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A PORTFOLIO REBALANCING PROBLEM

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ABSTRACT. In this paper we consider a portfolio consisting of three types of assets, a bond, a liquid risky asset, and an illiquid risky asset. Liquid assets can be traded continuously whereas an illiquid asset can only be traded at specific pre-specified times or randomly. The investor's liquid and illiquid wealth are modeled as stochastic differential equations (SDE). We consider a control problem governed by these SDEs. We will model the rebalancing of the portfolio, transferring between liquid and illiquid wealth, as an impulsive control problem. Using a dynamic programming approach and a stochastic programming approach we will determine the the amount to transfer between the liquid and illiquid assets. Numerical examples are given to validate the correctness for our results.

AMS (MOS) Subject Classification. 91G10, 91B32, 91B10,60H30, 60H35.

Key Words and Phrases. Portfolio rebalancing, Asset allocation, HJB equation.

1. Introduction

We examine the optimal portfolio rebalancing of an investor able to trade a risky liquid and illiquid asset as well as a risk-free liquid asset. The illiquid asset can only be traded at infrequently occurring random times. When a trading opportunity arrives, the investor chooses the optimal mix of liquid and illiquid assets. We model the arrival of trading opportunities as random. This captures two key features of real-world markets. First, certain securities are only periodically marketable, as opposed to always marketable at a cost reflecting transactions fees or price impact. This illiquidity can arise as a result of a limited number of market participants, possibly due to the specialized skills or systems needed to trade these assets. Second, for some illiquid assets, one may want to choose when to rebalance.

We find that illiquidity causes the investor to behave in a more risk-averse manner with respect to both liquid and illiquid holdings. Uncertain trading opportunities create an unhedgeable source of risk, which causes the investor to reduce allocation to the illiquid asset. A second effect comes from the investor's immediate obligations (consumption), which can be financed only through liquid wealth. If the investor's liquid wealth drops to zero, these obligations cannot be met until after the next rebalancing opportunity. This matches real world settings in which investors or investment funds are insolvent, not because their assets under management have hit zero, but because they cannot fund their immediate obligations. As a result, the investor alters asset allocation to minimize low liquid wealth states by holding fewer risky liquid securities. This effect causes the portfolio policy of the liquid risky asset to be affected by illiquidity even when the liquid and illiquid asset returns are uncorrelated and when the investor has log utility. The investor's effective level of risk aversion endogenously

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increases in the fraction of wealth held in illiquid securities [9]. This is because taking offsetting positions in perfectly correlated liquid and illiquid assets causes the investor's liquid wealth to drop to zero with positive probability and only liquid wealth funds current consumption. In Longstaff and Ang et all [9,15] model rebalancing using jump processes such a compound poisson process is presented. We will model rebalancing as an impulse control problem.

Our analysis falls into a long literature dealing with asset choice and various aspects of the investor's unwillingness or inability to continuously rebalance part of the total endowment. Longstaff [15] investigates the effects of periodic market closures where the entire market is closed deterministically. Our portfolio choice is simultaneously done over liquid and illiquid assets and we are able to trade the illiquid asset at deterministic time as well as random times. Viewing rebalancing has not been studied to this extent to model rebalancing times.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbation whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulse [8]. For such an idealization, it becomes necessary to study dynamical system with discontinuous trajectories, and they might be called differential equations with impulses or impulsive differential equations [7]. Theories involving impulsive control systems have been widely studied [18]. Impulsive control problems arise in many modeling problems such as treatment of diseases, production planning and inventory management, pest control, engineering, economics, finances, management science and the physical sciences

2. **Model**

Consider an investor who has a portfolio composed of riskless bond, a risky liquid asset and a risky illiquid assets. Following [9], the first asset is a riskless bond,

$$\frac{dB_t}{B_t} = rdt$$

with interest rate r. The second is a liquid risky asset,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t^1$$

with drift μ and volatility σ. These two assets are considered liquid assets because they can be rebalanced continuously. Our third asset is a risky illiquid asset

$$\frac{dP_t}{P_t} = vdt + \psi\rho dZ_t^1 + \psi\sqrt{1-\rho^2}dZ_t^2$$

with drift v, volatility ψ, and ρ represents the correlation between the return of the two risky assets. We will assume Z_t^1 and Z_t^2 are independent Brownian Motions. The third asset is considered illiquid because the asset can only be rebalanced at certain times.

Let W_t and X_t be in the investor's liquid and illiquid wealth respectively and θ_t be the fraction of wealth invested in the liquid risky asset and the riskless bond. Since we are considering the case when we are not rebalancing our model of our portfolio is,

$$\begin{aligned}\frac{dW_t}{W_t} &= (1-\theta_t)\frac{dB_t}{B_t} + \theta_t\frac{dS_t}{S_t} - \frac{C_t}{W_t}dt\\ \frac{dX_t}{X_t} &= \frac{dP_t}{P_t}\end{aligned}$$

Certain utility functions are commonly used such as quadratic utility function, exponential utility function and power utility function [Gerber and Pafumi (1999)]. For our purposes we will use a quadratic utility function. In [Ang 2010] they used a constant relative risk aversion (CRRA) utility function,

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \gamma > 1\\ \log(C) & \gamma = 1\end{cases}$$

Under constant relative risk aversion the fraction of wealth optimally placed in the risky asset is independent of the level of initial wealth. When using a dynamic programming approach it's beneficial to be able to isolate our consumption, $c_t = C_t/W_t$. Using the CRRA utility function makes it difficult to isolate and derive our Hamilton Jacobi Bellman (HJB) equation. Therefore, we chose a quadratic utility function,

$$U(C_t) = \alpha_3 + \alpha_1 C_t - \alpha_2 C_t^2 \tag{2.1}$$

where α_1, α_2 and α_3 are constants.

Next, we will derive the Hamilton-Jacobi-Bellman Equation (HJB) for our portfolio and we will find an explicit form for our controls, c_t and θ_t.

First, we will derive our HJB equation. For

$$\begin{aligned}\frac{dW_t}{W_t} &= rdt + (\mu - r)\theta_t dt - c_t dt + \theta_t\sigma dZ_t^1\\ \frac{dX_t}{X_t} &= vdt + \psi\rho dZ_t^1 + \psi\sqrt{1-\rho^2}dZ_t^2\end{aligned}$$

where, $c_t = \frac{C_t}{W_t}$ is the ratio of consumption to liquid wealth.

Let $\phi(t_0, W_{t_0}, X_{t_0}) = \phi(t,w,x) = \mathbb{E}\Big[\int_t^\tau e^{-\beta t}U(C_t)dt + K(\tau, X(\tau), W(\tau))\Big]$, where τ is a stopping time. Later in this section, we will discuss $\phi(t, W_t, X_t)$ more fully. Since we're considering a finite interval, we will ignore $e^{-\beta t}$. As mentioned in the introduction, we will use a quadratic utility function,

$$U(C_t) = \alpha_1 C_t - \alpha_2 C_t^2 + \alpha_3$$

where α_1, α_2 and α_3 are constants.

Proposition 2.1. *The Hamilton Jacobi Bellman (HJB) Equation is given by*

$$0 = \max_{c_t,\theta_t} U(c_t w) + \phi_t + (r + (\mu - r)\theta_t - c_t)w\phi_W + vx\phi_X + \frac{1}{2}\sigma^2\theta_t^2 w^2\phi_{WW} + \sigma\rho\psi\theta_t wx\phi_{WX} + \frac{1}{2}\psi^2 x^2\phi_{XX} \tag{2.2}$$

We consider a value function of the form,

$$\phi(t,w,x) = A(t) + B(t)w + D(t)x + E(t)w^2 + F(t)x^2 \tag{2.3}$$

with terminal condition,

$$\phi(T,w,x) = (k_0 w + k_1)^2 + (k_2 x + k_3)^2. \tag{2.4}$$

Our objective is to look for explicit solution for the value function that is consistent with the idea of rebalancing.

Proposition 2.2. *From the HJB equation the optimal decision variables are*

$$c_t = -\frac{1}{\alpha_2}E(t) \text{ and } \theta_t = \frac{-(\mu-r)(\alpha_1 + 2E(t)w)}{2\sigma^2 E(t) w}, \tag{2.5}$$

where $0 \le c_t \le 1$ *and* $0 \le \theta_t \le 1$.

Proof. From proposition 2.1 we have the following relationships,

$$\begin{aligned} U'(c_t W) - w\phi_W &= 0 \\ U'(c_t W) &= w\phi_W \\ w(\alpha_1 - 2\alpha_2 c_t w) &= w(B(t) + 2E(t)w) \end{aligned}$$

Therefore,

$$B(t) = \alpha_1 \text{ and } c_t = -\frac{1}{\alpha_2}E(t).$$

To ensure that, $0 \le c_t \le 1$, α_2 needs to be large enough to ensure c_t is less than 1. Also, either α_2 or $E(t)$ needs to be negative, but not both. Solving for control variable θ_t we have

$$\begin{aligned} (\mu - r)w\phi_W + \sigma^2 w^2 \phi_{WW}\theta_t + \sigma\psi\rho w x \phi_{WX} &= 0 \\ \theta_t &= \frac{-(\mu-r)\phi_W}{\sigma^2 w \phi_{WW}} \qquad (\phi_{WX} = 0) \\ \theta_t &= \frac{-(\mu-r)(\alpha_1 + 2E(t)w)}{2\sigma^2 E(t) w} \qquad (B(t) = \alpha_1) \end{aligned}$$

Recall that, $0 \le \theta_t \le 1$. Thus, if $E(t)$ is positive use α_1 to ensure that θ_t is positive and less than 1. □

We now proceed to find explicit solution for the HJB equation. Using Proposition 2.1 our HJB equation becomes

$$\begin{aligned} 0 = A'(t) - \frac{(\mu-r)\alpha_1^2}{4\sigma^2 E(t)} + \alpha_3 + \left[r\alpha_1 - \frac{(\mu-r)^2}{\sigma^2}\alpha_1\right]w + \Big[D'(t) + vD(t)\Big]x \\ + \left[E'(t) + \frac{1}{\alpha_2}E(t)^2 + 2rE(t) - \frac{(\mu-r)^2}{\sigma^2}E(t)\right]w^2 + \Big[F'(t) + 2vF(t) + \psi^2 F(t)\Big]x^2 \end{aligned} \tag{2.6}$$

Thus we obtain the following system of nonlinear differential equations,

$$\begin{cases} A'(t) = \frac{(\mu-r)^2\alpha_1^2}{4\sigma^2 E} - \alpha_3 \\ 0 = \left(r - \frac{(\mu-r)^2}{\sigma^2}\right)\alpha_1 \\ D'(t) = -vD(t) \\ E'(t) = \frac{(\mu-r)^2}{\sigma^2}E(t) - 2rE(t) - \frac{1}{\alpha_2}E(t)^2 \\ F'(t) = -2vF(t) - \psi^2 F(t). \end{cases}$$

Using standard methods to solve ordinary differential equations with terminal conditions (2.4) we have the following solutions,

$$
\begin{aligned}
&\sigma = \sqrt{\frac{\mu^2}{r} - 2\mu + r}\\
&A(t) = \big(\frac{\alpha_1}{2k_0}\big)^2 + k_3^2 + \alpha_3(T-t) - \frac{(\mu-r)^2\alpha_1^2}{4\sigma^2\big(\frac{(\mu-r)^2}{\sigma^2} - 2r\big)}\Bigg[\frac{\frac{(\mu-r)^2-2r}{k_0^2} - \frac{1}{\alpha_2}}{2r - \frac{(\mu-r)^2}{\sigma^2}}\Big(1 - \exp\big(\big(\frac{(\mu-r)^2}{\sigma^2} - 2r\big)(T-t)\big)\Big)\\
&\qquad + \frac{1}{\alpha_2}(T-t)\Bigg]\\
&B(t) = \alpha_1\\
&D(t) = 2k_2k_3\exp(v(T-t))\\
&E(t) = \frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{\Big(\frac{\frac{(\mu-r)^2}{\sigma^2}-2r}{k_0^2} - \frac{1}{\alpha_2}\Big)\exp\Big(-\Big(\frac{(\mu-r)^2}{\sigma^2} - 2r\Big)t\Big) + \frac{1}{\alpha_2}}\\
&F(t) = k_2^2\exp((2v+\psi^2)(T-t))
\end{aligned}
$$

Note that, the k_1 coefficient of the terminal condition has to be specified in order to ensure $B(t) = \alpha_1$, and from (2.4) there is no need to impose restrictions on k_2 and k_3.

3. Predetermined Rebalancing Times

We will first, consider the case when we introduce predetermined rebalancing times, and introduce impulses/jumps, ξ_i to and from the liquid and illiquid assets. Consider the time horizon $[t_0, t_3]$ with impulses at t_1 and t_2. We will start at the last interval, $[t_2, t_3]$.

The control problem for this interval

$$\max_{c_{3,t},\theta_{3,t}} \mathbb{E}\Big[\int_t^{t_3} U(c_{3,s}w_{3,s})ds + \Big(k_0 w_{t_3} + \frac{\alpha_1^3}{2k_0}\Big)^2 + (k_2 x_{t_3} + k_3)^2\Big] \tag{3.1}$$

$$\text{subject to}\quad \begin{aligned} dW_{3,t} &= (r + (\mu - r)\theta_{3,t} - c_{3,t})W_{3,t}dt + \theta_{3,t}\sigma W_{3,t}dZ_{3,t}^1\\ dX_{3,t} &= vX_{3,t}dt + \psi\rho X_{3,t}dZ_{3,t}^1 + \psi\sqrt{1-\rho^2}X_{3,t}dZ_{3,t}^2 \end{aligned} \tag{3.2}$$

Let $\phi_3(t, w, x)$ be the value function for this interval. Then,

$$\phi_3(t, w, x) = \max_{c_{3,t},\theta_{3,t}} \mathbb{E}\Big[\int_t^{t_3} U_3(c_{3,s}w_{3,s})ds + \Big(k_0 w_{t_3} + \frac{\alpha_1}{2k_0}\Big)^2 + (k_2 x_{t_3} + k_3)^2\Big]$$

The value function, $\phi_3(t, w, x)$ has the form,

$$\phi_3(t, w_{3,t}, x_{3,t}) = A_3(t) + \alpha_{1,3}w_{3,t} + D_3(t)x_{3,t} + E_3(t)w_{3,t}^2 + F_3(t)x_{3,t}^2.$$

The utility function for this interval,

$$U_3(c_{3,t}w_{3,t}) = -\alpha_{2,3}c_{3,t}^2 w_{3,t}^2 + \alpha_{1,3}c_{3,t}w_{3,t} + \alpha_{3,3}. \tag{3.3}$$

Our HJB equation becomes

$$
\begin{aligned}
0 = A_3'(t) - \frac{(\mu-r)^2\alpha_{1,3}}{4\sigma^2 E_3(t)} + \alpha_{3,3} + [r\alpha_{1,3} - \frac{(\mu-r)^2}{\sigma^2}\alpha_{1,3}]w_3 + [D_3'(t) + vD_3(t)]x_3 \\
+ [E_3'(t) + \frac{1}{\alpha_{2,3}}E_3^2(t) - \frac{(\mu-r)^2}{\sigma^2}E_3(t)]w_{3,t}^2 + [F_3'(t) + 2vF_3(t) \\
+ \psi^2 F_3(t)]x_3^2
\end{aligned}
$$

Optimal control variables for this interval

$$
c_{3,t} = -\frac{1}{\alpha_{2,3}}E_3(t) \text{ and } \theta_{3,t} = \frac{-(\mu-r)(\alpha_{1,3} + 2E_3(t)w_3)}{2\sigma^2 E_3(t)w_3}. \tag{3.4}
$$

We now have the following system of nonlinear differential equations,

$$
\begin{cases}
A_3'(t) = \frac{(\mu-r)^2(\alpha_{1,3})^2}{4\sigma^2 E_3(t)} - \alpha_{3,3} \\
0 = \left(r - \frac{(\mu-r)^2}{\sigma^2}\right)\alpha_{1,3} \\
D_3'(t) = -vD_3(t) \\
E_3'(t) = \frac{(\mu-r)^2}{\sigma^2}E_3(t) - 2rE_3(t) - \frac{1}{\alpha_{2,3}}E_3(t)^2 \\
F_3'(t) = -2vF_3(t) - \psi^2 F_3(t).
\end{cases}
$$

where,

$$
\begin{aligned}
A_3(t_3) &= \left(\frac{\alpha_1^3}{2k_0}\right)^2 + k_3^2, \\
D_3(t_3) &= 2k_2k_3, \\
E_3(t_3) &= k_0^2, \\
F_3(t_3) &= k_2^2.
\end{aligned}
$$

The coefficients become

$$
\begin{aligned}
\sigma &= \sqrt{\frac{\mu^2}{r} - 2\mu + r} \\
A_3(t) &= \left(\frac{\alpha_{1,3}}{2k_0}\right)^2 + k_3^2 + \alpha_{3,3}(t_3 - t) - \frac{(\mu-r)^2(\alpha_{1,3})^2}{4\sigma^2\left(\frac{(\mu-r)^2}{\sigma^2} - 2r\right)}\left[\frac{\frac{\frac{(\mu-r)^2}{\sigma^2}-2r}{k_0^2} - \frac{1}{\alpha_{2,3}}}{2r - \frac{(\mu-r)^2}{\sigma^2}}\Big(1\right. \\
&\quad \left. - \exp\left(\left(\frac{(\mu-r)^2}{\sigma^2} - 2r\right)(t_3 - t)\right)\Big) + \frac{1}{\alpha_{2,3}}(t_3 - t)\right] \\
D_3(t) &= 2k_2k_3\exp(v(t_3 - t)) \\
E_3(t) &= \frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{\left(\frac{\frac{(\mu-r)^2}{\sigma^2}-2r}{k_0^2} - \frac{1}{\alpha_{2,3}}\right)\exp\left(\left(\frac{(\mu-r)^2}{\sigma^2} - 2r\right)(t_3 - t)\right) + \frac{1}{\alpha_{2,3}}} \\
F_3(t) &= k_2^2\exp((2v - \psi^2)(t_3 - t))
\end{aligned}
$$

Going back to the second interval, $[t_1, t_2]$,

$$\max_{c_{2,t},\theta_{2,t}} \mathbb{E}[\int_t^{t_2} U_2(c_{2,s}w_{2,s})ds + \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)] \tag{3.5}$$

$$\text{subject to} \quad \begin{aligned} dW_{2,t} &= (r + (\mu - r)\theta_{2,t} - c_{2,t})W_{2,t}dt + \theta_{2,t}\sigma W_{2,t}dZ^1_{2,t} \\ dX_{2,t} &= vX_{2,t}dt + \psi\rho X_{2,t}dZ^1_{2,t} + \psi\sqrt{1-\rho^2}X_{2,t}dZ^2_{2,t} \end{aligned} \tag{3.6}$$

Let $\phi_2(t, w_{2,t}, x_{2,t})$ be the value function for this interval. Then,

$$\phi_2(t, w_{2,t}, x_{2,t}) = \max_{c_{2,t},\theta_{2,t}} \mathbb{E}[\int_t^{t_2} U_2(c_{2,s}w_{2,s})dt + \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)]$$

The value function, $\phi_2(t, w, x)$ has the form,

$$\phi_2(t, w_2, x_2) = A_2(t) + \alpha_{1,2}w_2 + D_2(t)x_2 + E_2(t)w_2^2 + F_2(t)x_2^2.$$

The utility function for this interval,

$$U_2(c_{2,t}w_{2,t}) = -\alpha_{2,2}c^2_{2,t}w^2_{2,t} + \alpha_{1,2}c_{2,t}w_{2,t} + \alpha_{3,2}. \tag{3.7}$$

Optimal decision variables for this interval are

$$c_{2,t} = -\frac{1}{\alpha_{2,2}}E_2(t) \text{ and } \theta_{2,t} = \frac{-(\mu - r)(\alpha_{1,2} + 2E_2(t)w_2)}{2\sigma^2 E_2(t)w_2}. \tag{3.8}$$

Then our HJB equation for this interval becomes

$$\begin{aligned} 0 = A_2'(t) &- \frac{(\mu - r)^2\alpha_{1,2}(t)}{4\sigma^2 E_2(t)} + \alpha_{3,2} + [r\alpha_{1,2} - \frac{(\mu - r)^2}{\sigma^2}\alpha_{1,2}]w_2 + [D_2'(t) + vD_2(t)]x_2 \\ &+ [E_2'(t) + \frac{1}{\alpha_{2,2}}E_2^2(t) - \frac{(\mu - r)^2}{\sigma^2}E_2(t)]w_2^2 + [F_2'(t) + 2vF_2(t) + \psi^2 F_2(t)]x_2^2 \end{aligned}$$

We have the following system of nonlinear differential equations,

$$\begin{cases} A_2'(t) = \frac{(\mu-r)^2(\alpha_{1,2})^2}{4\sigma^2 E_2(t)} - \alpha_{3,2} \\ 0 = \left(r - \frac{(\mu-r)^2}{\sigma^2}\right)(\alpha_{1,2}) \\ D_2'(t) = -vD_2(t) \\ E_2'(t) = \frac{(\mu-r)^2}{\sigma^2}E_2(t) - 2rE_2(t) - \frac{1}{\alpha_{2,2}}E_2(t)^2 \\ F_2'(t) = -2vF_2(t) - \psi^2 F_2(t). \end{cases}$$

where,

$$\begin{aligned} A_2(t_2) &= \bar{A}_3, \\ D_2(t_2) &= \bar{D}_3, \\ E_2(t_2) &= \bar{E}_3, \\ F_2(t_2) &= \bar{F}_3. \end{aligned}$$

We will determine, $\bar{A}_3, \bar{D}_3, \bar{E}_3$, and $\bar{F}_3$. Recall, $w_{3,t}(t_2^+) = w_2(t_2-) - \xi_2$ and $x_3(t_2^+) = x_2(t_2^-) + \xi_2$. Then,

$$\begin{aligned}\phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2) &= \phi_3(t_2, w_{2,t}(t_2^-) - \xi_2, x_{2,t}(t_2^-) + \xi_2)\\ &= [A_3(t_2) - \alpha_{1,3}\xi_2 + D_3(t_2)\xi_2 + E_3(t_2)\xi_2^2 + F_3(t_2)\xi^2] + [\alpha_{1,3} - 2E_3(t_2)\xi_2]w_2(t_2)\\ &+ [D_3(t_2) + 2F_3(t_2)\xi_2]x_2(t_2) + E_3(t_2)w_2^2(t_2) + F_3(t_2)x_2^2(t_2)\end{aligned}$$

Therefore our terminal conditions are

$$\begin{aligned}\bar{A}_3 &= A_3(t_2) - \alpha_{1,3}\xi_2 + D_3(t_2)\xi_2 + E_3(t_2)\xi_2^2 + F_3(t_2)\xi_2^2,\\ \bar{D}_3 &= D_3(t_2) + 2F_3(t_2)\xi_2,\\ \bar{E}_3 &= E_3(t_2),\\ \bar{F}_3 &= F_3(t_2),\\ \alpha_{1,2} &= \alpha_{1,3} - 2E_3(t_2)\xi_2.\end{aligned}$$

Therefore the nonlinear differential equations have the following solutions,

$$\begin{aligned}A_2(t) &= A_3(t_2) - \alpha_{1,3}\xi_2 + D_3(t_2)\xi_2 + E_3(t_2)\xi_2^2 + F_3(t_2)\xi_2^2 + \alpha_{3,2}(t_2 - t)\\ &\quad - a_1\left[a_2\Big(1 - \exp\Big(\frac{(\mu - r)^2}{\sigma^2} - 2r\Big)(t_2 - t)\Big) + \frac{1}{\alpha_{2,2}}(t_2 - t)\right]\\ D_2(t) &= (D_3(t_2) + 2F_3(t_2)\xi_2)\exp(v(t_2 - t))\\ E_2(t) &= \frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{\Big(\frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{E_3(t_2)} - \frac{1}{\alpha_{2,2}}\Big)\exp\Big(\Big(\frac{(\mu-r)^2}{\sigma^2} - 2r\Big)(t_2 - t)\Big) + \frac{1}{\alpha_{2,2}}}\\ F_2(t) &= F_3(t_2)\exp((2v - \psi^2)(t_2 - t))\end{aligned}$$

where

$$\begin{aligned}a_1 &= \frac{(\mu - r)^2(\alpha_{1,3} - 2E_3(t_2)\xi_2)^2}{4\sigma^2\big(\frac{(\mu-r)^2}{\sigma^2} - 2r\big)},\\ a_2 &= \frac{\frac{(\mu-r)^2 - 2r}{E_3(t_2)^2} - \frac{1}{\alpha_{2,2}}}{2r - \frac{(\mu-r)^2}{\sigma^2}}.\end{aligned}$$

Going back to the first and final interval, $[t_0, t_1]$.

$$\max_{c_{1,t},\theta_{1,t}} \mathbb{E}[\int_t^{t_1} U_1(c_{1,s}w_{1,s})ds + \phi_2(t_1, w_1(t_1) - \xi_1, x_1(t_1) + \xi_1)] \tag{3.9}$$

$$\begin{aligned}\text{subject to}\quad dW_1 &= (r + (\mu - r)\theta_{1,t} - c_{1,t})W_{1,t}dt + \theta_{1,t}\sigma W_{1,t}dZ_{1,t}^1\\ dX_{1,t} &= vX_{1,t}dt + \psi\rho X_{1,t}dZ_{1,t}^1 + \psi\sqrt{1-\rho^2}X_{1,t}dZ_{1,t}^2\end{aligned} \tag{3.10}$$

Let $\phi_1(t, w_1, x_1)$ be the value function for this interval. Then,

$$\phi_1(t, w_1, x_1) = \max_{c_{1,t},\theta_{1,t}} \mathbb{E}[\int_t^{t_1} U_1(c_{1,s}w_{1,s})ds + \phi_2(t_1, w_1(t_1) - \xi_1, x_1(t_1) + \xi_1)]$$

The value function, $\phi_1(t, w_1, x_1)$ has the form,

$$\phi_1(t, w_1, x_1) = A_1(t) + \alpha_{1,1}w_1 + D_1(t)x_1 + E_1(t)w_1^2 + F_1(t)x_1^2.$$

The utility function for this interval,

$$U_1(c_{1,t}w_{1,t}) = -\alpha_{2,1}c_{1,t}^2 w_{1,2}^2 + \alpha_{1,1}c_{1,t}w_{1,t} + \alpha_{3,1}. \tag{3.11}$$

Optimal control variables for this interval

$$c_{1,t} = -\frac{1}{\alpha_{2,1}}E_1(t) \text{ and } \theta_{1,t} = \frac{-(\mu - r)(\alpha_{1,1} + 2E_1(t)w_1)}{2\sigma^2 E_1(t)w_1}. \tag{3.12}$$

Then our HJB equation for this interval becomes

$$\begin{aligned}0 = A_1'(t) - \frac{(\mu-r)^2(\alpha_{1,1})^2}{4\sigma^2 E_1(t)} + \alpha_{3,1} + [r\alpha_{1,1} - \frac{(\mu-r)^2}{\sigma^2}\alpha_{1,1}]w_1 + [D_1'(t) + vD_1(t)]x_1 \\ + [E_1'(t) + \frac{1}{\alpha_{2,1}}E_1^2(t) - \frac{(\mu-r)^2}{\sigma^2}E_1(t)]w_1^2 + [F_1'(t) + 2vF_1(t) + \psi^2 F_1(t)]x_1^2\end{aligned}$$

We have the following system of nonlinear differential equations,

$$\begin{cases} A_1'(t) = \frac{(\mu-r)^2(\alpha_{1,1})^2}{4\sigma^2 E_1(t)} - \alpha_{3,1} \\ 0 = \left(r - \frac{(\mu-r)^2}{\sigma^2}\right)\alpha_{1,1} \\ D_1'(t) = -vD_1(t) \\ E_1'(t) = \frac{(\mu-r)^2}{\sigma^2}E_1(t) - 2rE_1(t) - \frac{1}{\alpha_{2,1}}E_1(t)^2 \\ F_1'(t) = -2vF_1(t) - \psi^2 F_1(t). \end{cases}$$

where,

$$\begin{aligned} A_1(t_1) &= \bar{A}_2, \\ D_1(t_1) &= \bar{D}_2, \\ E_1(t_1) &= \bar{E}_2, \\ F_1(t_1) &= \bar{F}_2. \end{aligned}$$

We will determine, $\bar{A}_2, \bar{D}_2, \bar{E}_2$, and $\bar{F}_2$. Recall, $w_{2,t}(t_1^+) = w_{1,t}(t_1^-) - \xi_1$ and $x_{2,t}(t_1^+) = x_{1,t}(t_1^-) + \xi_1$. Then,

$$\begin{aligned}\phi_2(t_1, w_1(t_1) - \xi_1, x_1(t_1) + \xi_1) &= \phi_2(t_1, w_1(t_1^-) - \xi_1, x_1(t_1^-) + \xi_1) \\ &= [A_2(t_1) - \alpha_{1,1}\xi_1 + D_2(t_1)\xi_1 + E_2(t_1)\xi_1^2 + F_2(t_1)\xi_1] \\ &+ [\alpha_{1,1} - 2E_2(t_1)\xi_1]w_1(t_1) + [D_2(t_1) + 2F_2(t_1)\xi_1]x_1(t_1) \\ &+ E_2(t_1)w_1^2(t_1) + F_2(t_1)x_1^2(t_1)\end{aligned}$$

Then our terminal conditions are

$$\begin{aligned} \bar{A}_2 &= A_2(t_1) - \alpha_{1,2}\xi_1 + D_2(t_1)\xi_1 + E_2(t_1)\xi_1^2 + F_2(t_1)\xi_1^2, \\ \bar{D}_2 &= D_2(t_1) + 2F_2(t_1)\xi_1, \\ \bar{E}_2 &= E_2(t_1), \\ \bar{F}_2 &= F_2(t_1), \\ \alpha_{1,1} &= \alpha_{1,2} - 2E_2(t_1)\xi_1. \end{aligned}$$

Therefore the nonlinear differential equations have the following solutions,

$$
\begin{aligned}
A_1(t) &= A_2(t_1) - \alpha_{1,2}\xi_1 + D_2(t_1)\xi_1 + E_2(t_1)\xi_1^2 + F_2(t_1)\xi_1^2 + \alpha_{3,1}(t_1 - t) \\
&\quad - a_1\left[a_2\Big(1 - \exp\Big(\frac{(\mu-r)^2}{\sigma^2} - 2r\Big)(t_1 - t)\Big) + \frac{1}{\alpha_{2,1}}(t_1 - t)\right] \\
D_1(t) &= (D_2(t_1) + 2F_2(t_1)\xi_1)\exp(v(t_1 - t)) \\
E_1(t) &= \frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{\Big(\frac{\frac{(\mu-r)^2}{\sigma^2}-2r}{E_2(t_1)} - \frac{1}{\alpha_{2,1}}\Big)\exp\Big(\Big(\frac{(\mu-r)^2}{\sigma^2} - 2r\Big)(t_1 - t)\Big) + \frac{1}{\alpha_{2,1}}} \\
F_1(t) &= F_2(t_1)\exp((2v - \psi^2)(t_1 - t))
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{(\mu - r)^2(\alpha_{1,2} - 2E_2(t_1)\xi_1)^2}{4\sigma^2\big(\frac{(\mu-r)^2}{\sigma^2} - 2r\big)}, \\
a_2 &= \frac{\frac{(\mu-r)^2-2r}{E_2(t_1)^2} - \frac{1}{\alpha_{2,1}}}{2r - \frac{(\mu-r)^2}{\sigma^2}}.
\end{aligned}
$$

4. Determine the Size of the Impulses/Jumps

First, we will consider when the time of the impulse is known but the size of the impulse needs to be determined. There are two ways we can do this. One way is by finding the optimal size of the jump by solving a small nonlinear optimization problem after each interval using the value function of the previous interval. For every impulse there's one mini optimization problem. The other way is by setting up one nonlinear optimization problem to determine the size of the jump only using the value function from the first interval. In the following subsections, we outline both methods and provide numerical examples for both.

5. Determine ξ_1 and ξ_2

The optimal ξ_2 is determined by

$$
\max_{\xi_2} \mathbb{E}\Big[\phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)\Big] \tag{5.1}
$$

The optimal ξ_1 is determined by

$$
\begin{gathered}
\max_{\xi_1} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\
\text{subject to} \\
\phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)
\end{gathered} \tag{5.2}
$$

Since ξ_1 is based on ξ_2 we can determine ξ_1 and ξ_2 using one optimization problem at the end of the first interval. So, we have

$$\begin{gathered}\max_{\xi_1 \xi_2} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\ \text{subject to} \\ \phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)\end{gathered} \tag{5.3}$$

where ξ_1 and ξ_2 appear in $\mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big]$, which can be seen below.

$$\begin{aligned}\mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] &= A_2(t_1) - \alpha_{1,2}\xi_1 + D_2(t_1)\xi_1 + E_2(t_1)\xi_1^2 + F_2(t_1)\xi_1^2 \\ &\quad + \Big(\alpha_{1,2} - 2E_2(t_1)\xi_1\Big)\mathbb{E}\big[W_1(t_1)\big] + \Big(D_2(t_1) + 2F_2(t_1)\xi_1\Big)\mathbb{E}\big[X_1(t_1)\big] \\ &\quad + E_2(t_1)\mathbb{E}\big[W_1^2(t_1)\big] + F_2(t_1)\mathbb{E}\big[X_1^2(t_1)\big]\end{aligned} \tag{5.4}$$

Note, ξ_2 is present in (5.3) and (5.4) by the following.

$$\begin{aligned}A_2(t_1) &= A_3(t_2) - \alpha_{1,3}\xi_2 + D_3(t_2)\xi_2 + E_3(t_2)\xi_2^2 + F_3(t_2)\xi_2^2 + \alpha_{3,2}(t_2 - t_1) \\ &\qquad - a_1\left[a_2\Big(1 - \exp\Big(\frac{(\mu - r)^2}{\sigma^2} - 2r\Big)(t_2 - t_1)\Big) + \frac{1}{\alpha_{2,2}}(t_2 - t_1)\right] \\ \alpha_{1,2} &= \alpha_{1,3} - 2E_3(t_2)\xi_2 \\ D_2(t_1) &= (D_3(t_2) + 2F_3(t_2)\xi_2)\exp(v(t_2 - t_1)) \\ E_2(t_1) &= \frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{\Big(\frac{\frac{(\mu-r)^2}{\sigma^2} - 2r}{E_3(t_2)} - \frac{1}{\alpha_{2,2}}\Big)\exp\Big(\Big(\frac{(\mu-r)^2}{\sigma^2} - 2r\Big)(t_2 - t_1)\Big) + \frac{1}{\alpha_{2,2}}} \\ F_2(t_1) &= F_3(t_2)\exp((2v - \psi^2)(t_2 - t_1))\end{aligned}$$

Also, recall, our optimal control variable $0 \leq \theta_t \leq 1$. So our optimization problem becomes.

$$\begin{gathered}\max_{\xi_1 \xi_2} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\ \text{subject to} \\ 0 \leq \theta_{1,t} \leq 1 \\ 0 \leq \theta_{2,t} \leq 1 \\ 0 \leq \theta_{3,t} \leq 1 \\ \phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)\end{gathered} \tag{5.5}$$

Using (3.3), (3.7), and (3.11) we have

$$
\begin{aligned}
&\max_{\xi_1\xi_2} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\
&\text{subject to} \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,1} + 2E_1(t)\mathbb{E}\big[W_{1,t}\big])}{2\sigma^2 E_1(t)\mathbb{E}\big[W_{1,t}\big]} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,2} + 2E_2(t)\mathbb{E}\big[W_{2,t}\big])}{2\sigma^2 E_2(t)\mathbb{E}\big[W_{2,t}\big]} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} + 2E_3(t)\mathbb{E}\big[W_{3,t}\big])}{2\sigma^2 E_3(t)\mathbb{E}\big[W_{3,t}\big]} \leq 1 \\
&\phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)
\end{aligned}
\tag{5.6}
$$

We can rewrite α_1^1 and α_1^2 explicitly using ξ_1 and ξ_2.

$$
\begin{aligned}
&\max_{\xi_1\xi_2} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\
&\text{subject to} \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} - 2E_3(t_2)\xi_2 - 2E_2(t_1)\xi_1 + 2E_1(t)\mathbb{E}\big[W_{1,t}\big])}{2\sigma^2 E_1(t)\mathbb{E}\big[W_{1,t}\big]} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} - 2E_3(t_2)\xi_2 + 2E_2(t)\mathbb{E}\big[W_{2,t_2}\big])}{2\sigma^2 E_2(t)\mathbb{E}\big[W_{2,t}\big]} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} + 2E_3(t)\mathbb{E}\big[W_{3,t}\big])}{2\sigma^2 E_3(t)\mathbb{E}\big[W_{3,t}\big]} \leq 1 \\
&\phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)
\end{aligned}
\tag{5.7}
$$

Assume $E_1(t)W_{1,t}, E_2(t)W_{2,t}$, and $E_3(t)W_{3,t}$ are all bounded by γ. Then our constraints become

$$
\begin{aligned}
&\max_{\xi_1\xi_2} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\
&\text{subject to} \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} - 2E_3(t_2)\xi_2 - 2E_2(t_1)\xi_1 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} - 2E_3(t_2)\xi_2 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&\phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)
\end{aligned}
\tag{5.8}
$$

Lastly, we want to ensure that we do not transfer more than our liquid wealth at the end of each interval. So, our optimization problem to determine the size of the jumps is stated below

$$
\begin{aligned}
&\max_{\xi_1\xi_2} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\
&\text{subject to} \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} - 2E_3(t_2)\xi_2 - 2E_2(t_1)\xi_1 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} - 2E_3(t_2)\xi_2 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&0 \leq \frac{-(\mu - r)(\alpha_{1,3} + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&\xi_1 < \mathbb{E}\big[W_{1,t_1}\big] \\
&\xi_2 < \mathbb{E}\big[W_{2,t_2}\big] \\
&\phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2)
\end{aligned} \tag{5.9}
$$

6. **Numerical Examples**

6.1. **Procedure for Numerical Examples.**

1. Solve the objective function (5.9) to determine the size of the jumps ξ_1 and ξ_2 using *fmincon* in MATLAB. [Using *syms* for ξ_1 and ξ_2.]
2. Simulate the following N times.
   (a) For the first interval, $t \in [t_0, t_1]$, determine $A_1(t), \alpha_1^1(t), D_1(t), E_1(t),$ and $F_1(t)$ for this interval. Generate wealth paths for $W_{1,t}$ and $X_{1,t}$ for this interval. Use (3.11) to determine optimal control variables, $\theta_{1,t}$ and $c_{1,t}$.
   (b) For the second interval, $t \in [t_1, t_2]$, determine $A_2(t), \alpha_1^2(t), D_2(t), E_2(t),$ and $F_2(t)$ for this interval. Generate wealth paths for $W_{2,t}$ and $X_{2,t}$ for this interval. Use (3.7) to determine optimal control variables, $\theta_{2,t}$ and $c_{2,t}$.
   (c) For the last interval, $t \in [t_2, t_3]$, determine $A_3(t), \alpha_1^3(t), D_3(t), E_3(t),$ and $F_3(t)$ for this interval. Generate wealth paths for $W_{3,t}$ and $X_{3,t}$ for this interval. Use (3.3) to determine optimal control variables, $\theta_{3,t}$ and $c_{3,t}$.

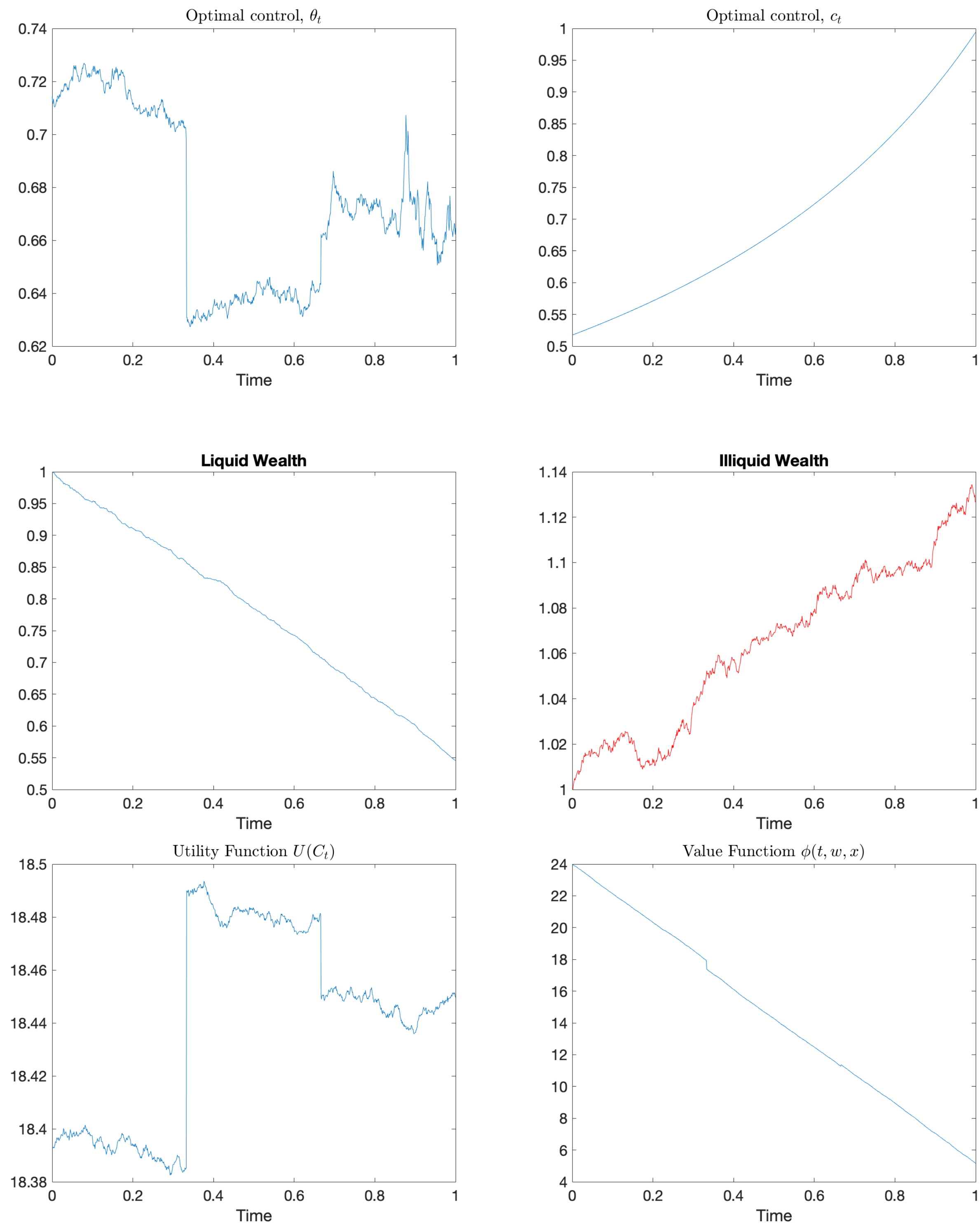
6.1.1. *Two Impulses.* On the interval $[0, 1]$ we will assume the following:

- Riskless Bond : $r = 0.05$
- Liquid Asset: $\mu = 0.12$ and $\sigma = \sqrt{\frac{\mu^2}{r} - 2\mu + r} = 0.313$
- Illiquid Asset: $v = 0.12, \psi = 0.313,$ and $\rho = 0$
- Initial Investment: $w_0 = 1$ and $x_0 = 1$
- For the final interval:
  - Utility Function: $U(C_t^3) = 2(C_t^3)^2 - 4C_t^3 + 20$
  - Terminal Condition: $\phi_3(1, w, x) = (1.41w - 1.4184)^2 + (x + 1)^2$

On the interval $[0, 1]$ two impulses ξ_1 and ξ_2 occur when $t_1 = 0.3333$ and $t_2 = 0.6667$. With $\gamma = 1$ optimal ξ_1 and ξ_2 determined by (5.9) are

$$\xi_1 = 0.0798 \text{ and } \xi_2 = -0.0185. \tag{6.1}$$

The following results are after $N = 100$ simulations. Using (3.3), (3.7), and (3.11) our optimal θ_t and c_t are determined.



The value of the value function is,

$$\phi_1(0,1,1) = 23.9903.$$

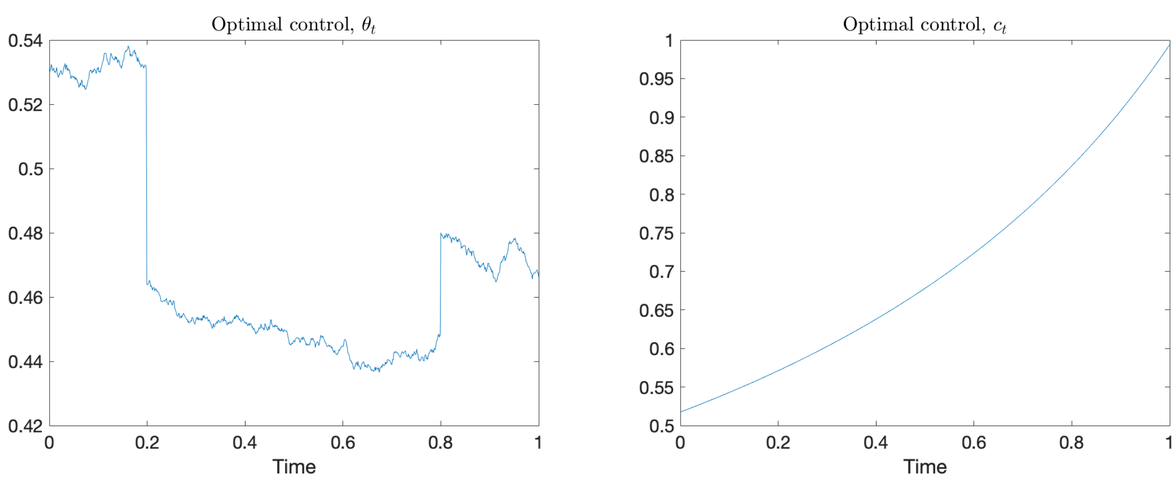
6.1.2. *Four Impulses.* On the interval $[0,1]$ we will assume the following:

- Riskless Bond : $r = 0.05$
- Liquid Asset: $\mu = 0.12$ and $\sigma = \sqrt{\frac{\mu^2}{r} - 2\mu + r} = 0.313$
- Illiquid Asset: $v = 0.12, \psi = 0.313,$ and $\rho = 0$
- Initial Investment: $w_0 = 1$ and $x_0 = 1$
- For the final interval:
  - Utility Function: $U(C_t^3) = 2(C_t^3)^2 - 3.5C_t^3 + 20$
  - Terminal Condition: $\phi_3(1,w,x) = (1.41w - 1.4184)^2 + (x+1)^2$

On the interval $[0,1]$ four impulses $\xi_1, \xi_2, \xi_3,$ and ξ_4 occur when $t_1 = 0.2, t_2 = 0.4, t_3 = 0.6,$ and $t_4 = 0.8$. With $\gamma = 1$ optimal $\xi_1, \xi_2, \xi_3,$ and ξ_4 determined by applying (5.9) to four impulses are

$$\xi_1 = 0.0858, \xi_2 = 0.0, \xi_3 = 0.0, \text{ and } \xi_4 = -0.0272. \tag{6.2}$$

The following results are after $N = 100$ simulations. Using (3.3), (3.7), and (3.11) our optimal θ_t and c_t are determined.



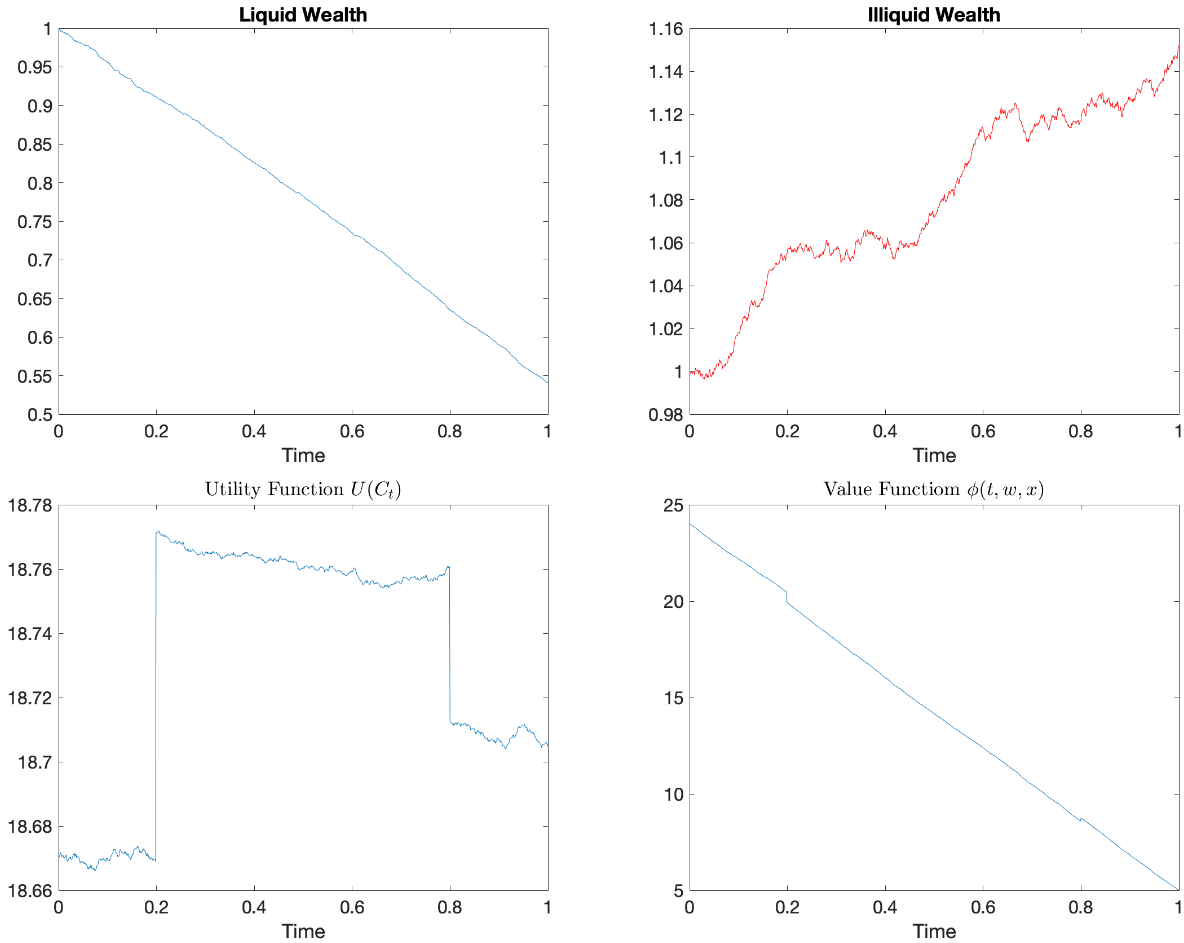
The value of the value function is,

$$\phi_1(0,1,1) = 24.0264.$$

7. Determine Size of Impulses and their Occurrences

Now, we will consider, if the number of rebalancing times is known, but when they occur and and size of the impulse is random.

Let $t_1 = 0.333 + a$ and $t_2 = 0.6667 + b$. The optimal t_1, t_2, ξ_1 and ξ_2 are determined by



$$
\begin{aligned}
&\max_{\xi_1,\xi_2,a,b} \mathbb{E}\Big[\phi_2(t_1, w_{1,t_1} - \xi_1, x_{1,t_1} + \xi_1)\Big] \\
&\text{subject to} \\
&\qquad 0 \leq \frac{-(\mu - r)(\alpha_1^3 - 2E_3(t_2)\xi_2 - 2E_2(t_1)\xi_1 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&\qquad 0 \leq \frac{-(\mu - r)(\alpha_1^3 - 2E_3(t_2)\xi_2 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&\qquad 0 \leq \frac{-(\mu - r)(\alpha_1^3 + 2\gamma)}{2\sigma^2\gamma} \leq 1 \\
&\qquad \xi_1 < \mathbb{E}\big[W_{1,t_1}\big] \\
&\qquad \xi_2 < \mathbb{E}\big[W_{2,t_2}\big] \\
&\phi_2(t_2, w_{2,t_2}, x_{2,t_2}) = \phi_3(t_2, w_{2,t_2} - \xi_2, x_{2,t_2} + \xi_2) \\
&\qquad -0.3333 < a < 0.3333 \\
&\qquad -0.3333 < b < 0.3333
\end{aligned} \tag{7.1}
$$

See the derivation for (5.9) on how to determine γ.

7.0.1. *Two Impulses.* On the interval $[0,1]$ we will assume the following:

- Riskless Bond : $r = 0.05$
- Liquid Asset: $\mu = 0.12$ and $\sigma = \sqrt{\frac{\mu^2}{r} - 2\mu + r} = 0.313$
- Illiquid Asset: $v = 0.12, \psi = 0.313,$ and $\rho = 0$
- Initial Investment: $w_0 = 1$ and $x_0 = 1$
- For the final interval:
  - Utility Function: $U(C_t^3) = 2(C_t^3)^2 - 4C_t^3 + 20$
  - Terminal Condition: $\phi_3(1, w, x) = (1.41w - 1.4184)^2 + (x+1)^2$

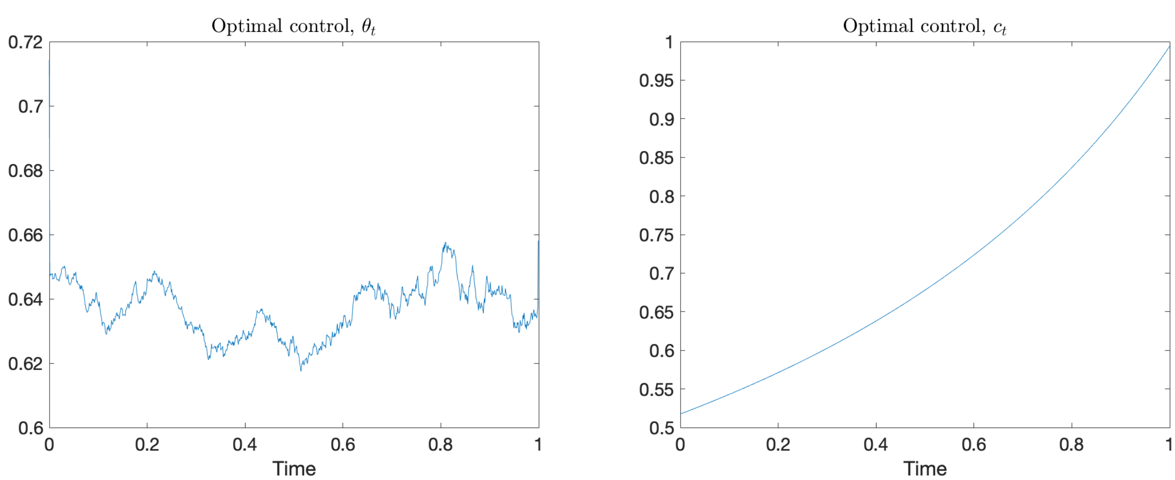
On the interval $[0,1]$ two impulses ξ_1 and ξ_2 occur at t_1 and t_2. With $\gamma = 1$ optimal $\xi_1, \xi_2, t_1,$ and t_2 determined by (7.1) are

$$\xi_1 = 0.0946 \text{ and } \xi_2 = -0.0141 \tag{7.2}$$

and

$$t_1 = 0.0010 \text{ and } t_2 = 0.9990. \tag{7.3}$$

The following results are after $N = 100$ simulations. Using (3.3), (3.7), and (3.11) our optimal θ_t and c_t are determined.

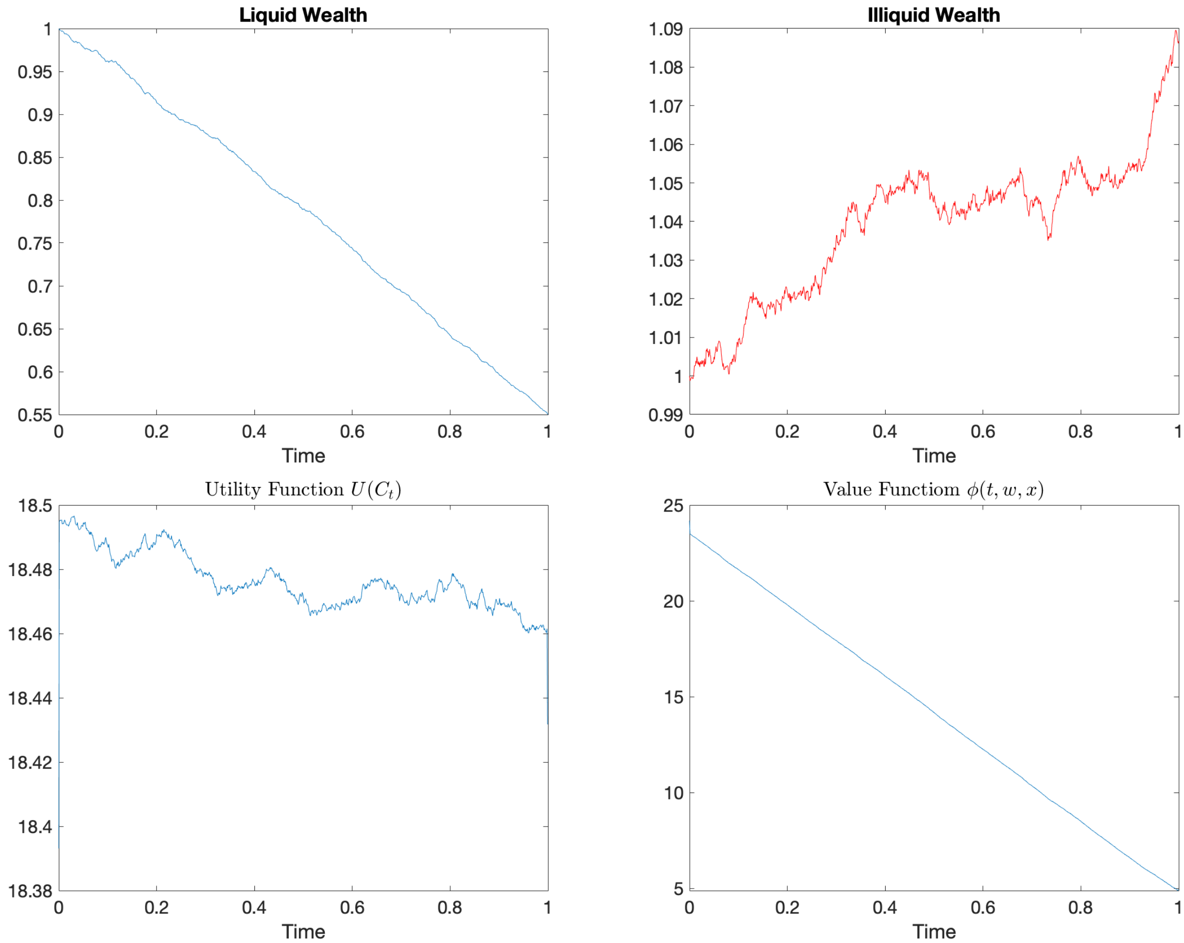


The value of the value function is,

$$\phi_1(0, 1, 1) = 24.1845.$$

7.0.2. *Four Impulses.* On the interval $[0,1]$ we will assume the following:

- Riskless Bond : $r = 0.05$
- Liquid Asset: $\mu = 0.12$ and $\sigma = \sqrt{\frac{\mu^2}{r} - 2\mu + r} = 0.313$
- Illiquid Asset: $v = 0.12, \psi = 0.313,$ and $\rho = 0$



- Initial Investment: $w_0 = 1$ and $x_0 = 1$
- For the final interval:
  - Utility Function: $U(C_t^3) = 2(C_t^3)^2 - 3.5C_t^3 + 20$
  - Terminal Condition: $\phi_3(1, w, x) = (1.41w - 1.4184)^2 + (x + 1)^2$

On the interval $[0, 1]$ four impulses $\xi_1, \xi_2, \xi_3,$ and ξ_4 occur at $t_1, t_2, t_3,$ and t_4. With $\gamma = 1$ optimal $\xi_1, \xi_2, \xi_3, \xi_4, t_1, t_2, t_3,$ and t_4 are determined by applying (7.1) to four impulses are

$$\xi_1 = 0.0945, \xi_2 = -0.0015, \xi_3 = 0.0015, \text{ and } \xi_4 = -0.0229 \tag{7.4}$$

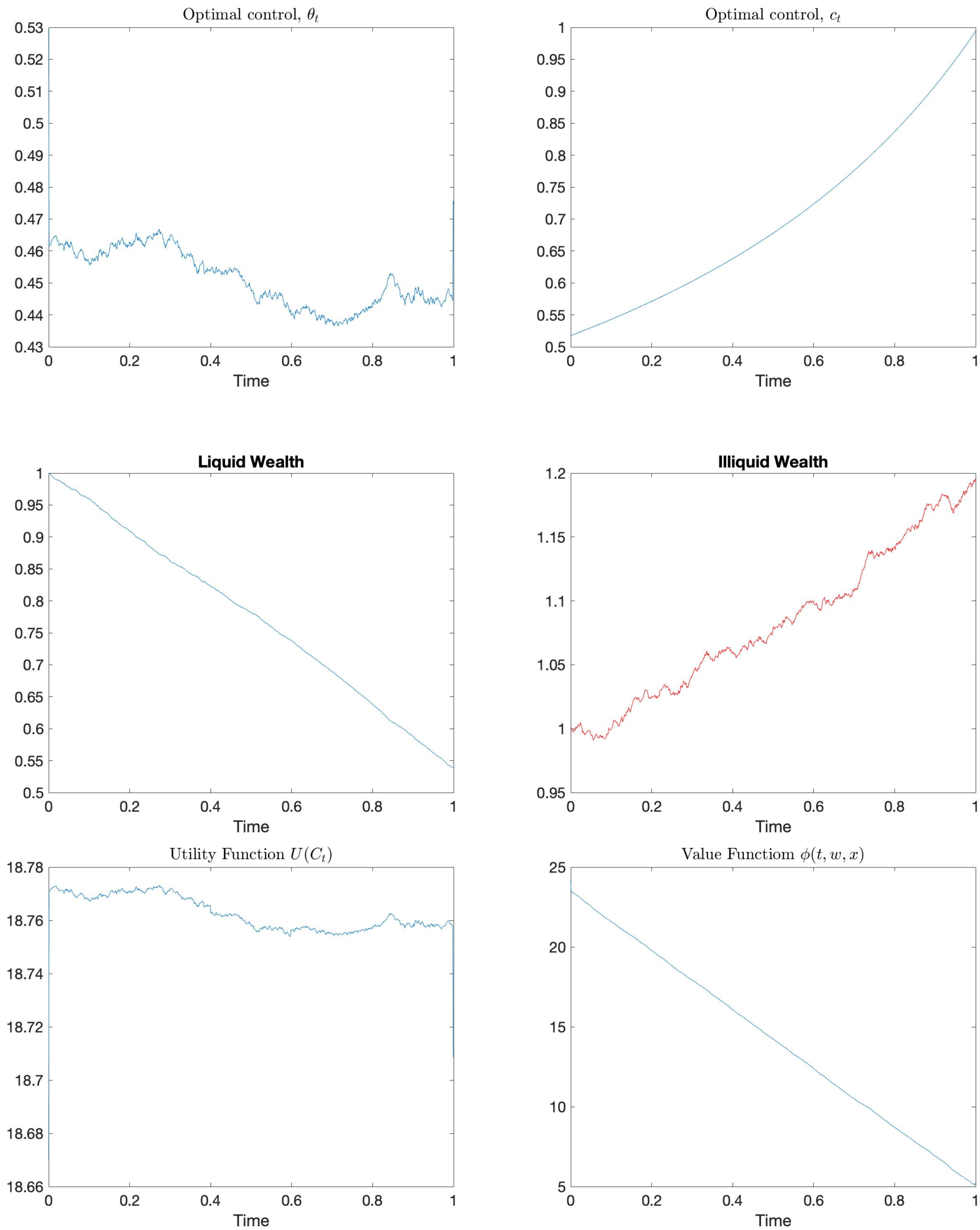
and

$$t_1 = 0.0010, t_2 = 0.4896, t_3 = 0.4929, \text{ and } t_4 = 0.9977. \tag{7.5}$$

The following results are after $N = 100$ simulations. Using (3.3), (3.7), and (3.11) our optimal θ_t and c_t are determined.

The value of the value function is,

$$\phi_1(0, 1, 1) = 24.1393.$$



8. Conclusion

We note that illiquid wealth increases in each numerical example as desired. The consumption also increases in each case. These are two desired features. These features are desirable goals of the rebalancing problem.

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