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A STUDY OF COLLECTIVELY COINCIDENCE POINTS AND MAXIMAL TYPE ELEMENTS

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ABSTRACT. We present some new collectively fixed and coincidence results for families of maps which enables us to establish maximal element type results.

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1. INTRODUCTION

Motivated by ideas in [4, 12, 13] in this paper we present some new collectively fixed and coincidence type results. Then using these results we establish some new maximal type element theorems for families of majorized type maps [4, 6] in the compact setting. In this paper we discuss the $\Phi^\star$ maps from the literature [2] and also admissible maps in the sense of of Gorniewicz [9]. We present collectively coincidence results bewteen classes of maps (the first result is between the same classes and the second result is between different classes).

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

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Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i). p is a Vietoris map

and

(ii). $q\left(p^{-1}(x)\right) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [9]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to Kakutani (and we write $\phi \in Kak(X, Y)$); here $K(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

The following class of maps will play a major role in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in \Phi^\star(Z, W)$ [2] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : \, w \in S(z)\}$ is open (in Z) for each $w \in W$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V\,(K)$ there exists a continuous functions $\pi_\alpha : K \to V$ such that

(i). π_α and $i : K \to V$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

Our first result is taken from [1, 10].

Theorem 1.1. *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in AD(X,X)$ a compact map. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

Remark 1.2. Other variations of Theorem 1.1 can be found in [11].

We recall that a point $x \in X$ is a maximal element of a set valued map F from a topological space X to another topological space Y if $F(x) = \emptyset$.

2. MAXIMAL ELEMENT RESULTS

We begin by establishing a new collectively fixed point result (motivated in part from [12, 14]).

Theorem 2.1. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,...,N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Finally suppose for each $x \in X$ there exists a $i \in \{1,...,N\}$ with $S_i(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in \{1,..,N\}$ with $x_i \in F_i(x)$ (here x_i is the projection of x on X_i).*

Proof. Note $A_i = \{x \in X : S_i(x) \neq \emptyset\}, i \in \{1,..,N\}$ is an open covering of X (recall the fibres of S_i are open). Now since X is compact (so in particular paracompact) then from [8, Lemma 5.1.6, pp301] there exists a covering $\{B_i\}_{i=1}^N$ of X where B_i is closed and $B_i \subset A_i$ for all $i \in \{1,...,N\}$. For each $i \in \{1,..,N\}$ let $G_i : X \to X_i$ and $T_i : X \to X_i$ be given by

$$G_i(x) = \begin{cases} F_i(x), \ x \in B_i \\ X_i, \ x \in X \backslash B_i \end{cases} \quad \text{and} \quad T_i(x) = \begin{cases} S_i(x), \ x \in B_i \\ X_i, \ x \in X \backslash B_i. \end{cases}$$

We claim for $i \in \{1,...,N\}$ that $G_i \in \Phi^\star(X, X_i)$. Note first for $i \in \{1,...,N\}$ that $T_i(x) \neq \emptyset$ for $x \in X$. Also for $x \in X$ and $i \in \{1,...,N\}$ then if $x \in B_i$ we have $T_i(x) = S_i(x) \subseteq F_i(x) = G_i(x)$ whereas if $x \in X \backslash B_i$ we have $T_i(x) = X_i = G_i(x)$. Also note if $y \in X_i$ then

$$\begin{aligned} T_i^{-1}(y) &= \{z \in X : y \in T_i(z)\} \\ &= \{z \in X \backslash B_i : y \in T_i(z) = X_i\} \cup \{z \in B_i : y \in T_i(z)\} \\ &= (X \backslash B_i) \cup \{z \in B_i : y \in S_i(z)\} = (X \backslash B_i) \cup [B_i \cap \{z \in X : y \in S_i(z)\}] \\ &= (X \backslash B_i) \cup \left[B_i \cap S_i^{-1}(y)\right] = X \cap \left[(X \backslash B_i) \cup S_i^{-1}(y)\right] = (X \backslash B_i) \cup S_i^{-1}(y) \end{aligned}$$

which is open in X (note $S_i^{-1}(y)$ is open in X and B_i is closed in X). Thus for $i \in \{1,...,N\}$ we have $G_i \in \Phi^\star(X, X_i)$.

Now since X is compact for each $i \in \{1, ..., N\}$ from [2, 5] there exists a continuous (single valued) selection $f_i : X \to X_i$ of G_i with $f_i(x) \in T_i(x) \subseteq G_i(x)$ for $x \in X$ and also there exists a finite set C_i of X_i with $f_i(X) \subseteq co\,(C_i) \equiv D_i$. Let

$$D = \prod_{i=1}^{N} D_i \quad \text{and} \quad f(x) = \prod_{i=1}^{N} f_i(x),\ x \in D.$$

Now D is compact and convex, $f : D \to D$ and $f(D)$ lies in a finite dimensional subspace of $E = \prod_{i=1}^{N} E_i$. Brouwer's fixed point theorem guarantees that there exists a $x \in D$ with $x = f(x)$ i.e. $x_j = f_j(x) \in T_j(x) \subseteq G_j(x)$ for each $j \in \{1, .., N\}$. Now since $\{B_i\}_{i=1}^{N}$ is a covering of X there exists a $j_0 \in \{1, .., N\}$ with $x \in B_{j_0}$ so $x_{j_0} \in G_{j_0}(x) = F_{j_0}(x)$. □

Remark 2.2. Note one could replace $\{X_i\}_{i=1}^{N}$ in Theorem 2.1 with $\{X_i\}_{i \in I}$ where I is an index set. In Theorem 2.1 since $X \equiv \prod_{i \in I} X_i$ is compact (since we assume each X_i is compact) then we could assume in the statement of Theorem 2.1 that there exists a finite subset I_0 of I and for each $x \in X$ there exists a $i \in I_0$ with $S_i(x) \neq \emptyset$ so as a result one could rewrite the statement of Theorem 2.1. This remark could also be applied to the other results in this paper.

Next we will rewrite Theorem 2.1 as a maximal type element result.

Theorem 2.3. *Let $\{X_i\}_{i=1}^{N}$ be a family of convex compact sets each in a Hausdorff topological vector space. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Now suppose for all $i \in \{1, .., N\}$ that $x_i \notin F_i(x)$ for each $x \in X$. Then there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $S_i(x) \neq \emptyset$. Now Theorem 2.1 guarantees a $x \in X$ and a $i \in \{1, .., N\}$ with $x_i \in F_i(x)$, a contradiction. □

We now discuss a generalization of majorized mappings in the literature (see [3, 4, 6, 15]). Let Z and W be sets in a Hausdorff topological vector space with W convex and Z compact. Suppose $H : Z \to W$, $J : Z \to W$ and for each $y \in Z$ assume there exists a map $A_y : Z \to W$ and an open set U_y containing y with $H(z) \subseteq A_y(z)$ for every $z \in U_y$, A_y is convex valued, $(A_y)^{-1}(x)$ is open (in Z) for each $x \in W$ and $J(w) \cap A_y(w) = \emptyset$ for $w \in Z$. We now claim that there exists a map $T : Z \to W$ with $H(z) \subseteq T(z)$ for $z \in Z$, T is convex valued, $T^{-1}(x)$ is open (in Z) for each $x \in W$ and $J(w) \cap T(w) = \emptyset$ for $w \in Z$. To see this note $\{U_y\}_{y \in Z}$ is an open covering of Z and since Z is compact there exists [7, 8] a finite set $\{y_1, ..., y_n\}$ (with $y_i \in Z$ for

$i \in \{1, ..., n\}$) and an open covering $\{V_{y_i}\}_{i=1}^n$ of Z with $y_i \in V_{y_i}$ and $\Omega_{y_i} = \overline{V_{y_i}} \subseteq U_{y_i}$ for $i \in \{1, ..., n\}$. Fix $i \in \{1, ..., n\}$ and let

$$Q_{y_i}(z) = \begin{cases} A_{y_i}(z), & z \in \Omega_{y_i} \\ W, & z \in Z \backslash \Omega_{y_i}. \end{cases}$$

Now Q_{y_i} is convex valued and $H(z) \subseteq Q_{y_i}(z)$ for every $z \in Z$ (note if $z \in \Omega_{y_i}$ then since $\Omega_{y_i} \subseteq U_{y_i}$ and since $H(w) \subseteq A_{y_i}(w)$ for $w \in U_{y_i}$ we have $H(z) \subseteq Q_{y_i}(z)$ whereas if $z \in Z \backslash \Omega_{y_i}$ then it is immediate since $Q_{y_i}(z) = W$). Also note the argument in Theorem 2.1 guarantees for any $x \in W$ that

$$(Q_{y_i})^{-1}(x) = (Z \backslash \Omega_{y_i}) \cup (A_{y_i})^{-1}(x)$$

which is open in Z. Let $T : Z \to W$ be given by

$$T(z) = \bigcap_{i=1}^n Q_{y_i}(z) \quad \text{for} \quad z \in Z.$$

Now T is convex valued, $H(z) \subseteq T(z)$ for every $z \in Z$ and for $x \in W$ we have

$$\begin{aligned} T^{-1}(x) &= \{z \in Z : x \in T(z)\} = \left\{ z \in Z : \; x \in \bigcap_{i=1}^n Q_{y_i}(z) \right\} \\ &= \bigcap_{i=1}^n \{z \in Z : \; x \in Q_{y_i}(z)\} = \bigcap_{i=1}^n (Q_{y_i})^{-1}(x) \end{aligned}$$

which is open in Z. Finally we note $J(w) \cap T(w) = \emptyset$ for $w \in Z$. To see this let $w \in Z$ and note there exists a $k \in \{1, ..., n\}$ with $y_k \in Z$ and $w \in \Omega_{y_k}$, so

$$T(w) = \bigcap_{i=1}^n Q_{y_i}(w) \subseteq Q_{y_k}(w) = A_{y_k}(w)$$

(since $w \in \Omega_{y_k}$) and thus $J(w) \cap T(w) \subseteq J(w) \cap A_{y_k}(w) = \emptyset$.

Now we will combine the above discussion with Theorem 2.3.

Theorem 2.4. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space. For each $i \in \{1, ..., N\}$ suppose $H_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and for each $x \in X$ assume there exists a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x}$ is convex valued, $(A_{i,x})^{-1}(z)$ is open (in X) for each $z \in X_i$ and $w_i \notin A_{i,x}(w)$ for each $w \in X$. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. Let $i \in \{1, ..., N\}$. From the discussion after Theorem 2.3 (with $Z = X$, $W = X_i$, $H = H_i$, $J =$ Projection of X on X_i, $A_y = A_{i,x}$) there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i is convex valued, $(T_i)^{-1}(z)$ is open for each $z \in X_i$ and $w_i \notin T_i(w)$ for each $w \in X$; here for a $i \in \{1, ..., N\}$ we have that $\{U_{i,x}\}_{x \in X}$ is an open covering of X so there exists a finite set $\{y_{i,1}, ..., y_{i,n_i}\}$ (with $y_{i,j} \in X$ for

$j \in \{1, ..., n_i\}$) and an open covering $\{V_{i,y_{i,j}}\}_{i=1}^{n_i}$ of X and $\Omega_{i,y_{i,j}} = \overline{V_{i,y_{i,j}}} \subseteq U_{i,y_{i,j}}$ for $j \in \{1, ..., n_i\}$ and for fixed $j \in \{1, ..., n_i\}$,

$$Q_{i,y_{i,j}}(z) = \begin{cases} A_{i,y_{i,j}}(z), & z \in \Omega_{i,y_{i,j}} \\ X_i, & z \in X \backslash \Omega_{i,y_{i,j}} \end{cases}$$

and

$$T_i(z) = \bigcap_{j=1}^{n_i} Q_{i,y_{i,j}}(z) \quad \text{for} \quad z \in X.$$

Now we will apply Theorem 2.3 with $F_i = S_i = T_i$ and so there exists a $x \in X$ with $T_j(x) = \emptyset$ for all $j \in \{1, ..., N\}$. Now since $H_j(w) \subseteq T_j(w)$ for $w \in X$ then we have $H_j(x) = \emptyset$ for all $j \in \{1, ..., N\}$. □

Next we will discuss collectively coincidence points motivated in part by [4, 13].

Theorem 2.5. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i with $\prod_{i=1}^{N} X_i$ paracompact and in addition $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, ...N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally suppose for each $x \in X$ there exists a $i \in \{1, ..., N_0\}$ with $T_i(x) \neq \emptyset$ and suppose for each $y \in Y$ there exists a $j \in \{1, ..., N\}$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$, a $j_0 \in \{1, ..., N_0\}$ and a $i_0 \in \{1, ..., N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. Note $A_i = \{x \in X : T_i(x) \neq \emptyset\}, i \in \{1, .., N_0\}$ is an open covering of X so from [8, Lemma 5.1.6, pp301] there exists a covering $\{B_i\}_{i=1}^{N_0}$ of X where B_i is closed in X and $B_i \subset A_i$ for all $i \in \{1, ..., N_0\}$. Also $C_i = \{y \in Y : S_i(y) \neq \emptyset\}, i \in \{1, .., N\}$ is an open covering of Y and from [8, Lemma 5.1.6, pp301] there exists a covering $\{D_i\}_{i=1}^{N}$ of Y where D_i is closed in Y and $D_i \subset C_i$ for all $i \in \{1, ..., N\}$. Now for each $i \in \{1, .., N_0\}$ let $H_i : X \to Y_i$ and $J_i : X \to Y_i$ be given by

$$H_i(x) = \begin{cases} F_i(x), & x \in B_i \\ Y_i, & x \in X \backslash B_i \end{cases} \quad \text{and} \quad J_i(x) = \begin{cases} T_i(x), & x \in B_i \\ Y_i, & x \in X \backslash B_i. \end{cases}$$

Also for each $i \in \{1, .., N\}$ let $M_i : Y \to X_i$ and $L_i : Y \to X_i$ be given by

$$M_i(y) = \begin{cases} G_i(y), & y \in D_i \\ X_i, & y \in Y \backslash D_i \end{cases} \quad \text{and} \quad L_i(y) = \begin{cases} S_i(y), & y \in D_i \\ X_i, & y \in Y \backslash D_i. \end{cases}$$

The reasoning in Theorem 2.1 guarantees that $H_i \in \Phi^{\star}(X, Y_i)$ for $i \in \{1, ..., N_0\}$ and $M_i \in \Phi^{\star}(Y, X_i)$ for $i \in \{1, ..., N\}$.

Now since Y is compact for each $i \in \{1, ..., N\}$ from [2, 5] there exists a continuous (single valued) selection $q_i : Y \to X_i$ of M_i with $q_i(y) \in L_i(y) \subseteq M_i(y)$ for $y \in Y$ and there exists a finite subset R_i of X_i with $q_i(Y) \subseteq co\,(R_i) \equiv Q_i$. Let $Q = \prod_{i=1}^{N} Q_i$ ($\subseteq X$) and note Q is compact. Let $H_i^\star$ (respectively, $J_i^\star$) denote the restriction of H_i (respectively, J_i) to Q. Note for $i \in \{1, ..., N_0\}$ that $H_i^\star \in \Phi^\star(Q, Y_i)$ since for $y \in Y_i$ we have

$$\begin{aligned}(J_i^\star)^{-1}(y) &= \{z \in Q : y \in J_i^\star(z)\} = \{z \in Q : y \in J_i(z)\} \\ &= Q \cap \{z \in X : y \in J_i(z)\} = Q \cap J_i^{-1}(y)\end{aligned}$$

which is open in $Q \cap X = Q$. Now since Q is compact (in particular paracompact) for each $i \in \{1, ..., N_0\}$ from [2, 5] there exists a continuous (single valued) selection $h_i : Q \to Y_i$ of $H_i^\star$ with $h_i(x) \in J_i^\star(x) \subseteq H_i^\star(x)$ for $x \in Q$. Let

$$h(x) = \prod_{i=1}^{N_0} h_i(x) \;\text{ for }\; x \in Q \;\text{ and }\; q(y) = \prod_{i=1}^{N} q_i(y) \;\text{ for }\; y \in Y$$

and note $h : Q \to Y$ and $q : Y \to Q$ are continuous. Consider the continuous map $\theta : Q \to Q$ given by $\theta(x) = q(h(x))$ for $x \in Q$. Note Q is a compact convex subset in a finite dimensional subspace of $E = \prod_{i=1}^{N} E_i$ so Brouwer's fixed point theorem guarantees that there exists a $x \in Q$ with $x = \theta(x) = q(h(x))$. Let $y = h(x)$ so $x = q(y)$. Then since $x \in Q$ we have $y_j = h_j(x) \in J_j^\star(x) \subseteq H_j^\star(x)$ i.e. $y_j \in H_j(x)$ for $j \in \{1, ..., N_0\}$ and $x_i = q_i(y) \in L_i(y) \subseteq M_i(y)$ for $i \in \{1, ..., N\}$. Next since $\{B_i\}_{i=1}^{N_0}$ is a covering of X there exists a $j_0 \in \{1, .., N_0\}$ with $x \in B_{j_0}$ so $y_{j_0} \in H_{j_0}(x) = F_{j_0}(x)$. Finally we note since $\{D_i\}_{i=1}^{N}$ is a covering of Y there exists a $i_0 \in \{1, .., N\}$ with $x \in D_{i_0}$ so $x_{i_0} \in M_{i_0}(y) = G_{i_0}(y)$. □

Theorem 2.6. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N} X_i$ paracompact and in addition $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, ...N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Now suppose either for all $j \in \{1, ..., N_0\}$ we have $y_j \notin F_j(x)$ for each $(x, y) \in X \times Y$ or for all $i \in \{1, ..., N\}$ we have $x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$. Then either there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N_0\}$ or there exists a $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists a $i \in \{1, ..., N_0\}$ with $T_i(x) \neq \emptyset$ and for each $y \in Y$ there exists a $j \in \{1, ..., N\}$ with $S_j(y) \neq \emptyset$. Now Theorem 2.5 guarantees a $x \in X$, a $y \in Y$, a $j_0 \in \{1, ..., N_0\}$ and a $i_0 \in \{1, ..., N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Theorem 2.7. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex compact sets each in a Hausdorff topological vector space. For each $i \in \{1, ..., N_0\}$ and for each $j \in \{1, ..., N\}$ suppose $H_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and for each $x \in X$ assume there exists a map $A_{i,x} : X \to Y_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x}$ is convex valued, $(A_{i,x})^{-1}(z)$ is open (in X) for each $z \in Y_i$ and for each $y \in Y$ assume there exists a map $B_{j,y} : Y \to X_j$ and an open set $O_{j,y}$ containing y with $\Psi_j(z) \subseteq B_{j,y}(z)$ for every $z \in O_{j,y}$, $B_{j,y}$ is convex valued, $(B_{j,y})^{-1}(z)$ is open (in Y) for each $z \in X_j$ and also assume either for all $i \in \{1, ..., N_0\}$ we have $v_i \notin A_{i,x}(u)$ for each $(u, v) \in X \times Y$ or for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. Then either there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, ..., N_0\}$ or there exists a $y \in Y$ with $\Psi_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$.*

Proof. We will modify slightly the ideas in the discussion after Theorem 2.3. Fix $i \in \{1, .., N_0\}$ (respectively, $j \in \{1, .., N\}$). Note $\{U_{i,x}\}_{x \in X}$ is an open covering of X (respectively, $\{O_{j,y}\}_{y \in Y}$ is an open covering of Y) so there exists a finite set $\{x_{i,1}, ..., x_{i,n_i}\}$ (with $x_{i,j} \in X$ for $j \in \{1, ..., n_i\}$) and an open covering $\{V_{i,x_{i,k}}\}_{k=1}^{n_i}$ of X with $x_{i,k} \in V_{i,x_{i,k}}$ and $\Omega_{i,x_{i,k}} = \overline{V_{i,x_{i,k}}} \subseteq U_{i,x_{i,k}}$ (respectively, a finite set $\{y_{j,1}, ..., y_{j,n_j}\}$ and an open covering $\{C_{j,y_{j,l}}\}_{l=1}^{n_j}$ of Y with $y_{j,l} \in C_{j,y_{j,l}}$ and $D_{j,y_{j,l}} = \overline{C_{j,y_{j,l}}} \subseteq O_{j,y_{j,l}}$) and for fixed $k \in \{1, ..., n_i\}$,

$$Q_{i,x_{i,k}}(z) = \begin{cases} A_{i,x_{i,k}}(z), & z \in \Omega_{i,x_{i,k}} \\ Y_i, & z \in X \backslash \Omega_{i,x_{i,k}} \end{cases}$$

and let $T_i : X \to Y_i$ be

$$T_i(z) = \bigcap_{k=1}^{n_i} Q_{i,x_{i,k}}(z),\ z \in X$$

(respectively, for fixed $l \in \{1, ..., n_j\}$,

$$R_{j,y_{j,l}}(z) = \begin{cases} B_{j,y_{j,l}}(z), & z \in D_{j,y_{j,l}} \\ X_j, & z \in Y \backslash D_{j,y_{j,l}} \end{cases}$$

and let $S_j : Y \to X_j$ be

$$S_j(w) = \bigcap_{l=1}^{n_j} R_{j,y_{j,l}}(w),\ w \in Y\ \ ).$$

The argument in the discussion after Theorem 2.3 guarantees that $H_i(z) \subseteq T_i(z)$ for every $z \in X$ (respectively, $\Psi_j(w) \subseteq S_j(w)$ for $w \in Y$), T_i (respectively, S_j) is convex valued and $T_i^{-1}(w)$ is open for each $w \in Y_i$ (respectively, $S_j^{-1}(z)$ is open for each $z \in X_j$).

There are two cases to consider (see the statement of Theorem 2.7). Suppose first that for each $x \in X$ for all $i \in \{1, ..., N_0\}$ we have $v_i \notin A_{i,x}(u)$ for each $(u, v) \in X \times Y$. Then for all $i \in \{1, ..., N_0\}$ we have $v_i \notin T_i(u)$ for each $(u, v) \in X \times Y$; to see this fix

$i \in \{1, ..., N_0\}$ and $(u, v) \in X \times Y$ and note there exists a $x_{i,m}$ ($m \in \{1, ..., n_i\}$) with $u \in \Omega_{i,x_{i,m}}$ so

$$T_i(u) = \bigcap_{k=1}^{n_i} Q_{i,x_{i,k}}(u) \subseteq Q_{i,x_{i,m}}(u) = A_{i,x_{i,m}}(u)$$

and as a result $v_i \notin T_i(u)$ since $v_i \notin A_{i,x_{i,m}}(u)$ and $T_i(u) \subseteq A_{i,x_{i,m}}(u)$. Next consider the case that for each $y \in Y$ for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. As in the first case (with S_j replacing T_i) we obtain for all $j \in \{1, ..., N\}$ we have $u_j \notin S_j(v)$ for each $(u, v) \in X \times Y$.

Now apply Theorem 2.6 (with $F_i = T_i$ and $G_j = S_j$) so either there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N_0\}$ or there exists a $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$, Now since $H_i(z) \subseteq T_i(z)$, $z \in X$ and $\Psi_j(w) \subseteq S_j(w)$, $w \in Y$ the conclusion follows. □

Remark 2.8. In Theorem 2.7 we could replace $\{X_i\}_{i=1}^N$ is a family of compact sets with the assumption that $X \equiv \prod_{i=1}^N X_i$ in paracompact. The proof is as in Theorem 2.7 (see Theorem 2.6) once we describe the map T_i ($i \in \{1, ..., N_0\}$ fixed) as follows: Note $\{U_{i,x}\}_{x \in X}$ is an open covering of X so there exists a locally finite open covering $\{V_{i,x}\}_{x \in X}$ of X with $x \in V_{i,x}$ and $\Omega_{i,x} = \overline{V_{i,x}} \subseteq U_{i,x}$ for each $x \in X$, and for each $x \in X$ let

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in \Omega_{i,x} \\ X_i, & z \in X \backslash \Omega_{i,x} \end{cases}$$

(it is easy to see that $Q_{i,x}$ is convex valued and $H_i(z) \subseteq Q_{i,x}(z)$ for $z \in X$) and let $T_i : X \to Y_i$ be

$$T_i(z) = \bigcap_{x \in X} Q_{i,x}(z) \quad \text{for} \quad z \in X.$$

Note $H_i(z) \subseteq T_i(z)$ for $z \in X$ and T_i is convex valued. It remains to show $T_i^{-1}(y)$ is open for each $y \in Y_i$. Fix $y \in Y_i$ and let $u \in T_i^{-1}(y)$. Since $\{V_{i,x}\}_{x \in X}$ is locally finite there exists an open neighborhood N_u of u such that $\{x \in X : N_u \cap V_{i,x} \neq \emptyset\} = \{x_{i,1}, ..., x_{i,m_i}\}$ (a finite set). Now if $x \notin \{x_{i,1}, ..., x_{i,m_i}\}$ then $\emptyset = V_{i,x} \cap N_u = \Omega_{i,x} \cap N_u$ so $Q_{i,x}(z) = Y_i$ for all $z \in N_u$, and as a result

$$T_i(z) = \bigcap_{x \in X} Q_{i,x}(z) = \bigcap_{j=1}^{m_i} Q_{i,x_{i,j}}(z) \quad \text{for all} \quad z \in N_u.$$

Now $T_i^{-1}(y) = \{z \in X : y \in T_i(z)\}$ but note

$$\{z \in N_u : y \in T_i(z)\} = \left\{ z \in N_u : y \in \bigcap_{j=1}^{m_i} Q_{i,x_{i,j}}(z) \right\} = N_u \cap \left[\cap_{j=1}^{m_i} (Q_{i,x_{i,j}})^{-1}(y) \right]$$

so

$$u \in N_u \cap \left[\cap_{j=1}^{m_i} (Q_{i,x_{i,j}})^{-1}(y) \right] \subseteq T_i^{-1}(y)$$

so $T_i^{-1}(y)$ is open. To finish the proof in Theorem 2.7 there is as before two cases to consider (see the statement of Theorem 2.7). Suppose first that for each $x \in X$ for all $i \in \{1, ..., N_0\}$ we have $v_i \notin A_{i,x}(u)$ for each $(u, v) \in X \times Y$. Fix $i \in \{1, ..., N_0\}$ and $(u, v) \in X \times Y$ and note there exists a $x^\star \in X$ with $u \in \Omega_{i,x^\star}$ so

$$T_i(z) = \bigcap_{x \in X} Q_{i,x}(z) \subseteq Q_{i,x^\star}(u) = A_{i,x^\star}(u)$$

so $v_i \notin T_i(u)$. Thus for all $i \in \{1, ..., N_0\}$ we have $v_i \notin T_i(u)$ for each $(u, v) \in X \times Y$. Next consider the case that for each $y \in Y$ for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. Then as in the proof of Theorem 2.7 we have for all $j \in \{1, ..., N\}$ that $u_j \notin S_j(v)$ for each $(u, v) \in X \times Y$. Now apply Theorem 2.6.

Now we consider a collectively coincidence result between the $\Phi^\star$ and Ad classes.

Theorem 2.9. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i and in addition $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, ...N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally suppose for each $y \in Y$ there exists a $j \in \{1, ..., N\}$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$, a $i_0 \in \{1, ..., N\}$ with $y_j \in F_j(x)$ for all $j \in \{1, ..., N_0\}$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For each $i \in \{1, ..., N\}$ let C_i, D_i, M_i and L_i be as in Theorem 2.5 and note $M_i \in \Phi^\star(Y, X_i)$. Now since Y is compact for each $i \in \{1, ..., N\}$ from [2, 5] there exists a continuous (single valued) selection $q_i : Y \to X_i$ of M_i with $q_i(y) \in L_i(y) \subseteq M_i(y)$ for $y \in Y$ and there exists a finite subset R_i of X_i with $q_i(Y) \subseteq co\,(R_i) \equiv Q_i$. Let $Q = \prod_{i=1}^N Q_i$ $(\subseteq X)$ and note Q is compact. Let $F_i^\star$ denote the restriction of F_i to Q and let $F^\star(x) = \prod_{i=1}^{N_0} F_i^\star(x)$ for $x \in Q$. Since Ad is closed under compositions and also since a finite product of Ad maps is an Ad map [9] then $F^\star \in Ad(Q, Y)$. Let $q(y) = \prod_{i=1}^N q_i(y)$ for $y \in Y$ and note $q : Y \to Q$ since $q_i : Y \to Q_i$ for $i \in \{1, ..., N\}$. Thus $q\,F^\star \in Ad(Q, Q)$ and note Q is a compact convex subset in a finite dimensional subspace of $E = \prod_{i=1}^N E_i$ so Theorem 1.1 guarantees a $x \in Q$ with $x \in q\,(F^\star(x))$. Now let $y \in F^\star(x)$ with $x = q(y)$. Note $y \in F(x)$ so $y_j \in F_j(x)$ for all $j \in \{1, ..., N_0\}$. Also since $x \in Q$ we have $x_i = q_i(y) \in L_i(y) \subseteq M_i(y)$ for $i \in \{1, ..., N\}$. Now since $\{D_i\}_{i=1}^N$ is a covering of Y then there exists a $i_0 \in \{1, ..., N_0\}$ with $y \in D_{i_0}$ so $x_{i_0} \in M_{i_0}(y) = G_{i_0}(y)$. □

Theorem 2.10. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space and in addition $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, ...N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Now suppose either for all $i \in \{1, ..., N\}$ we have*

$x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$ or for each $(x, y) \in X \times Y$ there exists a $j \in \{1, ..., N_0\}$ with $y_j \notin F_j(x)$. Then there exists a $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1, ..., N\}$.

Proof. Suupose the conclusion is false. Then for each $y \in Y$ there exists a $j \in \{1, ..., N\}$ with $S_j(y) \neq \emptyset$. Now Theorem 2.9 guarantees a $x \in X$, a $y \in Y$, a $i_0 \in \{1, ..., N\}$ with $y_j \in F_j(x)$ for all $j \in \{1, ..., N_0\}$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Theorem 2.11. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space and in addition $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, ...N\}$ suppose $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and for each $y \in Y$ assume there exists a map $B_{j,y} : Y \to X_j$ and an open set $O_{j,y}$ containing y with $\Psi_j(z) \subseteq B_{j,y}(z)$ for every $z \in O_{j,y}$, $B_{j,y}$ is convex valued and $(B_{j,y})^{-1}(z)$ is open (in Y) for each $z \in X_j$. Also suppose either for each $y \in Y$ for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$ or for each $(x, y) \in X \times Y$ there exists a $i \in \{1, ..., N_0\}$ with $y_i \notin F_i(x)$. Then there exists a $y \in Y$ with $\Psi_j(y) = \emptyset$ for all $j \in \{1, ..., N\}$.*

Proof. Let $j \in \{1, ..., N\}$ and create $\{y_{j,1}, ..., y_{j,n_j}\}$, $C_{j,y_{j,l}}$, $D_{j,y_{j,l}}$, $R_{j,y_{j,l}}$ ($l \in \{1, ..., n_j\}$) and S_j as in Theorem 2.7. We now claim that for all $j \in \{1, ..., N\}$ we have $u_j \notin S_j(v)$ for each $(u, v) \in X \times Y$ if in the statement of Theorem 2.11 we have for each $y \in Y$ for all $j \in \{1, ..., N\}$ we have $u_j \notin B_{j,y}(v)$ for each $(u, v) \in X \times Y$. Note for a fixed $j \in \{1, ..., N\}$ and $(u, v) \times X \times Y$ note there exists a $y_{j,m}$ ($m \in \{1, ..., n_j\}$) with $v \in D_{j,y_{j,m}}$ so

$$S_j(v) = \bigcap_{l=1}^{n_j} R_{j,y_{j,l}}(v) \subseteq R_{j,y_{j,m}}(v) = B_{j,y_{j,m}}(v)$$

so $u_j \notin S_j(v)$. Thus our claim is true. Now apply Theorem 2.10 (with $G_j = S_j$) so there exists a $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1, ..., N\}$. The result follows since $\Psi_j(w) \subseteq S_j(w)$, $w \in Y$. □

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