i (1-y)^{n-i} \text{ and } \pi_B = \sum_{i=0}^{n} b_i \binom{n}{i} y^i (1-y)^{n-i}$$

Internal equilibria in $(n+1)-$player two-strategy games can be derived using the definition of an evolutionary stable strategy. They are those points $0 < y < 1$ (note that y = 0 and y = 1 are trivial equilibria in the replicator dynamics) such that the fitnesses of the two strategies are the same $\pi_A = \pi_B$, that is

$$\sum_{i=0}^{n} \xi_i \binom{n}{i} y^i (1-y)^{n-i} = 0 \text{ where } \xi_i = a_i - b_i.$$

In the literature, the sequence of the difference of payoffs $\{\xi_i\}_i$ is called the gain sequence. Dividing the above equation by $(1-y)^n$ and using the transformation $x = y/(1-y)$, we obtain the following polynomial equation for $x(x > 0)$

$$F(x) = \sum_{i=0}^{n} \xi_i \binom{n}{i} x^i = 0.$$

In random games, the payoff entries $\{a_i\}_i$ and $\{b_i\}_i$ are random variables, thus so are the gain sequence $\{\xi_i\}_i$. Therefore, the expected number of internal equilibria in a $(n+1)-$player two-strategy random game is the same as the expected number of positive roots of the random polynomial F. The interesting and amazing fact is that it is half of the expected number of the real roots of F due to the symmetry of the distributions (cf. [14]). This connection between evolutionary game theory and random polynomial theory has been revealed and exploited in recent series of papers [23].[24],[25], [26],

19 DIRECTIONS FOR FUTURE RESEARCH

We have already stated some further investigation problems in the above sections. In addition, this is a compilation (by Sethuraman[86]) of the open problems posed by the participants of the AIM Workshop on Random Analytic functions. For the sake convenience, we reproduce the list of problems as stated in [86].

1. **Question 1: (Wenbo Li)** Consider random polynomials in one variable (real or complex). Find the asymptotic of the norm of the largest zero as the degree of the random polynomials tends to infinity.
2. **Question 2: (Yan Fyodorov)** What is the mean density of permanental polynomials ? This is unknown for random matrices of size greater than 5×5.
3. **Question 3: (Ashkan Nikeghbali)** What can one tell about the distribution of the zeros of the derivative of characteristic polynomial of random unitary matrix, especially near the boundary of the unit circle.Also what can one tell about $\mathbb{E}[|F_n'|s]$ as s tends to zero ?

4. **Question 4: (Maurice Rojas)** Investigate the connections between random sparse polygons and Newton polytopes. This should be extended to the case of random Viro diagrams.
5. **Question 5: (Balint Virag)** Find a Gaussian entire function with negatively correlated zeros. We know that there exists such a function on the unit disc. This is related to the repulsion properties of random polynomials.
6. **Question 6: (Bernard Shiffman)** Does the Fubini-Study metric on $\mathbb{P}^1$ minimize the expected number of critical points? There are reasons to conjecture that the answer is Yes.
7. **Question 7: (Steven Evans)** Is there a necessary and sufficient condition for a given correlation function to be the correlation function of an enire Gaussian function?
8. **Question 8: (Maurice Rojas)** What is the probability that a random Viro diagram contains no sphere?
9. **Question 9: (Ashkan Nikeghbali)** What are the natural physical examples of random functions with GUE zeros on the real line?
10. **Question 10: (Scott Sheffield)** Consider a function who Fourier transform is white noise on the unit circle. We aim to understand the web like appearance, that is the zero level lines of the Gaussian free field. Zelditch and Schramm make the above question more precise.
11. **Question 11: (Yan Fyodorov)** Given a random entire function on order 1, real on real line with given distribution of real zeros, what is the distribution of zeros of F_n ?

20 CONCLUSION

The purpose of this brief review is bring out a bird's eye view of basic techniques and possible applications in pure and applied mathematics, statistical physics, algebraic geometry, wireless communications, and mathematical cryptography. An attempt has been made here from quadratic equation to polynomial of general degree n with coefficients follow different probability laws. The degree of difficulty increases when moving from Gaussian to non-Gaussian random coefficients. In our next article, we propose to summarise the results on some more polynomials, system of polynomials, polynomials with several variables, irreducibility of random polynomials, polynomials whose coefficients from some algebraic structures, polynomials with complex coefficients, moments of number of complex zeros, etc.

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