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NEW RESULTS FOR IMPULSIVE FOURTH-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider the fourth-order differential equations with impulsive effects. By applying variational methods and critical point theory, some new criteria to guarantee the fourth-order impulsive differential equations have at least two solutions and infinitely many solutions will be given when the impulsive functions require to satisfy the superlinear growth conditions. Moreover, we improve and extend some previous results.

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1. INTRODUCTION

In this paper, we consider the following fourth-order impulsive differential equation:

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = f(t, u(t)), \quad t \neq t_j, \ a.e.\ t \in [0, T], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1, 2, \ldots, m, \\ -\Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1, 2, \ldots, m, \\ u(0) = u(T) = u''(0^+) = u''(T^-) = 0, \end{cases} \tag{1.1}$$

where A and B are real constants, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, the impulsive functions $I_{1j}, I_{2j} \in C(\mathbb{R}, \mathbb{R})$ for $1 \leq j \leq m$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$ and $\Delta(u''(t_j)) = u''(t_j^+) - u''(t_j^-)$, $\Delta(u'''(t_j)) = u'''(t_j^+) - u'''(t_j^-)$, where the left and right limits of $u''(t_j)$, $u'''(t_j)$ are represented by $u''(t_j^-)$, $u'''(t_j^-)$ and $u''(t_j^+)$, $u'''(t_j^+)$, at $t = t_j$, $j = 1, 2, \ldots, m$, respectively.

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Due to a wide range of significant applications in various sciences, such as physics, engineering, chemistry, biology, ecology, control, etc., impulsive fourth-order differential equations, have played the crucial role in above sciences. Their most outstanding advantage is that they can vividly describe the model which suffer sudden changes at certain instants which can not be modeled by the classical differential equations. Owing to this reason, more and more considerations by many people have been paid to study the existence of solutions for impulsive fourth-order differential equations. By means of variational methods and critical point theory, many interesting results on the fourth-order differential equations have presented to our sight, we refer the readers to [1, 2, 3, 7, 8, 9, 10].

More precisely, Sun *et al.* in [7] have investigated the fourth-order differential equation with impulsive effects as follow:

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = f(t, u(t)), \quad t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1, 2, \ldots, l, \\ \Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1, 2, \ldots, l, \\ u(0) = u(T) = u''(0^+) = u''(T^-) = 0. \end{cases} \tag{1.2}$$

They have proved that the problem (1.2) has at least one solution and infinitely many solutions by applying variational methods, the main results are as follow:

Theorem 1.1 ([7, Theorem 3.1]). *Assume that the following conditions hold:*

(H1) *There exists a constant* $\mu > 2$ *such that* $0 < \mu F(t, u) \leq uf(t, u)$ *for every* $t \in [0, T]$ *and* $u \in \mathbb{R} \setminus \{0\}$, *where* $F(t, u) = \int_0^u f(t, s)ds$;

(H2) *There exist constants* $L_j > 0$, $j = 1, 2, \ldots, l$, *such that*

$$|I_{1j}(u) - I_{1j}(v)| \leq L_j|u - v|, \quad \forall u, v \in \mathbb{R}, \ j = 1, 2, \ldots, l,$$

where L_j *satisfy* $0 < \sum_{j=1}^{l} L_j < \frac{\mu-2}{2M_1^2(\mu+1)}$, *and* M_1 *is defined in Lemma 2.2 of* [7];

(H3) *The impulsive functions* I_{2j} *satisfy sublinear growth, i.e., there exist constants* $a_{2j} > 0$, $b_{2j} > 0$ *and* $\gamma_{2j} \in [0, 1)$, $j = 1, 2, \ldots, l$ *such that*

$$|I_{2j}(u)| \leq a_{2j} + b_{2j}|u|^{\gamma_{2j}}, \quad \forall u \in \mathbb{R}, \ j = 1, 2, \ldots, l;$$

(H4) *The following inequality*

$$\frac{1}{2M_1^2} - \frac{MT^4}{\pi^4 M_1^2 \delta^2} - \sum_{j=1}^{l}(L_j + a_{2j} + b_{2j} + |I_{1j}(0)|) > 0$$

holds, where $M = \sup\{F(t, u) : t \in [0, T], |u| = 1\}$.

Then, the problem (1.2) *has at least one classical solution.*

Theorem 1.2 ([7, Theorem 3.2]). *Suppose that* (H1)-(H4) *hold. Moreover,* $f(t, u)$ *and impulsive functions* $I_{1j}(u)$, $I_{2j}(u)$ *are all odd in* u. *Then, the problem* (1.2) *has infinitely many classical solutions.*

In [9], Xie and Luo have further studied the following boundary value problem:

$$\text{(1.3)}\quad \begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = \lambda f(t, u(t)), \quad t \neq t_j, \text{ a.e. } t \in [0,1], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1, 2, \ldots, m, \\ -\Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1, 2, \ldots, m, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

They have proved that the problem (1.3) has at least one solution by applying variational methods, the main result is as follow:

Theorem 1.3 ([9, Theorem 3.1]). *Assume that the following condition holds:*

(H5) *There exist two positive constants k_1 and k_2 such that for each $u \in X$*

$$0 \leq \sum_{j=1}^{m} \int_0^{u'(t_j)} I_{1j}(s)ds \leq k_1 \max_{j\in\{1,2,\ldots,m\}} |u'(t_j)|^2$$

and

$$0 \leq \sum_{j=1}^{m} \int_0^{u(t_j)} I_{2j}(s)ds \leq k_2 \max_{j\in\{1,2,\ldots,m\}} |u(t_j)|^2.$$

If there exist constants c_1, c_2 and c satisfying $c_1 < \sqrt{2k_0}M_1 c < \sqrt{2k_3}M_1 c < c_2$ and $0 < a(c_2, c) < b(c_1, c)$, where $k_0 = 2 - \frac{A}{6} + \frac{B}{60}$, $k_3 = k_0 + k_1 + \frac{1}{4}k_2$,

$$a(c_2, c) = \frac{\int_0^1 \max_{|u|\leq c_2} F(t, u)dt - \int_0^1 F(t, u_1(t))dt}{c_2^2 - 2k_3 M_1^2 c^2},$$

$$b(c_1, c) = \frac{\int_0^1 F(t, u_1(t))dt - \int_0^1 \max_{|u|\leq c_1} F(t, u)dt}{2k_3 M_1^2 c^2 - c_1^2},$$

$u_1(t) = ct(1-t)$ and $M_1 = 1 + \frac{1}{\pi}$, then, for each $\lambda \in (\lambda_1, \lambda_2)$, the problem (1.3) has at least one solution u and $\|u\|_X < \frac{c_2}{M_1}$, where $\lambda_1 = \frac{1}{2M_1^2 b(c_1,c)}$ and $\lambda_2 = \frac{1}{2M_1^2 a(c_2,c)}$.

In [1], Afrouzi *et al.* also extended problem (1.3), they considered the following boundary value problem:

$$\text{(1.4)}\quad \begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = \lambda f(t, u(t)) + \mu g(t, u(t)), \quad t \neq t_j, \text{ a.e. } t \in [0,1], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1, 2, \ldots, m, \\ -\Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1, 2, \ldots, m, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

They have obtained that the problem (1.4) has at least three classical solutions under the above assumption (H5) by using variational methods.

In [8], Tian and Liu have considered the following Sturm-Liouville boundary value problem:

$$
\begin{cases}
u^{(iv)}(t) - u''(t) + u(t) = f(t, u(t)), \quad t \neq t_j, \text{ a.e. } t \in [0, T], \\
-\Delta(u'''(t_i)) = I_{1i}(u(t_i)), \quad i = 1, 2, \ldots, l, \\
\Delta(u''(t_i)) = I_{2i}(u'(t_i)), \quad i = 1, 2, \ldots, l, \\
au(0) - bu'(0) = 0, \quad cu(T) + du'(T) = 0, \\
au''(0) - bu'''(0) = 0, \quad cu''(T) + du'''(T) = 0.
\end{cases} \tag{1.5}
$$

They have proved that the problem (1.5) has at least one nontrivial solution by applying variational methods, the main result is as follow:

Theorem 1.4 ([8, Theorem 3.1]). *Assume that following conditions hold:*

(H6) *There exist constants $\mu > 2$ and $r \geq 0$ such that for $|\xi| \geq r$,*

$$0 < \mu F(t, \xi) \leq \xi f(t, \xi);$$

(H7) *The impulsive functions I_{1i} satisfy sublinear growth, that is, there exist constants $\alpha_i > 0$, $\beta_i > 0$ and $\gamma_i \in [0, 1)$, $i = 1, 2, \ldots, l$ such that*

$$|I_{1i}(u)| \leq \alpha_i + \beta_i |u|^{\gamma_i};$$

(H8) *The impulsive functions I_{2i}, $i = 1, 2, \ldots, l$ are bounded;*

(H9) *$f(t, u) = o(|u|)$, $I_{1i}(u) = o(|u|)$, $I_{2i}(u) = o(|u|)$ as $|u| \to 0$, $i = 1, 2, \ldots, l$.*

Then, the problem (1.5) *has at least one nontrivial solution.*

Motivated by above fact, it is clear to see that the impulsive functions in [1, 7, 8, 9] are required to satisfy the sublinear growth conditions, Lipschitz continuous conditions or bounded conditions. Compared with the corresponding conditions imposed on impulsive functions I_{1j} and I_{2j} in the above literatures, in our paper, we suppose that the impulsive functions I_{1j} and I_{2j} are required to satisfy the superlinear growth conditions. Under new certain assumptions, we prove that problem (1.1) has at least two solutions and infinitely many solutions by means of variational methods and critical point theory. It is worth pointing out that there is no paper which has considered the existence of solutions for fourth-order differential equations with superlinear impulsive growth conditions up to know, the aim of this paper is to close this gap. Moreover, we also improve and generalize some previous results.

The arrangement of the rest paper is as follows. In Section 2, some preliminaries and results which are applied in the later paper are presented. In Section 3, the main proof of theorems will be vividly shown. In Section 4, corresponding examples are given to illustrate the obtained results in Section 3.

2. PRELIMINARIES

We will assume for the remainder of the paper that $A \leq 0 \leq B$.

Let us recall some basic knowledge first. In the Sobolev space

$$X := H^2(0,T) \cap H_0^1(0,T),$$

we can consider the inner product

$$(u,v) = \int_0^T u''(t)v''(t)dt, \qquad \forall u, v \in X,$$

which induces the norm

$$\|u\| = \left[\int_0^T |u''(t)|^2 dt\right]^{\frac{1}{2}}.$$

Then we can define the following norm of X as follow

$$\|u\|_X = \left[\int_0^T (|u''(t)|^2 - A|u'(t)|^2 + B|u(t)|^2)dt\right]^{\frac{1}{2}}.$$

Since $A \leq 0 \leq B$, it is immediate that $\|u\| \leq \|u\|_X$.
The usual norm of $C^1([0,T])$ and $L^p(0,T)$ are defined as follow:

$$\|u\|_\infty = \max\left\{\max_{t\in[0,T]} |u(t)|, \max_{t\in[0,T]} |u'(t)|\right\}, \qquad \|u\|_{L^p} = \left(\int_0^T u^p(t)dt\right)^{\frac{1}{p}}.$$

Lemma 2.1 ([4]). *Let $u \in X$, then $\|u\|_\infty \leq M_1\|u\|_X$, where*

$$M_1 = \max\left\{\frac{T^{\frac{3}{2}}}{2\pi}, \sqrt{T}\right\}.$$

Remark 2.2. It is clear that the M_1 is more simple in our results than that in [7, 10], which is defined as $M_1 = \max\left\{\frac{T^{\frac{3}{2}}}{\pi\delta}, \frac{\sqrt{T}}{\delta}(1+\frac{1}{\pi})\right\}$, where $\delta := \sqrt{1-\sigma}$ and $\sigma := \max\left\{\frac{AT^2}{\pi^2}, -\frac{BT^4}{\pi^4}, \frac{AT^2}{\pi^2} - \frac{BT^4}{\pi^4}, 0\right\}$, we know if assume that $A \leq 0 \leq B$, then $\sigma = 0$ and $\delta = 1$, so $M_1 = \max\left\{\frac{T^{\frac{3}{2}}}{\pi}, \sqrt{T}(1+\frac{1}{\pi})\right\}$.

Remark 2.3. It is also obvious that the M_1 is better in our results than that in [1, 9], which is defined as $M_1 = 1 + \frac{1}{\pi}$. However, $M_1 = 1$ in our results if $T = 1$. Thus our results generalize and improve some known results.

Remark 2.4. It is also clear that the M_1 is better in our results than that in [8], which is defined as $M_1 = \frac{1}{\sqrt{T}} + \sqrt{T}$. When $T = 1$, $M_1 = 2$ in [8], however, $M_1 = 1$ in our results. Thus our results generalize and improve some known results.

Definition 2.5. A function $u \in X$ is a weak solution of the problem (1.1) if the following equality

$$\begin{aligned}\int_0^T (u''v'' - Au'v' + Buv)dt &= -\sum_{j=1}^m I_{1j}(u'(t_j))v'(t_j) - \sum_{j=1}^m I_{2j}(u(t_j))v(t_j) \\ &\quad + \int_0^T f(t,u(t))v(t)dt\end{aligned}$$

holds for every $v \in X$.

Then we consider the functional $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(t)dt + \sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(t)dt - \int_0^T F(t,u(t))dt, \tag{2.1}$$

where $F(t,u) = \int_0^u f(t,s)ds$.

Owing to the continuity of f and I_{ij}, $i = 1, 2$, we immediately deduce that φ is continuous and differentiable at any $u \in X$ and

$$\begin{aligned}\varphi'(u)(v) &= \int_0^T (u''v'' - Au'v' + Buv)dt + \sum_{j=1}^m I_{1j}(u'(t_j))v'(t_j) \\ &\quad + \sum_{j=1}^m I_{2j}(u(t_j))v(t_j) - \int_0^T f(t,u(t))v(t)dt, \qquad \forall v \in X.\end{aligned} \tag{2.2}$$

Hence, a critical point of φ, defined by (2.1), give us a weak solution of the problem (1.1).

Lemma 2.6. *If $u \in X$ is a weak solution of problem (1.1), then u is also a classical solution of (1.1).*

Proof. The proof is similar as to the Lemma 2.1 of [7], we omit it here. □

Lemma 2.7. *Assume that the sequence $\{u_k\}$ converges weakly to $u \in X$, i.e., $u_k \rightharpoonup u$. Then, we have that $\{u_k\}$ converges strongly to $u \in C([0,T],\mathbb{R})$, i.e., $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.*

Definition 2.8. Let X be a Banach space and $\varphi : X \to \mathbb{R}$. We say φ is sequentially weakly lower semi-continuous, if $\liminf_{k\to+\infty} \varphi(u_k) \geq \varphi(u)$ as $u_k \rightharpoonup u$ in X.

Lemma 2.9. *The functional φ is sequentially weakly lower semi-continuous.*

Proof. Since X is a reflexive real Banach space, we may choose a weakly convergent subsequence, we denote $\{u_k\} \in X$ and $u_k \rightharpoonup u$ in X. Also, we have $u_k \to u$ uniformly in $C([0,T],\mathbb{R})$ as $k \to \infty$; that is,

$$\|u_k - u\|_\infty \to 0, \quad \liminf_{k\to\infty} \|u_k\|_X \geq \|u\|_X, \qquad as\ k \to \infty.$$

From (2.1), we have

$$\begin{aligned}\liminf_{k\to\infty}\varphi(u_k) &= \liminf_{k\to\infty}\Big[\frac{1}{2}\|u_k\|_X^2+\sum_{j=1}^{m}\int_0^{u_k'(t_j)} I_{1j}(t)dt \\ &\quad +\sum_{j=1}^{m}\int_0^{u_k(t_j)} I_{2j}(t)dt-\int_0^T F(t,u_k(t))dt\Big] \\ &\geq \frac{1}{2}\|u\|_X^2+\sum_{j=1}^{m}\int_0^{u'(t_j)} I_{1j}(t)dt \\ &\quad +\sum_{j=1}^{m}\int_0^{u(t_j)} I_{2j}(t)dt-\int_0^T F(t,u(t))dt \\ &= \varphi(u),\end{aligned}$$

it implies that φ is sequentially weakly lower semi-continuous. □

Definition 2.10. Let X be a real Banach space. We say $\varphi\in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition (shortly (P.S.)), if every sequence $\{u_j\}\subset X$ for which $\{\varphi(u_j)\}$ is bounded and $\varphi'(u_j)\to 0$ as $j\to 0$, possesses a convergent subsequence in X.

Our main tools to prove the main results of this paper are the following theorems.

Theorem 2.11 ([5]). *Let $\varphi\in C^1(X,\mathbb{R})$, and φ satisfies the (P.S.) condition. Assume that there exist $u_0,u_1\in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1\notin\Omega$ and*

$$\inf_{v\in\partial\Omega}\varphi(v)>\max\{\varphi(u_0),\varphi(u_1)\}.$$

Then, there exists a critical point u of φ, i.e., $\varphi'(u)=0$, with $\varphi(u)>\max\{\varphi(u_0),\varphi(u_1)\}$.

Theorem 2.12 ([11]). *For the functional $\varphi: M\subseteq X\to\mathbb{R}$ with $M\neq\emptyset$, $\min_{u\in M}\varphi(u)=\alpha$ has a solution in case the following conditions hold:*

(i) *X is a real reflexive Banach space;*
(ii) *M is bounded and weak sequentially closed;*
(iii) *φ is sequentially weakly lower semi-continuous on M.*

Theorem 2.13 ([6]). *Let X be an infinite dimensional Banach space, and let $\varphi\in C^1(X,\mathbb{R})$ be even, satisfy the (P.S.) condition, and $\varphi(0)=0$. If $X=V\bigoplus\Upsilon$, where V is finite dimensional, and φ satisfies:*

(i) *There exist constants $\rho,\eta>0$ such that $\varphi(u)\geq\eta$ for all $u\in\Upsilon$ with $\|u\|_X=\rho$;*
(ii) *For each finite dimensional subspace $W\subset X$, there is an $R=R(W)$ such that $\varphi\leq 0$ on $W\backslash B_R$,*

then φ possesses an unbounded sequence of critical values.

3. MAIN RESULTS

In this section we present our main results. To be precise, we establish an existence result of at least two solutions, Theorem 3.1, which is based on Theorems 2.11 and 2.12. Finally, we present another existence result of infinitely many solutions, Theorem 3.3, which is based in turn on Theorem 2.13, and we point out a consequence, Corollary 3.4.

Theorem 3.1. *Assume that the following conditions hold:*

(A1) *There exist* $\mu > 2$, $\xi_{ij} > 0$, $i = 1, 2$, $j = 1, 2, \ldots, m$, *such that*
  (i) $I_{ij}(u)u \le \mu \int_0^u I_{ij}(s)ds < 0$, *for* $u \in \mathbb{R} \setminus \{0\}$,
  (ii) $\int_0^u I_{ij}(s)ds \ge -\xi_{ij}|u|^\mu$, *for* $u \in \mathbb{R} \setminus \{0\}$;

(A2) *There exist positive constants* c_1, c_2 *and* $\delta \in [0, 1)$ *such that*

$$|f(t, u)| \le c_1 + c_2|u|^\delta,$$

for all $(t, u) \in [0, T] \times \mathbb{R}$;

(A3) $F(t, u) \le 0$ *for all* $(t, u) \in [0, T] \times \mathbb{R}$.

Then, the problem (1.1) *has at least two solutions.*

Proof. It is easy to see that $\varphi \in C^1(X, \mathbb{R})$ is an even functional and $\varphi(0) = 0$.

Firstly, we need to prove that φ satisfies the (P.S.) condition. Let $\{u_k\} \subset X$ such that $\{\varphi(u_k)\}$ be a bounded sequence and $\lim_{k\to\infty} \varphi'(u_k) = 0$. Assume that there exists a constant $Z_1 > 0$ such that

$$|\varphi(u_k)| \le Z_1, \qquad \|\varphi'(u_k)\|_X \le Z_1. \tag{3.1}$$

By (2.1) and (A1), we have

$$\begin{aligned} \|u_k\|_X^2 &= 2\varphi(u_k) + 2\int_0^T F(t, u_k(t))dt \\ &\quad -2\sum_{j=1}^m \int_0^{u_k'(t_j)} I_{1j}(t)dt - 2\sum_{j=1}^m \int_0^{u_k(t_j)} I_{2j}(t)dt \\ &\le 2Z_1 + 2\int_0^T F(t, u_k(t))dt \\ &\quad -\frac{2}{\mu}\sum_{j=1}^m I_{1j}(u_k'(t_j))u_k'(t_j) - \frac{2}{\mu}\sum_{j=1}^m I_{2j}(u_k(t_j))u_k(t_j). \end{aligned}$$

From (2.2), together with the above inequality one has

$$\begin{aligned} \big(1 - \frac{2}{\mu}\big)\|u_k\|_X^2 &\le 2Z_1 + 2\int_0^T F(t, u_k(t))dt \\ &\quad -\frac{2}{\mu}\varphi'(u_k(t))(u_k(t)) - \frac{2}{\mu}\int_0^T f(t, u_k(t))u_k(t)dt. \end{aligned}$$

By (A2), (3.1) and Lemma 2.1, we have

$$(3.2)\quad \Big(1-\frac{2}{\mu}\Big)\|u_k\|_X^2 \le 2Z_1+\frac{2}{\mu}Z_1M_1\|u_k\|_X+\Big(2+\frac{2}{\mu}\Big)T\Big[c_1M_1\|u_k\|_X+c_2M_1^{\delta+1}\|u_k\|_X^{\delta+1}\Big].$$

Since $\mu > 2$, (3.2) implies that $\{u_k\}$ is bounded in X. Since X is a reflexive space, we may choose a weakly convergent subsequence, we denote $\{u_k\}$ and $u_k \rightharpoonup u$ in X. Then we will prove that $u_k \to u$ in X. In fact, from Lemma 2.7, we have $u_k \to u$ in $C([0,T],\mathbb{R})$ and $u_k \to u$, *a.e.* $t \in [0,T]$, so

$$\begin{cases} (\varphi'(u_k)-\varphi'(u))(u_k-u) \to 0, \\ \sum_{j=1}^m [I_{1j}(u_k'(t_j)-u'(t_j))][u_k'(t_j)-u'(t_j)] \to 0, \\ \sum_{j=1}^m [I_{2j}(u_k(t_j)-u(t_j))][u_k(t_j)-u(t_j)] \to 0, \\ \int_0^T [f(t,u_k(t))-f(t,u(t))][u_k(t)-u(t)]dt \to 0, \quad as\ k\to\infty. \end{cases}$$

Thus, by (2.2) we have

$$\begin{aligned} \|u_k-u\|_X^2 &= (\varphi'(u_k)-\varphi'(u))(u_k-u) \\ &\quad -\sum_{j=1}^m [I_{1j}(u_k'(t_j)-u'(t_j))][u_k'(t_j)-u'(t_j)] \\ &\quad -\sum_{j=1}^m [I_{2j}(u_k(t_j)-u(t_j))][u_k(t_j)-u(t_j)] \\ &\quad +\int_0^T [f(t,u_k(t))-f(t,u(t))][u_k(t)-u(t)]dt \to 0, \qquad as\ k\to\infty. \end{aligned}$$

We immediately deduce that $\|u_k-u\|_X \to 0$ as $k\to\infty$, and this implies that $\{u_k\}$ converges strongly to $u \in X$. So φ satisfies the (P.S.) condition.

Next, we verify that there exists $M_2 > 0$ such that the functional φ has a local minimum $u_0 \in B_{M_2} = \{u \in X : \|u\|_X < M_2\}$.

In fact, let $\{u_k\} \subseteq \bar{B}_{M_2}$ and $u_k \rightharpoonup u$ as $k\to\infty$, from Mazur Theorem, there exists a sequence of convex combinations

$$v_k = \sum_{j=1}^k \alpha_{kj}u_j, \quad \sum_{j=1}^k \alpha_{kj} = 1, \quad \alpha_{kj} \ge 0, \quad k \in \mathbb{N},$$

such that $v_k \to u$ in X. Since $\bar{B}_{M_2}$ is a closed convex set, we have $\{v_k\} \subseteq \bar{B}_{M_2}$ and $u \in \bar{B}_{M_2}$, thus we proved that $\bar{B}_{M_2}$ is bounded and weak sequentially closed.

Since X is a reflexive Banach space and φ is sequentially weakly lower semi-continuous on $\bar{B}_{M_2}$, by means of Theorem 2.12, we deduce that φ has a local minimum $u_0 \in \bar{B}_{M_2}$. Without loss of generality, we assume that $\varphi(u_0) = \min_{u\in\bar{B}_{M_2}} \varphi(u)$. Then we will verify

$$(3.3)\qquad \varphi(u_0) < \inf_{u\in\partial B_{M_2}} \varphi(u).$$

In fact, let $\varepsilon = M_2 > 0$ satisfying

$$\frac{1}{2}\varepsilon^2 - \sum_{j=1}^{m}(\xi_{1j} + \xi_{2j})M_1^{\mu}\varepsilon^{\mu} > 0. \tag{3.4}$$

For any $u = \varepsilon\tau$ with $\tau \in X$ and $\|\tau\|_X = 1$, one has $u \in \partial B_{M_2}$. By (2.1), (3.4), (A1) and (A3), we have

$$\begin{aligned}\varphi(u) = \varphi(\varepsilon\tau) &= \frac{1}{2}\|\varepsilon\tau\|_X^2 + \sum_{j=1}^{m}\int_0^{\varepsilon\tau'(t_j)} I_{1j}(t)dt \\ &\quad + \sum_{j=1}^{m}\int_0^{\varepsilon\tau(t_j)} I_{2j}(t)dt - \int_0^T F(t, \varepsilon\tau(t))dt \\ &\geq \frac{1}{2}\varepsilon^2 - \sum_{j=1}^{m}(\xi_{1j} + \xi_{2j})M_1^{\mu}\varepsilon^{\mu} > 0.\end{aligned}$$

Thus, $\varphi(u) > 0 = \varphi(0) \geq \varphi(u_0)$ for $u \in \partial B_{M_2}$, that is, (3.3) holds and $u_0 \in B_{M_2}$.

Finally, we verify that there exists u_1 with $\|u_1\| > M_2$ such that $\varphi(u_1) < \inf_{u\in\partial B_{M_2}} \varphi(u)$.

In fact, from (i) of (A1), we deduce that there exist positive constants d_{1j}, d_{2j}, D_1 and D_2 such that

$$\int_0^{u'} I_{1j}(s)ds \leq -d_{1j}|u'|^{\mu} + D_1 \tag{3.5}$$

and

$$\int_0^{u} I_{2j}(s)ds \leq -d_{2j}|u|^{\mu} + D_2 \tag{3.6}$$

for $u \in X$. Now, let $u_1 = \sigma\theta$, $\sigma > 0$ with $\theta \in X$ and $\|\theta\|_X = 1$, then by (2.1), (3.5), (3.6) and (A2), we have

$$\begin{aligned}\varphi(u_1) = \varphi(\sigma\theta) &= \frac{1}{2}\|\sigma\theta\|_X^2 + \sum_{j=1}^{m}\int_0^{\sigma\theta'(t_j)} I_{1j}(t)dt \\ &\quad + \sum_{j=1}^{m}\int_0^{\sigma\theta(t_j)} I_{2j}(t)dt - \int_0^T F(t, \sigma\theta(t))dt \\ &\leq \frac{1}{2}\sigma^2 + T\Big[c_1M_1\sigma + c_2M_1^{\delta+1}\sigma^{\delta+1}\Big] \\ &\quad -\sigma^{\mu}\Big[\sum_{j=1}^{m} d_{1j}|\theta'(t)|^{\mu} + \sum_{j=1}^{m} d_{2j}|\theta(t)|^{\mu}\Big] + m(D_1 + D_2),\end{aligned}$$

then, we deduce that there exists sufficiently large $\sigma > M_2 = \varepsilon > 0$ such that $\varphi(\sigma\theta) < 0$. So, we have

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{u\in\partial B_{M_2}} \varphi(u),$$

that is, we obtain another critical point u^* by Theorem 2.11. Therefore, u_0, u^* are two critical points of φ, which are two solutions of (1.1). □

Remark 3.2. In our results, the impulsive functions I_{1j} and I_{2j} require to satisfy the superlinear growth conditions which are different from [1, 7, 8, 9, 10].

Theorem 3.3. *Assume that* (A1)-(A3) *hold and* $f(t,u)$ *and* $I_{ij}(u)$, $i = 1,2$, $j = 1,2,\ldots,m$, *are odd in* u. *Then, the problem* (1.1) *has infinitely many solutions.*

Proof. Bacause of the continuity of f and I_{ij}, $i = 1,2$, $j = 1,2,\ldots,m$, we clearly know that φ is continuous and differentiable. Since $f(t,u)$ and $I_{ij}(u)$, $i = 1,2$, $j = 1,2,\ldots,m$, are odd in u, we know φ is even and $\varphi(0) = 0$. As the similar proof of Theorem 3.1, we obtain that φ satisfies the (P.S.) condition. By (2.1), (A1) and (A3), we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{m}\int_0^{u'(t_j)} I_{1j}(t)dt \\ &\quad + \sum_{j=1}^{m}\int_0^{u(t_j)} I_{2j}(t)dt - \int_0^T F(t,u)dt \\ &\geq \frac{1}{2}\|u\|_X^2 - \sum_{j=1}^{m}(\xi_{1j}+\xi_{2j})M_1^{\mu}\|u\|_X^{\mu}.\end{aligned}$$

Since $\mu > 2$, the above inequality implies that there exists $\rho > 0$ small enough such that $\varphi(u) \geq \alpha > 0$ with $\|u\| = \rho$. That is, (i) of Theorem 2.13 holds.

For every $\eta > 0$ and any finite dimensional subspace $W \subset X$, by (2.1), (3.5), (3.6) and (A2), we have

$$\begin{aligned}\varphi(\eta u) &= \frac{1}{2}\|\eta u\|_X^2 + \sum_{j=1}^{m}\int_0^{\eta u'(t_j)} I_{1j}(t)dt \\ &\quad + \sum_{j=1}^{m}\int_0^{\eta u(t_j)} I_{2j}(t)dt - \int_0^T F(t,\eta u(t))dt \\ &\leq \frac{1}{2}\eta^2\|u\|_X^2 + T\Big[c_1M_1\eta\|u\|_X + c_2M_1^{\delta+1}\eta^{\delta+1}\|u\|_X^{\delta+1}\Big] \\ &\quad -\eta^{\mu}\Big[\sum_{j=1}^{m} d_{1j}|u'(t)|^{\mu} + \sum_{j=1}^{m} d_{2j}|u(t)|^{\mu}\Big] + m(D_1+D_2),\end{aligned}$$

for $u \in W$. Let $\vartheta(t) \in W$ with $\|\vartheta\|_X = 1$. From above inequality, we have

$$\begin{aligned}\varphi(\eta\vartheta(t)) &\leq \frac{1}{2}\eta^2 + T\Big[c_1M_1\eta + c_2M_1^{\delta+1}\eta^{\delta+1}\Big] \\ &\quad -\eta^{\mu}\Big[\sum_{j=1}^{m} d_{1j}|\vartheta'(t)|^{\mu} + \sum_{j=1}^{m} d_{2j}|\vartheta|^{\mu}\Big] + m(D_1+D_2).\end{aligned}$$

Since $\mu > 2$, the above inequality implies that there exists $\eta_1 > 0$ such that $\|\eta\vartheta(t)\|_X = \eta > \rho$ and $\varphi(\eta\vartheta(t)) < 0$ for every $\eta \geq \eta_1 > 0$. Since W is a finite dimensional subspace, we select an $R = R(W) > 0$ such that $\varphi(u) \leq 0$ for any $u \in W$ with $\|u\|_X \geq R$. That is, (ii) of Theorem 2.13 holds.

From Theorem 2.13, we know φ has infinitely many critical points, that is, the problem (1.1) has infinitely many solutions. □

By the similar methods as Theorem 3.3, we also have the following corollary.

Corollary 3.4. *Let* (A1) *and the following condition holds:*

(A4) *There exists a constant* $\beta \in (2, \mu]$ *such that*

$$0 < \beta F(t,u) \leq uf(t,u),$$

for every $t \in [0,T]$ *and* $u \in \mathbb{R}\backslash\{0\}$, *where* $F(t,u) = \int_0^u f(t,s)ds$. *Moreover,* $f(t,u)$ *and* $I_{ij}(u)$, $i = 1,2$, $j = 1,2,\ldots,m$, *are odd in* u.

Then, the problem (1.1) *has infinitely many solutions.*

4. EXAMPLES

In this part, we will give corresponding examples to illustrate the main results in our paper.

Example 4.1. Let $T = B = 1$, $A = -1$, $t_1 \in [0,1]$, then consider the following differential equation:

(4.1)
$$\begin{cases} u^{(iv)}(t) - u''(t) + u(t) = -\sin(u(t))e^{-\cos(u(t))} - u^{\frac{1}{3}}(t), \quad t \neq t_1, \text{ a.e. } t \in [0,1], \\ \Delta(u''(t_1)) = -\frac{1}{2}u^3(t_1), \\ -\Delta(u'''(t_1)) = -\frac{1}{3}u^3(t_1), \\ u(0) = u(1) = u''(0^+) = u''(1^-) = 0. \end{cases}$$

It is obvious to see $f(t,u) = -\sin(u(t))e^{-\cos(u(t))} - u^{\frac{1}{3}}(t)$, $I_{11}(u) = -\frac{1}{2}u^3$, and $I_{21}(u) = -\frac{1}{3}u^3$. Let $\mu = 4$, $\xi_{11} = \frac{1}{8}$, and $\xi_{21} = \frac{1}{12}$, that is (A1) holds. Since $|f(t,u)| \leq e + |u|^{\frac{1}{3}}$, then $F(t,u) = -e^{-\cos(u(t))} - \frac{3}{4}u^{\frac{4}{3}} \leq 0$, that is (A2) and (A3) hold. By Theorem 3.1, the problem (4.1) has at least two solutions.

Example 4.2. Let $T = B = 1$, $A = -1$, $t_1 \in [0,1]$, then consider the following differential equation:

(4.2)
$$\begin{cases} u^{(iv)}(t) - u''(t) + u(t) = -e^t u^{\frac{1}{5}}(t), \quad t \neq t_1, \text{ a.e. } t \in [0,1], \\ \Delta(u''(t_1)) = -eu^7(t_1), \\ -\Delta(u'''(t_1)) = -e^2u^7(t_1), \\ u(0) = u(1) = u''(0^+) = u''(1^-) = 0. \end{cases}$$

It is obvious to see $f(t,u) = -e^t u^{\frac{1}{5}}(t)$, $I_{11}(u) = -eu^7$, and $I_{21}(u) = -e^2u^7$. Let $\mu = 8$, $\xi_{11} = \frac{e}{8}$, and $\xi_{21} = \frac{e^2}{8}$, that is (A1) holds. Since $|f(t,u)| \leq e|u|^{\frac{1}{5}}$, then $F(t,u) = -\frac{5e^t}{6}u^{\frac{6}{5}}(t) \leq 0$, that is, (A2) and (A3) hold. Also, $f(t,u)$, $I_{11}(u)$ and $I_{21}(u)$ are odd in u. Thus, by Theorem 3.3, the problem (4.2) has infinitely many solutions.

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