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FIXED POINT RESULTS FOR MULTI-VALUED MAPPINGS IN G - METRIC SPACES

YAN SUN[1] AND CHENGLIN ZHAO[2]

Department of Mathematics, Shanghai Normal University,
Shanghai, 200234, P. R. China
[1] Corresponding author: ysun881@163.com
[2] 1294726415@qq.com

ABSTRACT. In this paper, we present new definitions of multi-valued mappings in G- complete G-metric spaces and establish a sharper sufficient condition for the existence of fixed points by making use of analysis technique. The results which are obtained improve, enrich, generalize and extend many known results. Some examples are also given to demonstrate the application of our main results

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1. Introduction

The existence of a solution (positive solution) to a theoretical or practical considerations physical problem is equivalent to the existence of a fixed point for a suitable map or operator in a wide range of mathematical, economical, computational, modeling and engineering problems (see, e. g., [1]-[2]). Fixed point theory in ordered metric space plays a key role in many fields of mathematical problems in applied and pure mathematics, and sciences (see, for example [3]-[9]), chemical and multidisciplinary application simulations such as variational and differential inequalities, optimization (see [1]-[28]) etc. The fixed point theory itself is a beautiful mixture of analysis, geometry and topology. For decades, the classical theory of fixed points in ordered metric space has been revealed as a very important and powerful tool in the study of nonlinear phenomenon. Fixed point iterative techniques have been successfully applied in various kinds of areas such as game theory, chemistry, physics and so on. One refers to see ([10]–[28]) and the references therein.

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In 1962, Edelstein obtained a interesting result on fixed and periodic points with the aid of contractive mapping. One can see [8] and the references therein. Edelstein presented the following results.

Theorem 1.1. Let E be a metric space, and g be a contractive selfmapping of E satisfying the following conditions

$$\exists\ x(\varepsilon E) : \{g^{n_i}\} \subset \{g^n\} \quad \text{with} \quad \lim_{i\to\infty} g^{n_i}(x)\varepsilon E$$

Then $\xi = \lim_{i\to\infty} g^{n_i}(x)$ is a unique fixed point.

Many researchers studied fixed point theory by weakening conditions and applying simultaneously enriching metric space structure with partial orders afterwards. One can referee [1]-[28], where excellent authors extended and generalized fixed point results. All in all, the results of fixed points of mappings have been a center on rigorous research for a long time.

It is well known that Picard iterative operator can lead to uniqueness of fixed point. Operator T is said to be a weakly Picard operator, if E is a nonempty, $T : E \longrightarrow E$ and the sequence of successive approximation for any initial value in E converges to a fixed point of T.

Recently, many fixed point theorems have been showed in G–metric space, see [9]-[11] and the references therein. Samet et al. [27] studied that a few of fixed point theorems in the circumstance of a G–metric space could be showed (by simple change) using associate existing results in the ascertained a (quasi–) metric space. That is to say, if the contraction condition of the nonlinear operator on G–metric space can be cut to two variables, then one can establish an equivalent nonlinear operator equation in the ascertained usual metric space. Very recently, Karapinar and Agarwal [13] obtained many nice results. They presented new contraction condition in G– metric space. Agarwal et al. ([3]-[6]) and Karapinar [15] considered some excellent results for a class of generalized contractions in ordered metric spaces.

In 2006, Mustafa and Sims (see [18]) constructed a novel structure of generalized G– metric space and gave some basic topological properties of D–metric spaces. They brought in a new kind of G–metric space (see [9]–[25]), which is called G–metric spaces as a generalization and extension of metric spaces. In the new G– metric spaces, Mustafa [25] considered new fixed point theorems of various mappings. Since then, many researchers have studied and expanded fixed point theory in G-metric spaces (see [14]–[22]). Kikkawa and Suzuki [17], Popescu [26] defined new multi-valued operators. For more interesting results, one can also refer to ([9], [10], [11], [18]–[25]), in which listing results of the fixed point theory in G–metric space. And in ([10]–[14]) some new fixed point theorems for operators satisfying various constractive conditions in G–metric spaces were obtained. Karapinar et al. [16] gave some coupled fixed point results in G–metric spaces.

The G–metric space from then on has paid the attention of mathematicians and natural philosopher and became a very popular topic especially in the sense of perspective of fixed point theory. The definition of G- metric space is as the following (see [18] and [2]–[7]):

Definition 1.1.(see [2]–[7] and [18]) Let E be a nonempty set. A function G is said a generalized metric, or a G-metric, and the pair (E, G) is called a G- metric space, For $u, v, w, c \in E$, if the function $G : E \times E \times E \to [0, +\infty)$ satisfies the following properties

(G1) $G(u, v, w) = G(u, w, v) = \cdots = G(w, u, v) = 0$ if $u = v = w$;

(G2) $G(u, u, w), G(u, v, v), G(w, v, w) > 0$ for $u \neq v, w \neq v, u \neq w$;

(G3) $G(u, u, w), G(u, v, v), G(w, v, w) \leq G(u, v, w)$ for $u \neq v, w \neq v, u \neq w$;

(G4) $G(u, v, w) = G(u, w, v) = G(v, w, u) = G(v, u, w) = G(w, u, v) = G(w, v, u)$;

(G5) $G(u, v, w) \leq G(u, v, c) + G(c, v, w)$(rectangle inequality).

(G6) $G(\alpha u) = |\alpha|G(u), \alpha \in R^1, G(u, u, w) = G(u, w, u) = G(w, u, u)$.

Definition 1.2.(see [2]–[7] and [18]) Let $(E,\, G)$ be a G-metric space. Suppose that $\{u_n\}_{n=1}^{+\infty} \subset E$ be a subsequence of points. We call that u_n is G–convergent to $u^* \in E$ if for any $\varepsilon > 0$, there exists positive integer $N \in \mathbb{N}$ such that $G(u^*, u_n, u_m) < \varepsilon$, for all $n, m \geq N$, that is $\lim\limits_{n,m\to+\infty} G(u^*, u_n, u_m) = 0$.

Definition 1.3.(see [2]–[7] and [18]) Let $(E,\, G)$ be a G-metric space. A sequence $\{u_n\}_{n=1}^{+\infty} \subset E$ is said to be a G– Cauchy sequence if $\lim\limits_{n,m,l\to+\infty} G(u_n, u_m, v_l) = 0$, that is, there exists positive integer $N \in \mathbb{N}$ such that $G(u_n, u_m, u_l) < \varepsilon$ for all $m, n, l \geq N$.

Definition 1.4.(see [2]–[7] and [18]) Let $(E,\, G)$ be a G-metric space. A G–metric space $(E,\, G)$ is said to be G–complete if every G–Cauchy sequence is G–convergent in (E, G).

Definition 1.5.(see [2]–[7] and [18]) Let $(E,\, G)$ be a G-metric space. A mapping $\mathfrak{T} : E \times E \times E \longrightarrow E$ is called G– metric continuous if for any three G–convergent sequences $\{u_n\}_{n=1}^{+\infty} \subset E$, $\{v_n\}_{n=1}^{+\infty} \subset E$, $\{w_n\}_{n=1}^{+\infty} \subset E$ satisfying $\lim\limits_{n\to+\infty} u_n = u$, $\lim\limits_{n\to+\infty} v_n = v$, $\lim\limits_{n\to+\infty} w_n = u$ respectively, such that $\lim\limits_{n,m\to+\infty} \mathfrak{T}(u_n, v_n, w_n) = \mathfrak{T}(u, v, w)$, for $u, v, w \in E$.

Definition 1.6. Let (E, G) be a G-metric space. A mapping $\mathfrak{T} : E \times E \times E \longrightarrow E$ is called G–completely continuous if $\mathfrak{T}$ is compact and G–metric continuous.

As we notice that the strategy cannot be always valid, when the contraction condition is nonlinear type. Motivated and inspired by above nice articles, we will overcome the difficulty and present a new technique. In this work, we extend, generalize, improve, enrich the above mentioned fixed points results of nonlinear contraction mapping in partially ordered G- metric spaces under some weaker conditions. Firstly, we give new fixed point results in a G- complete G- metric space. Then the aim of the article is also to redefine multi-valued mapping in G- complete G- metric spaces and give the corresponding fixed point results. We should address here that our new results extend and complement some known results.

The rest of the article is organized as follows. In section 2, some elementary definitions are introduced. Then the main results and proofs are presented and some examples are given in section 3 to demonstrate the application of our main results, followed by some discussion in section 4.

2. Preliminaries

In this section, we recall some elementary definitions from the asymmetric topology and the order theory, which are necessary for a good understanding of the work below.

The following definitions and results (cf. [14], [17], [18], [26], [28]) gives a comparison result about the G- convergent and some definitions of G- metric space. Now we review some basic concepts and results of G-metric spaces. In the following paper, we denote $E = (E, G), J = [a, b] \subset R^1$.

Definition 2.1. Let E be a G- metric space, and for $t \in J$, let $\{u_n\} = \{u_n(t)\}$ be a sequence of points of E. Then, we say that $\{u_n\}$ is G-convergent to $u = u(t) \in E$ for $t \in J$ if $\lim\limits_{n,m\to\infty} G(u, u_n, u_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(u, u_n, u_m) < \varepsilon$ for all $n, m \geq \mathbb{N}$. We call u is the limit of the sequence $\{u_n\}$ and denote $u_n \longrightarrow u$ $(n \to \infty)$ or $\lim\limits_{n\to\infty} u_n = u$.

Proposition 2.2. Let E be a G- metric space. For any $u = u(t), v = v(t) \in E$, $t \in J$, define on E the metric d_G by $d_G(u, v) = G(u, v, v) + G(u, u, v)$. Then for sequence $\{u_n\}_{n=1}^{\infty} \subseteq E$, the following statements are equivalent

(a) $\{u_n\}$ is G-convergent to u;

(b) $\lim\limits_{n\to\infty} G(u_n, u_n, u) = 0$;

(c) $\lim\limits_{n\to\infty} G(u_n, u, u) = 0$;

(d) $\lim\limits_{n,m\to\infty} G(u_n, u_m, u) = 0.$

Definition 2.3. A G-metric space E is called symmetric G- metric if $G(u,v,v) = G(v,u,u) = G(u,v,u) = G(v,u,v)$ for all $u = u(t), v = v(t) \in E$, $t \in J$.

Definition 2.4. Let E be a G-metric space. For $t \in J$ a sequence $\{u_n\}_{n=1}^{\infty} = \{u_n(t)\} \subset E$ is called a G- Cauchy sequence if for any $\varepsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $G(u_n, u_m, u_l) < \varepsilon$ for all $n, m, l \geq \mathbb{N}$, that is $\lim\limits_{n,m,l\to\infty} G(u_n, u_m, u_l) = 0$ in E.

Definition 2.5. A G-metric space E is said G- complete if every G- Cauchy sequence $\{u_n\}_{n=1}^{\infty} = \{u_n(t)\} \subset E$ is G- convergent in E.

It is well known that for $t \in J$, a sequence $\{u_n\}_{n=1}^{\infty} = \{u_n(t)\} \subset E$ in a G- metric space E is the G- Cauchy if and only if for any $\varepsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $G(u_n, u_m, u_l) < \varepsilon$ for all $n, m, l \geq \mathbb{N}$, that is $\lim\limits_{n,m,l\to\infty} G(u_n, u_m, u_l) = 0$ in E. It is worth mentioning that every G-metric space E defines a metric d_G on E given by

$$d_G(u,v) = G(u,v,v) + G(v,u,u) \quad \text{for all } u(t), v(t) \in E,\ t \in J. \tag{2.1}$$

Corollary 2.6. Let E be a G- metric space. Then E is complete metric space iff each G-Cauchy sequence of E is G-convergent in E.

Definition 2.7. Let E be a G- metric space. A mapping $\Phi : E \longrightarrow E$ is called to be **orbitally continuous** iff $\lim\limits_{i\to\infty} \Phi_{n_i} = \bar{x}$ implies $\lim\limits_{i\to\infty} \Phi\Phi_{n_i} x = \Phi\bar{x}$.

It follows from Definition 1.1 and Proposition 2.2 that the following corollary holds.

Corollary 2.8. Let E be a G- metric space, A metric d_G on E is given as (2.1). Then the metric $d_G(u,v,w)$ on E is continuous with respect to its three variables $u, v, w \in E$.

Now we introduce some notations.

We denote by $\mathcal{CB}(E)$ the family of all nonempty closed bounded subsets of E, for any subsets A, B of E. Let

$$\delta(A,B) := \inf\{dist(a,b) : a \in A, b \in B\}.$$

$\delta(A,B)$ is said to be a metric of A and B.

Denotes the gap between the subsets A and B of E. In particular, if $u \in E$, then

$$dist(u,B) := \inf_{v\in B} dist(u,v), \quad dist(v,A) := \inf_{u\in A} dist(u,v).$$

Let $H(\cdot,\cdot)$ be the Hausdorff metric, i.e.,

$$H(A,B)=\sup\left\{\sup_{u\in A}d(u,B),\sup_{v\in B}d(v,A)\right\},\ \text{ for all }\ A,B\in\mathcal{CB}(E).$$

Recently, Kaewcharoen and Kaewkhao [14] introduced the following concepts.

Let E be a G- metric space and $H(\cdot,\cdot,\cdot)$ be the Hausdorff G-distance on $\mathcal{CB}(E)$, i.e.

$$G(u,v,A):=\inf\{G(u,v,w),w\in A\}$$

$$H_G(A,B,F)=\sup\left\{\sup_{u\in A}G(u,B,F),\sup_{u\in B}G(A,u,F),\sup_{u\in F}G(A,B,u)\right\},$$

where

$$G(u,B,F)=d_G(u,B)+d_G(B,F)+d_G(u,F),\ d_G(u,B)=\inf\{d_G(u,v):v\in B\}$$

$$d_G(A,B)=\inf\{d_G(a,b):a\in A,b\in B\}.$$

Recall that $G(u,v,F):=\inf\{G(u,v,w),w\in F\}$. A mapping $T:E\to\mathcal{CB}(E)$ is said to be a multi-valued mapping. A point $u\in E$ is said to be a fixed point of T if $u\in Tu$.

Lemma 2.9. Let E be a G- metric space and $A,B\in\mathcal{CB}(E)$. Then for each $\alpha\in A$, we have

$$G(u,B,B)\le H_G(A,B,B).$$

Corollary 2.10. Suppose that $f(t)=\frac{1}{1+\beta(t)}$ is a real function for $t\in[0,+\infty)$, with $0\le\beta(t)<1$, and E is a metric space with $F\subseteq E$. Then $T:F\longrightarrow\mathcal{CB}(E)$ is said to be a $\beta(t)$-KS multi-valued operator if $0\le\beta(t)<1$, $t\in[0,+\infty)$ and $u,v\in E$ with $\delta(u,Tu)\le\frac{1}{f(t)}d(u,v)$ implies

$$H(Tu,Tv)\le\beta(t)d(u,v).$$

The following result is a refinement of $Nadler's$ theorem. One refers to see ([17], [19]-[25], [26]) and references therein.

Theorem 2.11. Suppose that $f(t)=\frac{1}{1+\beta(t)}$ is a real function for $t\in[0,+\infty)$ with $0\le\beta(t)<1$, and E is a complete metric space. Let $T:E\longrightarrow\mathcal{CB}(E)$ be a $\beta(t)$-KS multi-valued operator. Then there exists $w\in E$ such that $w\in Tw$.

Definition 2.12. Let E be a complete metric space. Suppose that $T:E\longrightarrow\mathcal{CB}(E)$. T is said to be an $(\gamma,\beta(t))$-contractive multi-valued operator if $0\le\beta(t)<1,\gamma\ge\beta(t)$ and $u,v\in E$ with $\delta(v,Tu)\le\gamma d(v,u)$ implies $H(Tu,Tv)\le\beta(t)M_T(u,v))$, where

$$M_T(u,v)=\max\left\{d(u,v),\delta(u,Tu),\delta(v,Tv),\frac{\delta(u,Tv)+\delta(v,Tu)}{2}\right\}.$$

When $\gamma=s,\beta(t)=r$, one can see Popescu [26] defined the (s,r)- contractive multi-valued operator is a special of $(\gamma,\beta(t))$. Therefore Definition 2.12 is novel.

3. **Main results**

Definition 3.1. Let E be a complete G-metric space and $T : E \longrightarrow \mathcal{CB}(E)$. Suppose that $f(t) = \frac{1}{1+\beta(t)}$ is a strictly decreasing real function for $t \in [0, +\infty)$. If there exists $0 \leq \beta(t) < 1, t \in [0, +\infty)$ such that $G(u, u, Tu) \leq \frac{1}{f(t)} G(u, u, v)$ implies

$$H_G(Tu, Tu, Tv) \leq \beta(t)G(u, u, v), \text{ for all } u, v \in E. \tag{3.1}$$

Then T is called a $\beta(t)$-KS multi-valued operator in G-metric space.

Definition 3.2. Let E be a complete G- metric space and $T : E \longrightarrow \mathcal{CB}(E)$. Assume that $0 \leq \beta(t) < 1, t \in [0, +\infty), \gamma \geq 1$ and $u, v \in E$ with $G(v, v, Tu) \leq \gamma G(v, v, u)$ implies

$$H_G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) \tag{3.2}$$

Where

$$M_T(u, u, v) = \max\left\{G(u, u, v), G(u, u, Tu), G(v, v, Tv), \frac{G(u, u, Tv) + G(v, v, Tu)}{2}\right\}.$$

Then T is called an $(\gamma, \beta(t))$- contractive multi-valued operator in G-metric space.

Remark 3.3. Let E be a G-metric space, $A, B \subseteq \mathcal{CB}(E)$, by lemma 2.8, for each $b \in B$, we have $G(A, A, b) \leq H_G(A, A, B)$, since $G(A, A, b) = 2d_G(b, A)$, then there exists $a \in A$ such that $G(A, A, b) = 2(G(b, b, a) + G(a, a, b))$, hence we have

$$G(a, a, B) \leq G(a, a, b) \leq G(A, A, b) \leq H_G(A, A, B).$$

Theorem 3.4. Let E be a complete G-metric space and T be a r-KS multi-valued operator from E into $\mathcal{CB}(E)$ in G- metric space. Then there exists $w \in E$ such that $w \in Tw$.

Proof. Take a real number $\beta_1(t)$ with $0 \leq \beta(t) < \beta_1(t) < 1$, $t \in [0, +\infty)$. Then, for each $x = x_0 \in E$ and there exists $x_1 \in Tx$, we have

$$G(x, x, Tx) = G(x, x, x_1) \leq \frac{1}{f(t)} G(x, x, x_1).$$

From (3.1) and remark 3.3 we have

$$G(x_1, x_1, Tx_1) \leq H_G(Tx, Tx, Tx_1) \leq \beta(t)G(x, x, x_1)$$

holds. So, there exists $x_2 \in Tx_1$ such that $G(x_1, x_1, x_2) \leq \beta_1(t)G(x, x, x_1)$. Thus, we have a sequence $\{x_n\}$ in E such that $x_{n+1} \in Tx_n$ and

(3.3)
$$\begin{aligned} G(x_{n-1}, x_{n-1}, x_n) &\leq \beta_1(t)G(x_{n-2}, x_{n-2}, x_{n-1}) \leq \beta_1^2(t)G(x_{n-3}, x_{n-3}, x_{n-2}) \leq \cdots \\ &\leq \beta_1^{n-1}(t)G(x, x, x_1). \end{aligned}$$

Then, for all $n, m \in \mathbb{N}, n < m$, we have by repeated use of the rectangle inequality and equation (3.3) that

$$\begin{aligned} G(x_n, x_n, x_m) &\leq G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\ &\leq (\beta_1^n(t) + \beta_1^{n+1}(t) + \cdots + \beta_1^{m+n-1}(t))G(x, x, x_1) \\ &\leq \frac{\beta_1^m(t)}{1-\beta_1(t)} G(x, x, x_1). \end{aligned}$$

Then, $G(x_n, x_n, x_m) \longrightarrow 0$ as $n, m \to \infty$, since $\lim_{m\to\infty} \frac{\beta_1^m(t)}{1-\beta_1(t)} G(x, x, x_1) = 0$. For $n, m, l \in \mathbb{N}$, $(G5)$ implies that

$$G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l),$$

taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \longrightarrow 0$. So $\{x_n\}$ is G-Cauchy sequence. By completeness of E, there exist $w \in E$ such that $\{x_n\}$ is G-converges to w.

Then we show that $G(w, w, Tu) \leq \beta(t)G(w, w, u)$ for all $u \in E \setminus \{w\}$.

Since $x_n \longrightarrow w$, there exist $k \in \mathbb{N}$ such that

$$G(w, w, x_n) \leq \frac{1}{3} G(w, w, u), \text{ and } G(w, x_n, x_n) \leq \frac{1}{3} G(w, w, u), \ \forall\, n \in \mathbb{N}, \ n \geq k.$$

Then we have

$$\begin{aligned} f(t)G(x_n, x_n, Tx_n) &\leq G(x_n, x_n, Tx_n) \leq G(x_n, x_n, x_{n+1}) \\ &\leq G(x_n, x_n, w) + G(w, w, x_{n+1}) \\ &\leq \frac{2}{3} G(w, w, u) = G(w, w, u) - \frac{1}{3} G(w, w, u) \\ &\leq G(w, w, u) - G(w, w, x_n) \leq G(x_n, x_n, u) \end{aligned}$$

Hence $H_G(Tx_n, Tx_n, Tu) \leq \beta(t)G(x_n, x_n, u)$. So it follows that

$$G(x_{n+1}, x_{n+1}, Tu) \leq \beta(t)G(x_n, x_n, u), \text{ for } n \in \mathbb{N}, \text{ with } n \geq v.$$

Letting $n \longrightarrow \infty$, we obtain $G(w, w, Tu) \leq \beta(t)G(w, w, u)$ for all $u \in E \setminus \{w\}$.

Next we prove that $H_G(Tu, Tu, Tw) \leq \beta(t)G(u, u, w)$, for all $u \in E$.

If $w = u$, then it obviously holds. So we assume that $w \neq u$, then for every $n \in \mathbb{N}$, there exists $v_n \in Tu$ such that $G(w, w, v_n) \leq G(w, w, Tu) + \frac{1}{n} G(u, u, w)$. We have

$$\begin{aligned}G(u,u,Tu) &\leq G(u,u,v_n) \leq G(u,u,w)+G(w,w,v_n)\\ &\leq G(u,u,w)+G(w,w,Tu)+\frac{1}{n}G(u,u,w)\\ &\leq G(u,u,w)+\beta(t)G(w,w,u)+\frac{1}{n}G(u,u,w)\\ &\leq G(u,u,w)+\beta(t)(G(w,u,u)+G(u,w,u))+\frac{1}{n}G(u,u,w)\\ &=(1+2\beta(t)+\frac{1}{n})G(u,u,w)\end{aligned}$$

Hence we have

$$\frac{1}{1+2\beta(t)}G(u,u,Tu)\leq G(u,u,w).$$

From (3.1), we have $H_G(Tu,Tu,Tw)\leq 2\beta(t)G(u,u,w)$.

Finally, since

$$\begin{aligned}G(w,w,Tw) &= \lim_{n\to\infty} G(x_{n+1},x_{n+1},Tw) \leq \lim_{n\to\infty} H_G(Tx_n,Tx_n,Tw)\\ &\leq \lim_{n\to\infty} 2\beta(t)G(x_n,x_n,w)=0\end{aligned}$$

and Tw is closed, we obtain $w\in Tw$.

Corollary 3.5. Suppose that E is a complete G-metric space and $T:E\longrightarrow \mathcal{CB}(E)$. If there exists $0\leq\beta(t)<1, t\in[0,+\infty)$ such that

$$H_G(Tu,Tu,Tv)\leq\beta(t)G(u,u,v)\quad\text{for all }\ u,v\in E.$$

then there exists $w\in E$ such that $w\in Tw$.

Theorem 3.6. Suppose that E is a complete G- metric space and $T:E\longrightarrow\mathcal{CB}(E)$ is a $(\gamma,\beta(t))$- contractive multi-valued operator with $\gamma>\beta(t)$. Then there exists $w\in E$ such that $w\in Tw$.

Proof. Let $\beta_1(t)$ be a real number such that $0\leq\beta(t)<\beta_1(t)<\gamma$ and $0<\beta_1(t)<1$. Let $x_1\in E$ and there exists $x_2\in Tx_1$. If $x_2=x_1$, then $x_1\in Tx_1$ and the proof is complete. So we assume that $x_2\neq x_1$, then $G(x_2,x_2,Tx_1)\leq\gamma G(x_2,x_2,x_1)$. From (3.2) and remark 3.3 we have

$$\begin{aligned}G(x_2,x_2,Tx_2) &\leq H_G(Tx_1,Tx_1,Tx_2)\\ &\leq \beta(t)\max\Big\{G(x_1,x_1,x_2),G(x_1,x_1,Tx_1),G(x_2,x_2,Tx_2),\\ &\qquad \frac{G(x_1,x_1,Tx_1)+G(x_2,x_2,Tx_2)}{2}\Big\}\end{aligned}$$

So

$$G(x_2,x_2,Tx_2)\leq\beta(t)\max\left\{G(x_1,x_1,x_2),G(x_1,x_1,Tx_1),\frac{G(x_1,x_1,x_2)+G(x_2,x_2,Tx_2)}{2}\right\}$$

Hence, from $0 \le \beta(t) < 1$, we have $G(x_2, x_2, Tx_2) \le G(x_1, x_1, x_2)$. Then there exists $x_3 \in Tx_2$ such that $G(x_2, x_2, x_3) \le \beta_1(t)G(x_1, x_1, x_2)$. Thus, we can construct a sequence $\{x_n\}$ in E such that $x_{n+1} \in Tx_n$ and

$$G(x_{n-1}, x_{n-1}, x_n) \le \beta_1(t)G(x_{n-2}, x_{n-2}, x_{n-1}) \le \cdots \le \beta_1^{n-2}(t)G(x_1, x_1, x_2) \tag{3.4}$$

Then, for all $n, m \in \mathbb{N}, n < m$, we have by repeated use of the rectangle inequality and equation (3.4) that

$$\begin{aligned} &G(x_n, x_n, x_m) \le G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\ &\le (\beta_1^n(t) + \beta_1^{n+1}(t) + \cdots + \beta_1^{m+n-1}(t))G(x_1, x_1, x_2) \le \frac{\beta_1^n(t)}{1-\beta_1(t)}G(x_1, x_1, x_2). \end{aligned}$$

since $\lim_{n\to\infty} \frac{\beta_1^n(t)}{1-\beta_1(t)}G(x_1, x_1, x_2) = 0$. Then, $\lim_{n,m\to\infty} G(x_n, x_n, x_m) = 0$. Thus for $n, m, l \in \mathbb{N}$, together with $(G5)$, it implies that

$$G(x_n, x_m, x_l) \le G(x_n, x_m, x_m) + G(x_m, x_m, x_l)$$

Thus we get $\lim_{n,m,l\to\infty} G(x_n, x_m, x_l) = 0$. So $\{x_n\}$ is a G-Cauchy sequence. By completeness of E, there exists $w \in E$ such that $\{x_n\}$ is G- converges to w.

Now, we will show that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$G(w, w, Tx_{n_k}) \le \gamma G(w, w, x_{n_k}), \quad \text{for all } k \in \mathbb{N}.$$

Arguing by contradiction, we assume that there exists a positive integer N such that

$$G(w, w, Tw_n) > \gamma G(w, w, x_n), \quad \text{for all } n \ge N.$$

This implies $G(w, w, x_{n+1}) > \gamma G(w, w, x_n)$, by induction and (G5), for all $n \ge N$, $p \ge 1$, we get that

$$\begin{aligned} \gamma^p G(w, w, x_n) < G(w, w, x_{n+p}) &\le G(w, x_{n+p}, x_{n+p}) + G(x_{n+p}, w, x_{n+p}) \\ &= 2G(x_{n+p}, x_{n+p}, w) \end{aligned} \tag{3.5}$$

by repeated use of the rectangle inequality and (3.4) we have

$$\begin{aligned} G(x_n, x_n, x_{n+p}) &\le G(x_n, x_n, x_{n+1}) + \cdots + G(x_{n+p-1}, x_{n+p-1}, x_{n+p}) \\ &\le G(x_n, x_n, x_{n+1})(1 + \beta_1(t) + \beta_1^2(t) + \cdots + \beta_1^{p-1}(t)) \\ &= \frac{1-\beta_1^p(t)}{1-\beta_1(t)}G(x_n, x_n, x_{n+1}) \end{aligned}$$

for all $n \ge N$, $p \ge 1$. Taking the limit as $p \to \infty$, we get

$$G(x_n, x_n, w) \le \frac{1}{1-\beta_1(t)}G(x_n, x_n, x_{n+1}), \text{ for all } n \ge 1.$$

Then we obtain

$$G(x_{n+p}, x_{n+p}, w) \le \frac{1}{1-\beta_1(t)}G(x_{n+p}, x_{n+p}, x_{n+p+1}) \le \frac{\beta_1^p(t)}{1-\beta_1(t)}G(x_n, x_n, x_{n+1}) \tag{3.6}$$

for all $n \geq N, p \geq 1$. From (3.5) and (3.6), we obtain

$$G(w,w,x_n) < \frac{2\left(\frac{\beta_1(t)}{\gamma}\right)^p}{1-\beta_1(t)} G(x_n,x_n,x_{n+1})$$

for all $n \geq N, p \geq 1$. Taking the limit as $p \to \infty$, we get that $G(w,w,x_n) = 0$, that is $w = x_n$ for all $n \geq N$. This contradicts with (3.5). Therefore there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$G(w,w,Tx_{n_k}) \leq \gamma G(w,w,x_{n_k})$$

for all $k \in \mathbb{N}$. By making use of (3.2), we have

$$\begin{aligned} &H_G(Tx_{n_k},Tx_{n_k},Tw) \leq \beta(t) \max\Big\{G(x_{n_k},x_{n_k},w), G(x_{n_k},x_{n_k},Tx_{n_k}), \\ &G(w,w,Tw), \frac{G(w,w,Tx_{n_k}) + G(x_{n_k},x_{n_k},Tw)}{2}\Big\} \end{aligned} \tag{3.7}$$

Hence there exists $x_{n_k+1} \in Tx_{n_k}$ such that

$$\begin{aligned} &G(x_{n_k+1},x_{n_k+1},Tw) \leq \beta(t) \max\Big\{G(x_{n_k},x_{n_k},w), G(x_{n_k},x_{n_k},x_{n_k+1}), \\ &G(w,w,Tw), \frac{G(w,w,x_{n_k+1}) + G(x_{n_k},x_{n_k},Tw)}{2}\Big\} \end{aligned} \tag{3.8}$$

Letting $k \to \infty$, we have

$$G(w,w,Tw) \leq \beta(t) \max\left\{G(w,w,Tw), \frac{G(w,w,Tw)}{2}\right\}.$$

Then we get $G(w,w,Tw) = 0$ and $Tw \in \mathcal{CB}(E), w \in Tw$.

Example 3.7. Let $E = \{1,2,3\}$ and $G : E \times E \times E \to [0,+\infty)$ be defined by

$$\begin{aligned} &G(1,1,2) = G(2,1,1) = G(1,2,1) = G(1,2,2) = G(2,1,2) = G(2,2,1) = 2; \\ &G(1,1,3) = G(3,1,1) = G(1,3,1) = G(1,3,3) = G(3,1,3) = G(3,3,1) = 1; \\ &G(2,2,3) = G(3,2,2) = G(2,3,2) = G(2,3,3) = G(3,2,3) = G(3,3,2) = 6; \\ &G(1,2,3) = G(1,3,2) = G(2,1,3) = G(2,3,1) = G(3,1,2) = G(3,2,1) = 3; \\ &G(u,u,u) = 0 \ \text{ for all } \ u \in E. \end{aligned}$$

E is a symmetric G-complete G-metric space.

Let $T : E \to \mathcal{CB}(E)$ be defined by $T1 = T2 = \{1,2\}, T3 = \{1,2,3\}$. Then

(i) T is an $(\gamma, \beta(t))$-contractive multi-valued operator with $\beta(t) = 0.8$ and $\gamma = 0.9$;

(ii) Every $u \in E$ is a fixed point of T.

We have

$$G(1,1,T2) = G(1,1,T3) = \{0,2\}, \quad H_G(T1,T1,T2) = H_G(T1,T2,T2) = 0$$
$$H_G(T1,T1,T3) = H_G(T2,T2,T3) = G(2,2,3) = 6.$$
$$2d_G(3,T2) = 2(G(3,1,1) + G(1,3,3)) = 4 < 0.8G(2,2,3)$$
$$H_G(T3,T3,T1) = H_G(T3,T3,T2) = H_G(3,3,2)$$
$$d_G(3,T2) = G(3,1,1) + G(1,3,3) = 2 < 0.8G(3,3,2)$$

and

$$0 = G(1,1,T2) \le \gamma G(1,1,2) = 1.6, \quad 2 = G(1,1,T2) \ge \gamma G(1,1,1) = 0,$$
$$0 = G(1,1,T3) < \gamma G(1,1,3) = 0.8, \quad 2 = G(1,1,T3) = G(1,1,2) > \gamma G(1,1,3) = 0.8,$$
$$1 = G(1,1,T3) = G(1,1,3) < \gamma G(1,1,2) = 1.6,$$
$$2 = G(1,1,T3) = G(1,1,2) > \gamma G(1,1,1) = 0,$$
$$2 = G(2,2,1) = G(2,2,T1) > \gamma G(2,2,2) = 0,$$
$$0 = G(2,2,2) = G(2,2,T1) < \gamma G(2,2,1) = 1.6,$$
$$2 = G(2,2,1) = G(2,2,T3) < \gamma G(2,2,3) = 4.8,$$
$$0 = G(2,2,2) = G(2,2,T3) < \gamma G(2,2,3) = 4.8,$$
$$6 = G(2,2,3) = G(2,2,T3) > \gamma G(2,2,1) = 1.6,$$
$$6 = G(2,2,3) = G(2,2,T3) > \gamma G(2,2,2) = 0,$$
$$1 = G(3,3,1) = G(3,3,T1) > \gamma G(3,3,1) = 0.8,$$
$$6 = G(3,3,2) = G(3,3,T1) > \gamma G(3,3,2) = 4.8,$$
$$1 = G(3,3,1) = G(3,3,T2) < \gamma G(3,3,2) = 4.8,$$
$$6 = G(3,3,2) = G(3,3,T2) > \gamma G(3,3,1) = 0.8$$

So T is an $(\gamma, \beta(t))$- contractive multi-valued operator with $\beta(t) = 0.8$ and $\gamma = 0.9$.

(ii) It is obvious.

Considering T as a single-valued mapping, then we have the following theorem.

Theorem 3.8. Let E be a complete G-metric space and $T : E \longrightarrow E$ be a $(\gamma, \beta(t))$ -contractive single-valued operator $u, v \in E$ with $G(v,v,Tu) \le \gamma G(v,v,u)$ implies

$$G(Tu,Tu,Tv) \le \beta(t) M_T(u,u,v), \tag{3.9}$$

where

$$M_T(u,u,v) = \max\left\{G(u,u,v),\ G(u,u,Tu),\ G(v,v,Tv),\ \frac{G(u,u,Tv) + G(v,v,Tu)}{2}\right\}.$$

Then T has a fixed point. Moreover, if $\gamma \geq 1$, then T has a unique point.

Proof. It follows from Theorem 3.6. that if $\gamma \geq 1$, there exists $u, v \in Fix(T), u \neq v$, then $G(v, v, Tu) = G(v, v, u) \leq \gamma G(v, v, u)$. By (3.7), we have

$$G(u, u, v) = G(Tu, Tu, Tv) \leq \max\left\{G(u, u, v), \frac{G(u, u, v) + G(v, v, u)}{2}\right\} \tag{3.10}$$

Consequently, there are the following two cases

Case (1) If $G(u, u, v) \leq \beta(t)G(u, u, v)$ which is a contradiction.

Case (2) If $G(u, u, v) \leq \beta(t)\frac{G(u,u,v)+G(v,v,u)}{2}$, then we have

$$G(u, u, v) \leq \frac{\beta(t)}{2 - \beta(t)}G(v, v, u).$$

By the symmetry of u and v, we obtain

$$G(u, u, v) \leq \frac{\beta(t)}{2 - \beta(t)}G(v, v, u) \leq \left(\frac{\beta(t)}{2 - \beta(t)}\right)^2 G(u, u, v) \tag{3.11}$$

Notice that $\beta(t) \in [0, 1)$, then we have $\left(\frac{\beta(t)}{2-\beta(t)}\right)^2 < 1$. Thus (3.11) is a contradiction. Hence, if $\gamma \geq 1$, then T has a unique fixed point.

Example 3.9. Let $E = \{1, 2, 3, 4\}$ and $G : E \times E \times E \to [0, +\infty)$ be defined by

$$G(1, 1, 2) = G(2, 1, 1) = G(1, 2, 1) = G(1, 2, 2) = G(2, 1, 2) = G(2, 2, 1) = 7;$$
$$G(1, 1, 3) = G(3, 1, 1) = G(1, 3, 1) = G(1, 3, 3) = G(3, 1, 3) = G(3, 3, 1) = 6;$$
$$G(1, 1, 4) = G(4, 1, 1) = G(1, 4, 1) = G(1, 4, 4) = G(4, 1, 4) = G(4, 4, 1) = 5;$$
$$G(2, 2, 3) = G(3, 2, 2) = G(2, 3, 2) = G(2, 3, 3) = G(3, 2, 3) = G(3, 3, 2) = 4;$$
$$G(2, 2, 4) = G(4, 2, 2) = G(2, 4, 2) = G(2, 4, 4) = G(4, 2, 4) = G(4, 4, 2) = 3;$$
$$G(3, 3, 4) = G(4, 3, 3) = G(3, 4, 3) = G(3, 4, 4) = G(4, 3, 4) = G(4, 4, 3) = 2;$$
$$G(1, 2, 3) = G(1, 3, 2) = G(2, 1, 3) = G(2, 3, 1) = G(3, 1, 2) = G(3, 2, 1) = 8;$$
$$G(2, 3, 4) = G(2, 4, 3) = G(3, 2, 4) = G(3, 4, 2) = G(4, 2, 3) = G(4, 3, 2) = 8;$$
$$G(1, 3, 4) = G(1, 4, 3) = G(3, 1, 4) = G(3, 4, 1) = G(4, 1, 3) = G(4, 3, 1) = 8;$$
$$G(1, 2, 4) = G(1, 4, 2) = G(2, 1, 4) = G(2, 4, 1) = G(4, 1, 2) = G(4, 2, 1) = 8;$$
$$G(u, u, u) = 0 \quad \text{for all} \quad u \in E.$$

E is a symmetric G-complete G-metric space.

Let $T : E \to E$ be such that $T1 = T2 = 3$, $T3 = T4 = 4$. Then

(a) T is a $(\gamma, \beta(t))$-contractive single-valued operator with $\beta(t) = 0.6$ and $\gamma = 1.1$;

(b) T has a unique fixed point.

Proof. **(a)** we have the following cases:

(i) If $u = 1$, $v = 2$ or $u = 2$, $v = 1$, then $G(Tu, Tu, Tv) = G(3, 3, 3) = 0$ and $M_T(u, u, v) = 7$, hence $0 = G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) = 4.2$.

(ii) If $u = 1$, $v = 3$ or $u = 3$, $v = 1$ then $G(Tu, Tu, Tv) = G(3, 3, 4) = 2$ and $M_T(u, u, v) = 6$, hence $2 = G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) = 3.6$.

(iii) If $u = 1$, $v = 4$ or $u = 4$, $v = 1$, then $G(Tu, Tu, Tv) = G(3, 3, 4) = 2$ and $M_T(u, u, v) = 6$, hence $2 = G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) = 3.6$.

(iv) If $u = 2$, $v = 3$ or $u = 3$, $v = 2$, then $G(Tu, Tu, Tv) = G(3, 3, 4) = 2$ and $M_T(u, u, v) = 4$, hence $2 = G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) = 2.4$.

(v) If $u = 2$, $v = 4$ or $u = 4$, $v = 2$, then $G(Tu, Tu, Tv) = G(3, 3, 4) = 2$ and $M_T(u, u, v) = 4$, hence $2 = G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) = 2.4$.

(vi) If $u = 3$, $v = 4$ or $u = 4$, $v = 3$, then $G(Tu, Tu, Tv) = G(4, 4, 4) = 0$ and $M_T(u, u, v) = 2$, hence $0 = G(Tu, Tu, Tv) \leq \beta(t)M_T(u, u, v) = 1.2$.

(b) It is obvious.

4. Discussions on the conditions of Theorems

We discuss the conditions in this paper. It is easy to see that the functions satisfying the conditions of the theorems are rather wide. For example, we can obtain the following corollary:

Corollary 4.1. Suppose that all $\beta_i(y)$ $(i = 0,\ 1,\ \cdots,\ m)$ are nonnegative continuous functions and satisfy $0 \leq \beta_i(t) < 1$ on $[0, +\infty)$. Let E be a complete G-metric space and $T : E \longrightarrow E$ be a $(\gamma, \beta(t))$-contractive single-valued operator with $G(v, v, Tu) \leq \gamma G(v, v, u)$ for u, $v \in E$, $\gamma > \beta(t)$ implies

$$G(Tu, Tu, Tv) \leq \beta_i(t)M_T(u, u, v). \tag{4.1}$$

where

$$M_T(u, u, v) = \max\left\{G(u, u, v),\ G(u, u, Tu),\ G(v, v, Tv),\ \frac{G(u, u, Tv) + G(v, v, Tu)}{2}\right\}.$$

Then T has a fixed point. Moreover, if $\gamma \geq 1$, then T has a unique fixed point.

Remark 4.1. The key condition in [11] requires the constant $0 < r < 1$. Here in our work, we only require the function $0 < \beta(t) < 1$, $t \in [0, +\infty)$. From above discussions, it is clear that our results improve and extend the results in [11] and [25].

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