

FACTORIZATION OF SCALAR PIECEWISE CONTINUOUS ALMOST PERIODIC FUNCTIONS

LIANGPING QI AND RONG YUAN

School of Science and Technology, Tianjin University of Finance and Economics,
300222, Tianjin, P. R. China.

Laboratory of Mathematics and Complex Systems (Ministry of Education), School
of Mathematical Sciences, Beijing Normal University, 100875, Beijing, P. R. China.

ABSTRACT. This paper is to characterize piecewise continuous almost periodic functions as the product of Bohr almost periodic functions and sequences. As applications, the result is used to discuss piecewise continuous almost periodic solutions of impulsive differential equations.

AMS (MOS) Subject Classification. 42A75, 34A37, 34C27.

Key Words and Phrases. Piecewise Continuous Almost Periodic Functions; Impulsive Differential Equations; Factorization.

1. Introduction

Impulsive differential equations can model many real processes and phenomena studied in mechanics, radio engineering, communication security, control theory, neural networks, etc. since the states of many evolutionary processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. Piecewise continuous almost periodic (p.c.a.p., for short) functions come from the study of impulsive differential equations and certain problems in real phenomena, see e.g. [2, 3, 6, 7, 13, 14, 17] and [1, 4, 10, 11, 19] respectively for various works devoted to impulsive differential equations and almost periodic differential equations.

The definition of p.c.a.p. functions is complex and consists of three conditions (Definition 1.3). The aim of this paper is to present a synthesis property of p.c.a.p. functions. This property can be used as not only a good characterization of p.c.a.p. functions, but also a useful tool in the discussion of p.c.a.p. solutions to impulsive differential equations. Since it is related to Bohr almost periodic functions and sequences, recall that

Definition 1.1 ([5, p. 45], [13, p. 183]). A continuous function $f : \mathbb{G} \rightarrow \mathbb{R}$, where $\mathbb{G} = \mathbb{R}$ or \mathbb{Z} , is called (Bohr) almost periodic if given any $\epsilon > 0$, the ϵ -almost periodic

set of f ,

$$T(f, \epsilon) := \{\tau \in \mathbb{G}; |f(t + \tau) - f(t)| < \epsilon, \forall t \in \mathbb{G}\}$$

is relatively dense, that is, there is a positive number $l = l(\epsilon) \in \mathbb{G}$ satisfying $[a, a + l] \cap T(f, \epsilon) \neq \emptyset$ for all $a \in \mathbb{G}$.

Denote by $AP(\mathbb{G}, \mathbb{R})$ the set of all Bohr almost periodic functions from \mathbb{G} to \mathbb{R} . $AP(\mathbb{G}, \mathbb{R})$ is a Banach space equipped with the norm $\|f\| = \sup_{t \in \mathbb{G}} |f(t)|$.

We call a sequence $\{\tau_j\}_{j \in \mathbb{Z}} \subset \mathbb{R}$ admissible if $\tau_j < \tau_{j+1}$ for all $j \in \mathbb{Z}$ and $\lim_{j \rightarrow \pm\infty} \tau_j = \pm\infty$. Put $\tau_j^k = \tau_{j+k} - \tau_j$, where $j, k \in \mathbb{Z}$.

Definition 1.2 ([13, p. 195]). An admissible sequence $\{\tau_j\}_{j \in \mathbb{Z}}$ is called a Wexler sequence if $\inf_{j \in \mathbb{Z}} \tau_j^1 > 0$ and the family of sequences

$$\{\{\tau_j^k\}\} := \{\{\tau_j^k\}_{j \in \mathbb{Z}}\}_{k \in \mathbb{Z}}$$

is equi-potentially almost periodic, i.e. for each $\epsilon > 0$ the common ϵ -almost periodic set of all the sequences $\{\{\tau_j^k\}\}$,

$$T(\{\{\tau_j^k\}\}, \epsilon) := \{p \in \mathbb{Z}; |\tau_{j+p}^k - \tau_j^k| < \epsilon \text{ for all } j, k \in \mathbb{Z}\}$$

is relatively dense.

Let $PC(\mathbb{R}, \mathbb{R})$ be the set of piecewise continuous functions $h : \mathbb{R} \rightarrow \mathbb{R}$ which have discontinuities of the first kind only at the points of a subset of an admissible sequence $\{\tau_j = \tau_j(h)\}_{j \in \mathbb{Z}}$ and are continuous from the left at $\{\tau_j\}_{j \in \mathbb{Z}}$.

Definition 1.3 ([13, p. 201]). A function $h \in PC(\mathbb{R}, \mathbb{R})$ is called piecewise continuous almost periodic (p.c.a.p.) if the following conditions hold.

- (i) There is an admissible sequence $\{\tau_j\}_{j \in \mathbb{Z}}$ containing possible discontinuities of h .
The family of sequences $\{\{\tau_j^k\}\}$ is equi-potentially almost periodic.
- (ii) For each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|h(s) - h(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$.
- (iii) For each $\epsilon > 0$, the ϵ -almost periodic set of h ,

$$T(h, \epsilon) := \{\tau \in \mathbb{R}; |h(t + \tau) - h(t)| < \epsilon \text{ for all } t \in \mathbb{R} \\ \text{such that } |t - \tau_j| > \epsilon, j \in \mathbb{Z}\}$$

is relatively dense.

Denote by $PCAP(\mathbb{R}, \mathbb{R})$ the set of all p.c.a.p. functions. Our main results are formulated as follows.

Theorem 1.4. *Suppose that $h \in PCAP(\mathbb{R}, \mathbb{R})$ has discontinuities at the points of a subset of a Wexler sequence $\{\tau_j\}_{j \in \mathbb{Z}}$ and*

$$(1.1) \quad \inf_{n \in \mathbb{Z}} |h(\tau_n)| > 0, \quad \inf_{n \in \mathbb{Z}} |h(\tau_n^+)| > 0.$$

If the difference equation

$$(1.2) \quad x(n) = \frac{h(\tau_n^+)}{h(\tau_n)} \cdot x(n - 1)$$

has a bounded solution v with $\inf_{n \in \mathbb{Z}} |v(n)| > 0$, then there exists a unique $f \in AP(\mathbb{R}, \mathbb{R})$ and $u \in AP(\mathbb{Z}, \mathbb{R})$, up to a nonzero multiplicative constant, such that

$$(1.3) \quad h(t) = f(t) \cdot u(n), \quad \tau_n < t \leq \tau_{n+1}, n \in \mathbb{Z}.$$

Theorem 1.5. *Suppose that $f \in AP(\mathbb{R}, \mathbb{R})$, $u \in AP(\mathbb{Z}, \mathbb{R})$ and $\{\tau_j\}_{j \in \mathbb{Z}}$ is a Wexler sequence. Then the function h defined by (1.3) is p.c.a.p.*

Theorems 1.4 and 1.5 give us a different understanding of p.c.a.p. functions by characterizing them as the synthesis of Bohr almost periodic functions and sequences. So it is convenient to generate new p.c.a.p. functions and investigate their properties from Bohr functions and sequences. In addition, (1.3) shows that p.c.a.p. functions have a delicate structure which deserves more study.

As for impulsive differential equations, Theorems 1.4 and 1.5 indicate the direction of looking for solutions in the factorization form and provide an answer to the almost periodicity of the functions in an assumption of some literature (see (A1) below). Changing variables is an important tool in dynamics. To simplify the analysis of the following impulsive dynamical system in \mathbb{R}^d ,

$$(1.4) \quad \begin{cases} y'_i = g_i(y_1, \dots, y_d, t), & t \neq \tau_n, \\ y_i(\tau_n^+) = [1 + b_i(n)]y_i(\tau_n), & n \in \mathbb{Z}_+, i = 1, \dots, d, \end{cases}$$

where $\{g_i\}_{i=1}^d$ are continuous functions and $0 < \tau_1 < \tau_2 < \dots$, it is convenient to make the following change of variables

$$(1.5) \quad \begin{cases} y_i(t) \mapsto \frac{y_i(t)}{\varphi_i(t)}, & t > 0, \\ \varphi_i(t) := \prod_{0 < \tau_k < t} [1 + b_i(t)], & i = 1, \dots, d, \end{cases}$$

which transforms (1.4) to the following system

$$(1.6) \quad y'_i = \frac{g_i(\varphi_1(t)y_1, \dots, \varphi_d(t)y_d, t)}{\varphi_i(t)}, \quad t \neq \tau_n, n \in \mathbb{Z}_+, i = 1, \dots, d,$$

so that (ϕ_1, \dots, ϕ_d) is a solution to (1.4) if and only if $(\phi_1/\varphi_1, \dots, \phi_d/\varphi_d)$ is a solution to (1.6). In order to obtain p.c.a.p. solutions to (1.4), [8, 9, 15, 16, 20, 21] make the assumption

(A1) The functions $\{\varphi_i\}_{i=1}^d$ are bounded and p.c.a.p.

However, there exists no theoretical reason for the almost periodicity of the functions in (1.5) in literature. Our Theorems 1.4 and 1.5 can solve this problem. To

show this, we first need a modification of the functions in (1.5) since they are only defined for $t > 0$. For $i = 1, \dots, d$, let

$$(1.7) \quad u_i(n) := \begin{cases} \prod_{k=1}^n [1 + b_i(k)], & n > 0; \\ 1, & n = 1; \\ \prod_{k=n+1}^0 \frac{1}{1 + b_i(k)}, & n < 0. \end{cases}$$

and

$$(1.8) \quad \begin{aligned} \psi_i(t) &:= \begin{cases} \prod_{\tau_0 < \tau_k < t} [1 + b_i(k)], & t \geq \tau_1; \\ 1, & \tau_0 < t \leq \tau_1; \\ \prod_{t \leq \tau_k \leq \tau_0} \frac{1}{1 + b_i(k)}, & t \leq \tau_0. \end{cases} \\ &= u_i(n), \quad \tau_n < t \leq \tau_{n+1}, n \in \mathbb{Z}, \end{aligned}$$

then u_i is a solution to

$$(1.9) \quad x(n) = [1 + b_i(n)]x(n - 1), \quad n \in \mathbb{Z},$$

and

$$\psi_i(\tau_n^+) = [1 + b_i(n)]\psi_i(\tau_n), \quad n \in \mathbb{Z}.$$

Instead of (A1), we propose the assumption

(A2) For every integer $i \in [1, d]$, the sequence b_i is almost periodic, $\inf_{n \in \mathbb{Z}} |1 + b_i(n)| > 0$, and (1.9) admits a bounded solution v_i with $\inf_{n \in \mathbb{Z}} |v_i(n)| > 0$.

Although (A1) looks simpler than (A2), (A1) is fruition rather than source and has no theoretical basis. (A2) concerns the essence and will be shown crucial in Theorem 2.7. Our answer to the almost periodicity of the functions in the transformation

$$(1.10) \quad y_i(t) \mapsto \frac{y_i(t)}{\psi_i(t)}, \quad t \in \mathbb{R}, i = 1, \dots, d$$

associated with impulsive differential equations reads as follows.

Theorem 1.6. *Suppose that $\{\tau_j\}_{j \in \mathbb{Z}}$ is a Wexler sequence and (A2) holds.*

(i) *If (ϕ_1, \dots, ϕ_d) is a p.c.a.p. solution to*

$$(1.11) \quad \begin{cases} y'_i = g_i(y_1, \dots, y_d, t), & t \neq \tau_n, \\ y_i(\tau_n^+) = [1 + b_i(n)]y_i(\tau_n), & n \in \mathbb{Z}, i = 1, \dots, d, \end{cases}$$

and $\phi_i(\tau_n) \neq 0$ for all $n \in \mathbb{Z}$ and $i = 1, \dots, d$, then $(\phi_1/\psi_1, \dots, \phi_d/\psi_d)$ is a Bohr almost periodic solution to

$$(1.12) \quad y'_i = \frac{g_i(\psi_1(t)y_1, \dots, \psi_d(t)y_d, t)}{\psi_i(t)}, \quad t \neq \tau_n, n \in \mathbb{Z}, i = 1, \dots, d.$$

(ii) *If $(\phi_1/\psi_1, \dots, \phi_d/\psi_d)$ is a Bohr almost periodic solution to (1.12), then (ϕ_1, \dots, ϕ_d) is a p.c.a.p. solution to (1.11).*

We shall study in Section 2 almost periodic solutions to homogeneous difference equations, which is crucial in proving our main results. In Section 3 we prove Theorems 1.4-1.6.

2. Homogeneous difference equations

To prove Theorems 1.4 and 1.5 we need some results on difference equations. We shall give an equivalent condition on the existence of almost periodic solutions to the following difference equation in \mathbb{R} ,

$$(2.1) \quad x(n) = a(n)x(n - 1), \quad n \in \mathbb{Z},$$

where $a(n) \neq 0$. It is easy to check that for each $x_0 \in \mathbb{R}$ the initial value problem

$$(2.2) \quad \begin{cases} x(n) = a(n)x(n - 1), & n \in \mathbb{Z}, \\ x(n_0) = x_0 \end{cases}$$

admits on \mathbb{Z} a unique solution ϕ given by

$$(2.3) \quad \phi(n) = \begin{cases} [\prod_{k=n_0+1}^n a(k)] \cdot x_0, & n > n_0; \\ x_0, & n = n_0; \\ [\prod_{k=n+1}^{n_0} \frac{1}{a(k)}] \cdot x_0, & n < n_0. \end{cases}$$

which is trivial (identically zero) if and only if it attains zero value at any point. Define a function, which is essentially the Cauchy matrix of (2.1), by

$$(2.4) \quad V(n, m) = \begin{cases} \prod_{k=m+1}^n a(k), & n > m; \\ 1, & n = m; \\ \prod_{k=n+1}^m \frac{1}{a(k)}, & n < m. \end{cases}$$

where $m, n \in \mathbb{Z}$. It is easy to see that

$$\begin{aligned} V(n, m) &= a(n) \cdot V(n - 1, m) = V(n, m - 1) \cdot \frac{1}{a(m)} \\ &= \frac{1}{V(m, n)} = V(n, l)V(l, m) \end{aligned}$$

for all $l, m, n \in \mathbb{Z}$. Therefore, $\phi(n) = V(n, n_0) \cdot x_0$.

The following result is obvious from the homogeneity.

Lemma 2.1. *Suppose that u and v are two nontrivial (not identically zero) solutions to (2.1), then there exists a constant $c \in \mathbb{R}$, $c \neq 0$ such that $u = cv$.*

We are interested in almost periodic solutions to (2.1). The following lemma provides a necessary condition.

Lemma 2.2. *Suppose that $a \in AP(\mathbb{Z}, \mathbb{R})$ and $0 < \vartheta := \inf_{n \in \mathbb{Z}} |a(n)| < \|a\|$. If (2.1) has a nontrivial almost periodic solution, then $\vartheta < 1 < \|a\|$.*

Proof. Assume the contrary that $\|a\| \leq 1$. It follows that

$$|a(n)| \leq 1, \frac{1}{|a(n)|} \geq 1, \quad n \in \mathbb{Z}.$$

If $u \in AP(\mathbb{Z}, \mathbb{R})$ is a nontrivial solution to (2.1), then

$$|u(n)| \leq |u(0)|, \quad n \geq 0; \quad |u(n)| \geq |u(0)|, \quad n \leq 0.$$

Since $\vartheta < \|a\|$, (2.1) admits no nontrivial constant solution. Therefore, u is not constant on \mathbb{Z} , and there is an $l \in \mathbb{Z}_+$ such that $|u(l)| < |u(0)|$. Given $\epsilon \in (0, |u(0)| - |u(l)|)$, the set $T(u, \epsilon)$ is relatively dense from the almost periodicity of u . Let $p \in T(u, \epsilon)$, $p + l < 0$. A direct calculation implies the contradiction

$$|u(0)| \leq |u(p + l)| < |u(l)| + \epsilon < |u(0)|.$$

The proof for the case of $\vartheta \geq 1$ is similar. So we omit it. □

The following lemma rules out the possibility that $\inf_{n \in \mathbb{Z}} |u(n)| = 0$ for a desired solution u .

Lemma 2.3. *Suppose that $a \in AP(\mathbb{Z}, \mathbb{R})$, $\inf_{n \in \mathbb{Z}} |a(n)| = \vartheta > 0$, and u is a nontrivial bounded solution to (2.1), $\inf_{n \in \mathbb{Z}} |u(n)| = 0$. Then $u \notin AA(\mathbb{Z}, \mathbb{R})$, the space of real almost automorphic sequences v such that from any sequence $\{n'_k\}_{k=1}^\infty \subset \mathbb{Z}$ we can extract a subsequence $\{n_k\}_{k=1}^\infty \subset \{n'_k\}_{k=1}^\infty$ with*

$$\lim_{k \rightarrow \infty} v(n + n_k) = w(n), \quad \lim_{k \rightarrow \infty} w(n - n_k) = v(n), \quad n \in \mathbb{Z},$$

for some sequence w .

Proof. Since u is nontrivial, $u(n) \neq 0$ for all $n \in \mathbb{Z}$. Let $\{n'_k\}_{k=1}^\infty \subset \mathbb{Z}$ be a sequence such that $\lim_{k \rightarrow \infty} u(n'_k) = 0$. Using the diagonal procedure, it follows from the boundedness of u that there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \{n'_k\}_{k=1}^\infty$ with $\lim_{k \rightarrow \infty} u(n + n_k) = \hat{u}(n)$ for all $n \in \mathbb{Z}$ and some sequence \hat{u} . In particular, $\hat{u}(0) = 0$ by assumption. Because $a \in AP(\mathbb{Z}, \mathbb{R})$, we may assume further that $\lim_{k \rightarrow \infty} a(n + n_k) = \hat{a}(n)$ uniformly for all $n \in \mathbb{Z}$ and some $\hat{a} \in AP(\mathbb{Z}, \mathbb{R})$. Consequently, the equality

$$u(n + n_k) = a(n + n_k)u(n + n_k - 1)$$

implies

$$\hat{u}(n) = \hat{a}(n)\hat{u}(n - 1)$$

for all $n \in \mathbb{Z}$. Moreover, from $\vartheta \leq |a(n + n_k)| \leq \|a\|$ it follows that $\vartheta \leq |\hat{a}(n)| \leq \|a\|$ for all $n \in \mathbb{Z}$. So, $\hat{u}(n) = 0$ for all $n \in \mathbb{Z}$. Hence $\lim_{k \rightarrow \infty} \hat{u}(n - n_k) = 0 \neq u(n)$ for all $n \in \mathbb{Z}$. Thus $u \notin AA(\mathbb{Z}, \mathbb{R})$. □

Remark 2.4. Lemma 4.7 in [18] proves that an almost periodic solution ϕ to $x(n + 1) = C(n)x(n)$, $n \in \mathbb{Z}$, where $C \in AP(\mathbb{Z}, \mathbb{R}^d)$, $d \in \mathbb{Z}_+$, satisfies either $\inf_{n \in \mathbb{Z}} |\phi(n)| > 0$ or $\phi(n) = 0$ for all $n \in \mathbb{Z}$. Our lemma shows that such a nontrivial bounded solution u with $\inf_{n \in \mathbb{Z}} |u(n)| = 0$ is even not almost automorphic.

Consider auxiliary equations

$$(2.5) \quad x(n) = |a(n)| \cdot x(n - 1),$$

$$(2.6) \quad x(n) = [\text{sgn } a(n)] \cdot x(n - 1),$$

of which the Cauchy matrices are respectively $|V(n, m)|$ and $\text{sgn } V(n, m)$, where $m, n \in \mathbb{Z}$. It is easy to prove the following result.

Lemma 2.5. *The following statements are true.*

- (i) *If u is a solution to (2.1), then $\{|u(n)|\}_{n \in \mathbb{Z}}$ is a solution to (2.5).*
- (ii) *If v is a solution to (2.5), then $\{v(n) \cdot \text{sgn } V(n, m)\}_{n \in \mathbb{Z}}$ is a solution to (2.1) for each $m \in \mathbb{Z}$.*
- (iii) *If u, v, w are respectively nontrivial solutions to (2.1), (2.5), (2.6), then $v/w = v \cdot w, u/w = u \cdot w, u/v = v/u$ are respectively solutions to (2.1), (2.5), (2.6).*
- (iv) *u is a solution, bounded or with $\inf_{n \in \mathbb{Z}} |u(n)| > 0$, to (2.1), if and only if v is a solution, bounded or with $\inf_{n \in \mathbb{Z}} |v(n)| > 0$, to (2.5), where $v(n) = |u(n)|, u(n) = v(n) \cdot [\text{sgn } V(n, m)] \cdot u(m)/|u(m)| = v(n) \cdot [\text{sgn } V(n, m)u(m)]$.*
- (v) *If u is an almost periodic solution to (2.1), then $\{|u(n)|\}_{n \in \mathbb{Z}}$ is an almost periodic solution to (2.5). If in addition, $\inf_{n \in \mathbb{Z}} |u(n)| > 0$, then $\{\text{sgn } u(n)\}_{n \in \mathbb{Z}}$ is an almost periodic solution to (2.6).*

One may expect integer-valued almost periodic sequences to be periodic.

Lemma 2.6. *Suppose that $a \in AP(\mathbb{Z}, \mathbb{R})$ and $\inf_{n \in \mathbb{Z}} |a(n)| = \vartheta > 0$, then all solutions to (2.6) are $|2p|$ -periodic, where $p \in T(a, \epsilon)$, $p \neq 0$ and $0 < \epsilon < 2\vartheta$.*

Proof. It suffices to consider nontrivial solutions u of (2.6). Let $\epsilon, 0 < \epsilon < 2\vartheta$, be given. For each $p \in T(a, \epsilon)$ and $n \in \mathbb{Z}$, the inequality

$$|a(n + p) - a(n)| < \epsilon < 2\vartheta$$

implies $\text{sgn } a(n + p) = \text{sgn } a(n)$. Consequently, from the equalities

$$V(n + 2p, n) = \begin{cases} \prod_{k=n+1}^{n+2p} a(k) = \prod_{k=n+1}^{n+p} a(k)a(k + p), & p > 0; \\ 1, & p = 0; \\ \prod_{k=n}^{n+2p+1} \frac{1}{a(k)} = \prod_{k=n}^{n+p+1} \frac{1}{a(k)a(k + p)}, & p > 0. \end{cases}$$

it follows that $\text{sgn } V(n + 2p, n) = 1$ for all $n \in \mathbb{Z}$. Therefore,

$$u(n + 2p) = [\text{sgn } V(n + 2p, n)] \cdot u(n) = u(n)$$

for all $n \in \mathbb{Z}$. □

We are in the position giving an equivalent condition for the existence of almost periodic solutions to (2.1).

Theorem 2.7. *Suppose that $a \in AP(\mathbb{Z}, \mathbb{R})$ and $\inf_{n \in \mathbb{Z}} |a(n)| = \vartheta > 0$, then a nontrivial solution u to (2.1) is almost periodic if and only if u is bounded on \mathbb{Z} and $\inf_{n \in \mathbb{Z}} |u(n)| > 0$.*

Proof. Necessity. Suppose that $u \in AP(\mathbb{Z}, \mathbb{R})$ is a nontrivial solution to (2.1). Clear, u is bounded on \mathbb{Z} . From Lemma 2.3 it follows that $\inf_{n \in \mathbb{Z}} |u(n)| > 0$.

Sufficiency. Let u be a bounded solution to (2.1) such that $\inf_{n \in \mathbb{Z}} |u(n)| > 0$. Then $\{|u(n)|\}_{n \in \mathbb{Z}}$ and $\{\ln |u(n)|\}_{n \in \mathbb{Z}}$ are respectively bounded solutions to (2.5) and the auxiliary equation

$$(2.7) \quad x(n) - x(n - 1) = \ln |a(n)|, \quad n \in \mathbb{Z}.$$

Because $\{\ln |a(n)|\}_{n \in \mathbb{Z}} \in AP(\mathbb{Z}, \mathbb{R})$, $\{\ln |u(n)|\}_{n \in \mathbb{Z}}$ is an almost periodic solution to (2.7). Therefore,

$$\{|u(n)| = e^{\ln |u(n)|}\}_{n \in \mathbb{Z}}$$

is an almost periodic solution to (2.5). By Lemma 2.6, $\{sgn V(n, 0)\}_{n \in \mathbb{Z}}$ is a periodic solution to (2.6). A direct calculation shows that

$$\begin{aligned} u(n) &= V(n, 0)u(0) \\ &= |V(n, 0)| \cdot |u(0)| \cdot sgn [V(n, 0)u(0)] \\ &= |u(n)| \cdot sgn V(n, 0) \cdot sgn u(0) \end{aligned}$$

for all $n \in \mathbb{Z}$. Thus $u \in AP(\mathbb{Z}, \mathbb{R})$. □

Remark 2.8. Under the conditions of Theorem 2.7, (2.1) has a nontrivial solution u with $\inf_{n \in \mathbb{Z}} |u(n)| = 0$ if and only if $\{\ln |u(n)|\}_{n \in \mathbb{Z}}$ is a solution to (2.7) such that $\inf_{n \in \mathbb{Z}} \ln |u(n)| = -\infty$, which may be true in many situations.

3. The factorization theorem

In this section, we prove the main result on the factorization of p.c.a.p. functions into Bohr almost periodic functions and sequences, and show the almost periodicity of the functions in (1.10). Elementary properties of p.c.a.p. functions are listed below.

Lemma 3.1 ([13, p. 206, 214]). *Suppose that $h, h_1 \in PCAP(\mathbb{R}, \mathbb{R})$ have discontinuities at the points of a subset of the same Wexler sequence $\{\tau_j\}_{j \in \mathbb{Z}}$, then (i) $hh_1 \in PCAP(\mathbb{R}, \mathbb{R})$; (ii) $h/h_1 \in PCAP(\mathbb{R}, \mathbb{R})$ if $\inf_{t \in \mathbb{R}} |h_1(t)| > 0$; (iii) $\{h(\tau_n)\}_{n \in \mathbb{Z}} \in AP(\mathbb{Z}, \mathbb{R})$.*

Proof of Theorem 1.4. If $a(n) := h(\tau_n^+)/h(\tau_n)$ for $n \in \mathbb{Z}$, then $a \in AP(\mathbb{Z}, \mathbb{R})$ and $\inf_{n \in \mathbb{Z}} |a(n)| > 0$ by (iii) of Lemma 3.1. Theorem 2.7 implies that the bounded solution v with $\inf_{n \in \mathbb{Z}} |v(n)| > 0$ to (1.2) is almost periodic. Define a function g piecewise on \mathbb{R} by

$$g(t) = v(n), \quad \tau_n < t \leq \tau_{n+1}, n \in \mathbb{Z}.$$

It is easy to check that $g \in PCAPW(\mathbb{R}, \mathbb{R})$ and $\inf_{t \in \mathbb{R}} |g(t)| > 0$. If

$$f(t) := \frac{h(t)}{g(t)}, \quad t \in \mathbb{R},$$

then $f \in PCAP(\mathbb{R}, \mathbb{R})$ by (ii) of 3.1. We shall prove that $f \in C(\mathbb{R}, \mathbb{R})$, which implies the almost periodicity of f by Theorem 3.8 in [12]. A direct calculation shows that

$$f(\tau_n^+) = \frac{h(\tau_n^+)}{v(n)} = \frac{h(\tau_n)}{v(n-1)} = f(\tau_n)$$

for all $n \in \mathbb{Z}$.

At last, the uniqueness of f and v up to a nonzero multiplicative constant follows from Lemma 2.1. □

Remark 3.2. From the proof above it follows that Theorem 1.4 remains true if (1.1) is replaced by

$$(3.1) \quad \left\{ \frac{h(\tau_n^+)}{h(\tau_n)} \right\}_{n \in \mathbb{Z}} \in AP(\mathbb{Z}, \mathbb{R}), \quad \inf_{n \in \mathbb{Z}} \left| \frac{h(\tau_n^+)}{h(\tau_n)} \right| > 0.$$

Note that (1.1) is proposed to generate the sequence in (3.1). This remark will be referenced in the applications of impulsive differential equations.

Proof of Theorem 1.5. Let h be given by (1.3). We first prove that h satisfies (ii) of Definition 1.3. Since f is uniformly continuous on \mathbb{R} , for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $|f(s) - f(t)| < \epsilon$ for all $s, t \in \mathbb{R}$ satisfying $|s - t| < \delta$. A straightforward computation shows that

$$|h(s) - h(t)| = |f(s)u(j) - f(t)u(j)| \leq \|u\| \cdot \epsilon$$

whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$.

Next we prove that h satisfies (iii) of Definition 1.3 by showing the set $T(h, \epsilon')$ to be relatively dense, where $\epsilon' = (\|u\| + \|f\|) \cdot \epsilon$. Consider the following inequalities in $(r, q) \in \mathbb{R} \times \mathbb{Z}$,

$$(3.2) \quad |f(t+r) - f(t)| < \epsilon, \quad t \in \mathbb{R},$$

$$(3.3) \quad |u(n+q) - u(n)| < \epsilon, \quad n \in \mathbb{Z},$$

$$(3.4) \quad |\tau_j^q - r| < \epsilon, \quad j \in \mathbb{Z}.$$

Using the method of common almost periods as in Lemma 35 in [13, p. 208], the following two sets

$$\Gamma = \{r \in \mathbb{R}; \text{ there exists } q \in \mathbb{Z} \text{ such that } (r, q) \text{ satisfies (3.2)-(3.4)}\},$$

$$Q = \{q \in \mathbb{Z}; \text{ there exists } r \in \mathbb{R} \text{ such that } (r, q) \text{ satisfies (3.2)-(3.4)}\}$$

are relatively dense. Let $(r, q) \in \Gamma \times Q$ satisfy (3.2)-(3.4). If $\tau_m + \epsilon < t < \tau_{m+1} - \epsilon$ for some $m \in \mathbb{Z}$, from (3.4) it follows that

$$\tau_{m+q} - \tau_m - \epsilon < r < \tau_{m+1+q} - \tau_{m+1} + \epsilon,$$

$$\tau_{m+q} < t + r < \tau_{m+q+1}.$$

A direct calculation shows that

$$\begin{aligned} |h(t+r) - h(t)| &= |f(t+r)u(m+q) - f(t)u(m)| \\ &\leq |[f(t+r) - f(t)]u(m+q)| + |f(t)[u(m+q) - u(m)]| \\ &< (\|u\| + \|f\|) \cdot \epsilon = \epsilon'. \end{aligned}$$

Hence $T(h, \epsilon')$ contains the relatively dense set Γ and h is p.c.a.p. □

Now we show the almost periodicity of the functions in (1.10).

Proof of Theorem 1.6. It is easy to check that (ϕ_1, \dots, ϕ_d) is a solution to (1.11) if and only if $(\phi_1/\psi_1, \dots, \phi_d/\psi_d)$ is a solution to (1.12). Next we prove the almost periodicity.

Let (ϕ_1, \dots, ϕ_d) be a p.c.a.p. solution to (1.11) such that $\phi_i(\tau_n) \neq 0$ for all $n \in \mathbb{Z}$ and $i = 1, \dots, d$. From

$$\frac{\phi_i(\tau_n^+)}{\phi_i(\tau_n)} = \frac{\psi_i(\tau_n^+)}{\psi_i(\tau_n)} = 1 + b_i(n), \quad n \in \mathbb{Z}.$$

it follows that (1.2) takes the form of (1.9) for $i = 1, \dots, d$. Therefore, the sequences $\{u_i\}_{i=1}^d$ defined by (1.7) are almost periodic by (A2) and Theorem 2.7, and the functions $\{\psi_i\}_{i=1}^d$ given by (1.8) are p.c.a.p. by Theorem 1.5 with $f = 1$. Since u_i is a nontrivial solution to (1.9) and

$$(3.5) \quad \phi_i(t) = \frac{\phi_i(t)}{\psi_i(t)} \cdot u_i(n), \quad \tau_n < t \leq \tau_{n+1}, n \in \mathbb{Z},$$

Remark 3.2 implies that ϕ_i/ψ_i is the unique Bohr almost periodic function such that (1.3) holds for $\phi_i, \phi_i/\psi_i$ and u_i .

The second conclusion follows from (3.5) and Theorem 1.5. □

Remark 3.3. Assumption of the existence of almost periodic solutions in Theorem 1.6 may hold in various well-known situations by using, e.g., Favard’s separation method, fixed point theory, semigroup theory, exponential dichotomy and Lyapunov functions etc. Clearly, additional conditions should be imposed on the equations so

that one of the tools mentioned above is applicable. See Example 3.4 for such a presentation.

At last, we discuss an example arising from [9] to enhance the contribution of the main results in this paper.

Example 3.4. Consider almost periodic competitive system subject to impulsive perturbations

$$(3.6) \quad \begin{cases} y_1'(t) = y_1(t) \left[\gamma_1(t) - \alpha_1(t)y_1(t) - \frac{\beta_2(t)y_2(t)}{1 + y_2(t)} \right], \\ y_2'(t) = y_2(t) \left[\gamma_2(t) - \alpha_2(t)y_2(t) - \frac{\beta_1(t)y_1(t)}{1 + y_1(t)} \right], & t \neq \tau_n, \\ y_1(\tau_n^+) = [1 + b_1(n)]y_1(\tau_n), \\ y_2(\tau_n^+) = [1 + b_2(n)]y_2(\tau_n), & n \in \mathbb{Z}. \end{cases}$$

With (1.8) the transformation $(x_1, x_2) = \left(\frac{y_1}{\psi_1}, \frac{y_2}{\psi_2} \right)$ reduces (3.6) to a continuous system with p.c.a.p. coefficients

$$(3.7) \quad \begin{cases} x_1'(t) = x_1(t) \left[\gamma_1(t) - \alpha_1(t)\psi_1(t)x_1(t) - \frac{\beta_2(t)\psi_2(t)x_2(t)}{1 + \psi_2(t)x_2(t)} \right], \\ x_2'(t) = x_2(t) \left[\gamma_2(t) - \alpha_2(t)\psi_2(t)x_2(t) - \frac{\beta_1(t)\psi_1(t)x_1(t)}{1 + \psi_1(t)x_1(t)} \right]. \end{cases}$$

Taking logarithms $(u_1, u_2) = (\ln x_1, \ln x_2)$ further transforms (3.7) into

$$(3.8) \quad \begin{cases} u_1'(t) = \gamma_1(t) - \alpha_1(t)\psi_1(t)e^{u_1(t)} - \frac{\beta_2(t)\psi_2(t)e^{u_2(t)}}{1 + \psi_2(t)e^{u_2(t)}}, \\ u_2'(t) = \gamma_2(t) - \alpha_2(t)\psi_2(t)e^{u_2(t)} - \frac{\beta_1(t)\psi_1(t)e^{u_1(t)}}{1 + \psi_1(t)e^{u_1(t)}}. \end{cases}$$

Under suitable conditions including (A2) on the coefficients of (3.6), a Lyapunov function exists for (3.8). We refer the readers to [9] for detailed calculations and numerical simulations. It follows the existence of a unique uniformly asymptotically stable positive almost periodic solution of (3.7). By (ii) of Theorem 1.6, (3.6) admits a unique p.c.a.p. solution with desired properties.

Acknowledgments

Supported by The National Nature Science Foundation of China (No. 12001397, 11871007), The Nature Science Foundation of Tianjin City (No. 20JCQNJC00970). The authors thank the anonymous referee for valuable comments.

REFERENCES

- [1] M.U. Akhmet, Almost periodic solutions of differential equations with piecewise constant argument of generalized type, *Nonlinear Anal. Hybrid Syst.*, 2 (2): 456–467, 2008.
- [2] J.O. Alzabut, G.T. Stamov, and E. Sermetlu, On almost periodic solutions for an impulsive delay logarithmic population model, *Math. Comput. Modelling*, 51 (5/6): 625–631, 2010.
- [3] D. Bainov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, volume 66 of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Harlow, 1993.
- [4] A. Boumenir, N.V. Minh, and V.K. Tuan, Frequency modules and nonexistence of quasi-periodic solutions of nonlinear evolution equations, *Semigroup Forum*, 76 (1): 58–70, 2008.
- [5] C. Corduneanu, *Almost Periodic Functions*, volume 22 of Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1989.
- [6] S. Hu and V. Lakshmikantham, Periodic boundary value problems for second order impulsive differential systems, *Nonlinear Anal.*, 13 (1): 75–85, 1989.
- [7] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov, *Theory of Impulsive Differential Equations*, volume 6 of Series in Modern Applied Mathematics. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.
- [8] Y. Li and Y. Ye, Multiple positive almost periodic solutions to an impulsive non-autonomous Lotka-Volterra predator-prey system with harvesting terms, *Commun. Nonlinear Sci. Numer. Simul.*, 18 (11): 3190–3201, 2013.
- [9] Z. Liu and Q. Wang, An almost periodic competitive system subject to impulsive perturbations, *Appl. Math. Comput.*, 231 (15): 377–385, 2014.
- [10] G.M. N’Guerekata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenum Publishers, New York, 2001.
- [11] R. Ortega and M. Tarallo, Almost periodic linear differential equations with non-separated solutions, *J. Funct. Anal.*, 237 (2): 402–426, 2006.
- [12] L. Qi and R. Yuan, A generalization of Bochner’s theorem and its applications in the study of impulsive differential equations, *J. Dyn. Differ. Equ.*, 31 (4): 1955–1985, 2019.
- [13] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations*, volume 14 of World Scientific Series on Nonlinear Science, Series A: Monographs and Treatises, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.
- [14] G.T. Stamov, *Almost Periodic Solutions of Impulsive Differential Equations*, volume 2047 of Lecture Notes in Mathematics. Springer, Heidelberg, 2012.
- [15] R. Tan, W. Liu, Q. Wang, and Z. Liu, Uniformly asymptotic stability of almost periodic solutions for a competitive system with impulsive perturbations, *Adv. Difference Equ.*, 2014 (2), 2014.
- [16] B. Tian, N. Chen, and Y. Qiu, Almost periodic solution for an epidemic prey-predator system with impulsive effects and multiple delays, *Discrete Dyn. Nat. Soc.*, 2015.
- [17] V. Tkachenko, Almost periodic solutions of evolution differential equations with impulsive action, *Mathematical Modeling and Applications in Nonlinear Dynamics*, 14: 161–205, 2016.
- [18] R. Yuan, On Favard’s theorems, *J. Differential Equations*, 249 (8): 1884–1916, 2010.
- [19] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press, Beijing, 2003.

- [20] H. Zhang, Y. Li, B. Jing, and W. Zhao, Global stability of almost periodic solution of multi-species mutualism system with time delays and impulsive effects, *Appl. Math. Comput.*, 232 (1): 1138–1150, 2014.
- [21] T. Zhang, Y. Li, and Y. Ye, On the existence and stability of a unique almost periodic solution of Schoener’s competition model with pure-delays and impulsive effects, *Commun. Nonlinear Sci. Numer. Simul.*, 17 (3): 1408–1422, 2012.