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EXISTENCE OF EXPONENTIALLY GROWING SOLUTIONS AND ASYMPTOTIC STABILITY RESULTS FOR HILFER FRACTIONAL (P_1,P_2...P_n) LAPLACIAN INITIAL VALUE PROBLEM WITH WEIGHTED DUFFING OSCILLATOR

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ABSTRACT. In this paper, we give existence of exponentially growing solutions and asymptotic stability results of the initial value problem with nonautonomous and variable coefficients weighted Duffing equation involving the (p_1,p_2...p_n)-Laplacian operator.

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1. Introduction

In this paper, we study the following initial value problem with weighted Duffing equation with variable coefficients

(1.1)
$$\begin{cases} -\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(a.u\right)'\right)\right)(t) + D_{0^+}^{\beta,\omega,\sigma}\left(\delta.u\right)(t) + p(t,u\left(t\right)) + q\left(t\right)f(t,u(t)) = 0, \ t>0, \\ u(0)=0, \end{cases}$$

where $\phi_{p_i}\left(x\right) = \left|x\right|^{p_i-2}.x$, $p_i>1$, for $i\in\{1,..,N\}$, $p\left(t,u\right)=\sum_{n=1}^{n=m}\eta_n\left(t\right)u^n$, $N,m\in\mathbb{N}^*$ with $\eta_n:\mathbb{R}^+\to\mathbb{R}^+$ is a function, $D_{0^+}^{\mu,\omega,\sigma}$is the $\sigma-$Hilfer fractional derivative of order $\mu\in\{\alpha,\beta\}$ and type $0\leq\omega\leq 1$ with $0<\beta<\alpha<1$.

The study of solutions of the Duffing equations has been of great interest recently. The Duffing equation named after George Duffing is a nonlinear second order differential equation $\ddot{x}+\delta\dot{x}+\alpha x+\beta x^3=\gamma\cos\left(\omega t\right)$ where the (unknown) function $x(t)$ is the displacement at time t, the first derivative $\dot{x}$ is the velocity, and the second time $\ddot{x}$ derivative is acceleration. It used to model certain damped and driven oscillators with a more complicated potential than in simple harmonic motion (which corresponds to the case $\delta=\beta=0$).

The Duffing equations present in the frequency response the jump resonance phenomenon that is a sort of frequency hysteresis behaviour, where δ controls the

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amount of damping, α controls the linear stiffness, β controls the amount of non-linearity in the restoring force, γ is the amplitude of the periodic driving force and ω is the angular frequency of the periodic driving force.
In most physical oscillation systems, the amplitude of excitation (force or moment) usually varies over time, and some external and internal excitation impulses can occur. For details, see [4, 5, 9, 10, 11, 12, 15], and references therein.

In [4], fractional Duffing's equations were discussed by using homotopy analysis method. In [15], asymptotic behavior of Solutions were studied by S. T. Wu. In [13], Yao and Zhang give existence results for p-Laplacian neutral damped Duffing equation.

Motivated by the cited papers, in the present article, we discuss the existence and asymptotic stability of positive and exponentially growing solutions for the problem (1.1).

Throughout the article, we assume that $\sigma \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is increasing with $\sigma(0) = 0$ and $\sigma'(t) \neq 0$ for all $t \geq 0$, $p : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a two variable polynomial function and $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and there exists $k \in \mathbb{N}^*$, $\lambda \geq 0, \epsilon > 0$ and $r > 0$ such that for all $x \in [0, r]$

$$0 < f(t, e^{kt}x) \leq \lambda.x + \epsilon,\ t \geq 0 \tag{1.2}$$

$\eta_n : \mathbb{R}^+ \to \mathbb{R}^+$ is the measurable function for all $n \in \{1, 2...m\}$, a, h are absolutely continuous on $\mathbb{R}^+$ with $a(x) \geq 1\ \forall x \geq 0$, δ, $q \in C(\mathbb{R}^+)$ do not vanish identically on any subinterval of $\mathbb{R}^+$ and there exists $k > 0$ such that

$$\frac{1}{h} \in L^1_{loc}(\mathbb{R}^+, \mathbb{R}^+), \lim_{x \to +\infty} e^{-kx} \int_0^x \frac{ds}{h(s)} = 0,\ \hat{\delta}_k \in L^{\alpha-\beta}_\sigma(\mathbb{R}^+, \mathbb{R}^+) \text{ and } \bar{\eta}_{n,k}, q \in L^\alpha_\sigma(\mathbb{R}^+, \mathbb{R}^+) \tag{1.3}$$

where $\bar{\eta}_{n,k}(s) = e^{nks}\eta_n(s)$ and $\hat{\delta}_k(s) = e^{ks}\delta(s)$, and for $\mu > 0$

$$L^\mu_\sigma(\mathbb{R}^+, \mathbb{R}^+) = \left\{ u : \mathbb{R}^+ \to \mathbb{R}^+, \sup_{x \geq 0} \int_0^x \sigma'(s)(\sigma(x) - \sigma(s))^{\mu-1} u(s)\, ds < \infty \right\}.$$

The rest of the paper is organized as follows. In Section 2, some preliminary materials to be used later are stated. In Section 3, we present and prove our main results consisting of the existence of positive exponentially growing solution and asymptotic stability results of the initial value problem (1.1). Finally, examples are given to illustrate our results.

2. **Preliminaries**

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norm denoted $\|.\|$. A nonempty closed convex subset P of E is said to be a cone if $P \cap (-P) = 0$ and $(tP) \subset P$ for all $t \geq 0$.

It is well known that a cone P induces a partial order in the Banach space E. We write for all $x; y$:$\in E$; $x \leq y$ if $y - x \in P$.

The mapping $L : E \to E$ is said to be positive in P if $L(P) \subset P$, and compact if it is continuous and $L(B)$ is relatively compact in E for all bounded subset B of E.

Definition 2.1. [14] Let $a \in \mathbb{R}^+$ and $\alpha > 0$. Also, let $\sigma(x)$ be an increasing and positive function having a continuous derivative $\sigma'(x)$ on $(a, +\infty)$. Then the left-sided fractional integral of a function u with respect to another function σ on $\mathbb{R}^+$ is defined by

$$I_{a^+}^{\alpha,\sigma} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \sigma'(t) (\sigma(x) - \sigma(t))^{\alpha-1} u(t) dt.$$

In the case $\alpha = 0$, this integral is interpreted as the identity operator $I_{a^+}^{0,\sigma} u = u$.

Definition 2.2. [14] Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, and u, $\sigma \in C^n(\mathbb{R}^+, \mathbb{R})$ two functions such that σ is increasing and $\sigma'(t) \neq 0$, for all $t \in \mathbb{R}^+$. The σ-Hilfer fractional derivative $D_{a^+}^{\alpha,\omega,\sigma}$ of u of order $n-1 < \alpha < n$ and type $0 \leq \omega \leq 1$ is defined by

$$D_{a^+}^{\alpha,\omega,\sigma} u(x) = I_{a^+}^{\omega(n-\alpha),\sigma} \left(\frac{1}{\sigma'(x)} \frac{\partial}{\partial x} \right)^n I_{a^+}^{(1-\omega)(n-\alpha),\sigma} u(x).$$

Let's also recall the following important result :

Theorem 2.3. [14] *If $u \in C^n(\mathbb{R}^+)$, $n-1 < \beta < \alpha < n$, $0 \leq \omega \leq 1$ and*

$$\xi = \alpha + \omega(n - \alpha),$$

then

$$I_{a^+}^{\alpha,\sigma} . D_{a^+}^{\alpha,\omega,\sigma} u(x) = u(x) - \sum_{k=1}^{n} \frac{(\sigma(x) - \sigma(a))^{\xi-k}}{\Gamma(\xi - k + 1)} \left(\frac{1}{\sigma'(x)} \frac{\partial}{\partial x} \right)^{n-k} I_{a^+}^{(1-\omega)(n-\alpha),\sigma} u(a^+).$$

Moreover, $I_{a^+}^{\beta,\sigma} I_{a^+}^{\alpha-\beta,\sigma}(u) = I_{a^+}^{\alpha-\beta,\sigma} I_{a^+}^{\beta,\sigma}(u) = I_{a^+}^{\alpha,\sigma}(u)$ and ${}^H D_{a^+}^{\alpha,\omega,\sigma} I_{a^+}^{\alpha,\sigma}(u) = u$.

Remark 2.4. In this paper, we assume that $\sigma(x)$ is increasing and positive with $\sigma(0) = 0$, having a continuous derivative $\sigma'(x)$ on $\mathbb{R}^+$ and $\sigma'(x) \neq 0$, for all $x \in \mathbb{R}^+$. If $\alpha \in (0,1)$, then $n = 1$ and for $x > 0$

$$I_{0^+}^{\alpha,\sigma} . D_{0^+}^{\alpha,\omega,\sigma} u(x) = u(x) - \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0^+}^{(1-\omega)(1-\alpha),\sigma} u \right)(0^+).$$

Moreover, if $u : \mathbb{R}^+ \to \mathbb{R}$ is continuous, then

$$\lim_{x \to 0^+} \left(I_{0^+}^{(1-\omega)(1-\alpha),\sigma} u \right)(x) = 0$$

and so $I_{0^+}^{\alpha,\sigma} . {}^H D_{0^+}^{\alpha,\omega,\sigma} u(x) = u(x)$.

Definition 2.5. A positive solution u of problem (1.1) is said to be exponentially growing solution, if there exists the constants $c_1, c_2 > 0$ and a positive and increasing function v such that

$$u\left(x\right) \geq c_1 \exp(v\left(x\right)) \text{ for all } x \geq c_2.$$

In what follows, we use of the following Schauder Fixed-Point Theorem :

Theorem 2.6. [7] *Let E be a Banach space, let C be a nonempty bounded convex and closed subset of E, and let $T : C \to C$ be a compact and continuous map. Then T has at least one fixed point in C.*

We will use the following lemma concerning existence of fixed point for a compact map $T : P \cap \bar{B}\left(0, r\right) \to P$, where $r > 0$ and P is a cone in a Banach space F.

Lemma 2.7. [3] *If $\|Tu\| < \|u\|$ for all $u \in P \cap \partial B\left(0, r\right)$, then T asmits a fixed point u in $P \cap \bar{B}\left(0, r\right)$.*

Definition 2.8. [3] Solutions of ivp (1.1) are locally asymptotically stable in a cone K of a Banach space E, if there exits a nonempty bounded convex and open subset Ω of E such that, for any solutions $u, v \in K \cap \Omega$ of ivp (1.1), we can write

$$\lim_{x\to+\infty} \left(u(x) - v(x)\right) = 0 \tag{2.1}$$

uniformly with respect to $K\cap\Omega$. Moreover, if (2.1) is verified for all solutions $u, v \in K$, (1.1) is said to be asymptotically stable.

For $k \in \mathbb{N}^*$ given in (1.3), let E be a real Banach space defined as

$$E = \left\{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{|t|\to\infty} e^{-kt}u(t) = 0\right\}$$

equipped with the norm $\|\cdot\|$, where for $u \in E$, $\|u\| = \sup_{t\in\mathbb{R}^+} \left(e^{-kt} \left|u(t)\right|\right)$, and let

$$K = \left\{u \in E : u\left(0\right) = 0 \text{ and } u(t) \geq 0 \text{ for all } t \in \mathbb{R}^+\right\}$$

be the cone in E.

Lemma 2.9. [3] *A non empty subset M of E is relatively compact if the following conditions hold :*

1. M is bounded in E,
2. The set $\left\{e^{-kt}u, u \in M\right\}$ is locally equicontinuous on $[0, +\infty)$, and
3. The set $\left\{e^{-kt}u, u \in M\right\}$ is equiconvergent, that is, for any given $\epsilon > 0$, there exists $A > 0$ such that $\left|e^{-kx}u(x) - \lim_{y\to+\infty} e^{-ky}u(y)\right| < \epsilon$, for any $x > A$, $u \in M$.

3. Main results

We consider the operator $T : E \to C^1(\mathbb{R}^+)$ defined by

$$Tu(x) = \frac{1}{a(x)} \int_0^x \frac{1}{h(t)} \psi \left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u)) \right)(t)dt$$

where $\psi = \phi^{-1} : \mathbb{R} \to \mathbb{R}$ is the inverse function of sum of p$_i$-Laplacian operators $\phi = \sum_{i=1}^{i=N} \phi_{p_i}$, with $\phi_{p_i}(x) = |x|^{p_i-2}.x$ and ψ_{p_i} is the inverse function of ϕ_{p_i}.

Remark 3.1. Let $p^- = \min\{p_1, p_2...p_N\}$ and $p^+ = \max\{p_1, p_2...p_N\}$. For all $x \geq 0$, $i \in \{1, 2...N\}$

$$\phi_{p_i}(x) \leq \phi(x) \leq N.\phi^+(x)$$

where

$$\phi^+(x) = \begin{cases} \phi_{p^+}(x) & if\ x \geq 1 \\ \phi_{p^-}(x) & if\ x \leq 1 \end{cases}$$

and so, we conclude that

$$\psi^+\left(\frac{x}{N}\right) \leq \psi(x) \leq \psi_{p_i}(x) \tag{3.1}$$

where

$$\psi^+\left(\frac{x}{N}\right) = \begin{cases} \psi_{p^+}\left(\frac{x}{N}\right) & if\ x \geq 1 \\ \psi_{p^-}\left(\frac{x}{N}\right) & if\ x \leq 1. \end{cases}$$

For $p \in \{p_1, p_2...p_N\}$, let

$$\Lambda_r(p) = \sup_{x\geq 0} \psi_p \left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} \bar{\eta}_{n,k}.r^{n-1} + \left(\lambda + \frac{\epsilon}{r}\right).q \right) \right),$$

where $r > 0$ is the constant given in (1.2). Hypothesis (1.3) gives that that $\Lambda_r < \infty$.

Lemma 3.2. *$u \in C^1(\mathbb{R}^+)$ is solution of ivp (1.1) if and only if u is fixed point of T (i.e $Tu = u$).*

Proof. Let $u \in E$ be a fixed point of T, then $u \in C^1(\mathbb{R}^+)$, $u(0) = 0$ and

$$\phi\left(h.(a.u)'\right)(t) = I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(t,u) + q.f(.,u))(t)$$

then it follows from Theorem (2.3) that

$$D_{0^+}^{\alpha,\omega,\sigma}\phi\left(h.(a.u)'\right)(t) = D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)(t) + p(t,u) + q.f(.,u)(t) \tag{3.2}$$

The continuity of the function $t \mapsto \delta(t).u(t)$ gives that

$$\lim_{t\to 0^+} \left(I_{0^+}^{(1-\omega)(1-\beta),\sigma}(\delta.u) \right)(t) = 0$$

and so

$$\begin{aligned} D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) &= D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}\left(I_{0^+}^{\beta,\sigma} D_{0^+}^{\beta,\omega,\sigma}(\delta.u) + \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)}\left(I_{0^+}^{(1-\omega)(1-\alpha),\sigma}(\delta.u)\right)(0^+)\right) \\ &= D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}\left(I_{0^+}^{\beta,\sigma} D_{0^+}^{\beta,\omega,\sigma}(\delta.u)\right) = D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha,\sigma}\left(D_{0^+}^{\beta,\omega,\sigma}(\delta.u)\right) \\ &= D_{0^+}^{\beta,\omega,\sigma}(\delta.u) \end{aligned}$$

then equation (3.2) means that u is solution of ivp (1.1).
Conversely, assume that $u \in C^1(\mathbb{R}^+)$ is solution of the ivp (1.1) then $u(0) = 0$ and

$$D_{0^+}^{\alpha,\omega,\sigma}\left(\phi\left(h.(a.u)'\right)\right)(t) = D_{0^+}^{\beta,\omega,\sigma}(\delta.u)(t) + p(t, u(t)) + q(t) f(t, u(t)) = 0, \ t > 0.$$

By using the Theorem (2.3) and with the fact that $n = 1$, $I_{0^+}^{(1-\omega)(1-\alpha),\sigma}\left(\phi\left(h.(a.u)'\right)\right)(0^+) = 0$ and $I_{0^+}^{(1-\omega)(1-\beta),\sigma}(\delta.u)(0^+) = 0$ we obtain

$$\begin{aligned} I_{0^+}^{\alpha,\sigma}.D_{0^+}^{\alpha,\omega,\sigma}\left(\phi\left(h.(a.u)'\right)\right)(t) &= \phi(h.u')(t) = I_{0^+}^{\alpha,\sigma} D_{0^+}^{\beta,\omega,\sigma}(\delta.u)(t) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))(t) \\ &= I_{0^+}^{\alpha-\beta,\sigma} I_{0^+}^{\beta,\sigma} D_{0^+}^{\beta,\omega,\sigma}(\delta.u)(t) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))(t) \\ &= I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)(t) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))(t) \end{aligned}$$

leading to

$$(a.u)'(t) = \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)(t) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t).$$

Therefore, the solution of ivp (1.1) is

$$u(x) = \frac{1}{a(x)}\int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt,$$

which completes the proof. □ □

In what follows, the restriction mapping $T \backslash K \cap \bar{B}(0,r)$ is also denoted by T.

Lemma 3.3. *Assume that Hypothesis (1.2) and (1.3) hold true.*
Then the operator $T : K \cap \bar{B}(0,r) \to K$ is compact, where r is the constant given in (1.2).

Proof. We show that $Au = a.Tu$ is compact in $K \cap \bar{B}(0,r)$.
Let $M_r = A(\Omega_r)$, where $\Omega_r = K \cap \bar{B}(0,r)$.
It's clear that the continuity of the functions f and p and the hypothesis (1.3) make the operator $A : \Omega_r \to E$ continuous.

Now, we show that M_r is relatively compact.
In first, we show that the set $M_r = A(\Omega_r)$ is a subset of.E. Let $u \in \Omega_r$. For for $x > 0$

$$e^{-kx}Au(x) = e^{-kx}\int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt.$$

Hypothesis (1.2) and inequality (3.1) of Remark (3.1) lead that for all $i \in \{1, 2...N\}$

$$e^{-kx}Au(x) \le e^{-kx}\int_0^x \frac{1}{h(t)}\psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{u}_k\right) + I_{0^+}^{\alpha,\sigma}\left[\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}\left(\tilde{u}_k\right)^n + \lambda q\tilde{u}_k + \epsilon q\right)\right]\right)(t)dt$$

this is for all $x \ge 0$, where $\tilde{u}_k\left(s\right) = e^{-ks}u(s) \in [0, r]$. Then

$$\|Au\| \le R = \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).\sup_{x\ge 0}\left\{e^{-kx}\int_0^x \frac{dt}{h(t)}\right\}$$

proving the boundeness of M_r.

Let $b_1 \le t_1 < t_2 \le b_2$, $b_1, b_2 \in \mathbb{R}^+$ and set $w(t) = e^{-kt}$. For all $u \in \Omega_r$ we have

$$\begin{aligned}
|w(t_2)Au\left(t_2\right) - w(t_1)Au\left(t_1\right)| &\le w(t_2)\left|Au\left(t_2\right) - Au\left(t_1\right)\right| \\
&\quad + Au\left(t_1\right)\left|w(t_2) - w(t_1)\right| \\
&\le w(t_2)\left|Au\left(t_2\right) - Au\left(t_1\right)\right| \\
&\quad + e^{kb_2}R\left|w(t_2) - w(t_1)\right|
\end{aligned}$$

with

$$\begin{aligned}
w(t_2)\left|Au\left(t_2\right) - Au\left(t_1\right)\right| &= w(t_2)\int_{t_1}^{t_2}\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\delta.u\right) + I_{0^+}^{\alpha,\sigma}\left(p(t,u) + q.f(.,u)\right)\right)(t)dt \\
&\le \int_{t_1}^{t_2}\frac{1}{h(t)}\psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_{k,p_i}.\tilde{u}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k,p_i}.\left(\tilde{u}_k\right)^n + \check{q}_{k,p_i}\left(\lambda\tilde{u}_k + \epsilon\right)\right)\right) \\
&\le \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).\int_{t_1}^{t_2}\frac{dt}{h(t)}.
\end{aligned}$$

Because that w and $x \to \int_0^x \frac{dt}{h(t)}$ are uniformly continuous on compact intervals, the above estimates prove that $\left\{e^{-kt}u, u \in M_r\right\}$ is locally equicontinuous on $[0, +\infty)$.

Now, let $u \in \Omega_r$, $x \in \mathbb{R}^+$. For $y > x$

$$\begin{aligned}
\left|e^{-kx}A(u)\left(x\right) - e^{-ky}A(u)\left(y\right)\right| &\le e^{-ky}\left|Au\left(x\right) - Au\left(y\right)\right| \\
&\quad + Au\left(x\right)\left|e^{-kx} - e^{-ky}\right| \\
&\le \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).\int_x^y \frac{dt}{h(t)} \\
&\quad + \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).e^{-kx}\int_0^x \frac{dt}{h(t)}.\left|1 - e^{-k(y-x)}\right|
\end{aligned}$$

then

$$\begin{aligned}
\left|e^{-kt}A(u)\left(x\right) - \lim_{y\to+\infty}e^{-ky}A(u)\left(y\right)\right| &\le \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).\int_x^{+\infty}\frac{dt}{h(t)} \\
&\quad + \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).e^{-kx}\int_0^x \frac{dt}{h(t)}
\end{aligned}$$

with

$$\lim_{x\to+\infty}\int_x^{+\infty}\frac{dt}{h(t)} = \lim_{x\to+\infty}e^{-kx}\int_0^x \frac{dt}{h(t)} = 0,$$

so, the equiconvergence of $\left\{e^{-kt}u, u\in M_r\right\}$ holds. By Lemma (2.9), we deduce that M_r is relatively compact and so, T is compact.
Finally, we have from hypothesis (1.2) and (1.3) that for $u\in\Omega_r$ the functions $\frac{1}{a}, q.f(.,u), \eta.p(u)$ and $\delta.u$ are positive, and so $T\left(K\cap\bar{B}\left(0,r\right)\right)\subset K$.
Proving our claim. $\square$

Remark 3.4. For $u\in K$ and $x>0$

$$\begin{aligned}
I_{0^+}^{\alpha,\sigma}\left(u\right)\left(x\right) &= \frac{1}{\Gamma\left(\alpha\right)}\int_0^x\sigma'\left(t\right)\left(\sigma\left(x\right)-\sigma\left(t\right)\right)^{\alpha-1}\left(u\right)\left(t\right)dt\\
&= \frac{\Gamma\left(\alpha-\beta\right)}{\Gamma\left(\alpha\right).\Gamma\left(\alpha-\beta\right)}\int_0^x\sigma'\left(t\right)\left(\sigma\left(x\right)-\sigma\left(t\right)\right)^{\alpha-\beta-1}\left(\sigma\left(x\right)-\sigma\left(t\right)\right)^{\beta}\left(u\right)\left(t\right)dt\\
&\leq \nu\left(x\right).I_{0^+}^{\alpha-\beta,\sigma}\left(u\right)\left(x\right)
\end{aligned}$$

where

$$\nu\left(x\right)=\frac{\Gamma\left(\alpha-\beta\right)}{\Gamma\left(\alpha\right)}\left(\sigma\left(x\right)\right)^{\beta}$$

and so

$$\begin{aligned}
Tu(x) &= \int_0^x\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\delta.u\right)+I_{0^+}^{\alpha,\sigma}\left(p(.,u)+q.f(.,u)\right)\right)(t)dt\\
&\leq \int_0^x\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\delta.u\right)+\nu\left(t\right)I_{0^+}^{\alpha-\beta,\sigma}\left(p(.,u)+q.f(.,u)\right)\right)(t)dt.
\end{aligned}$$

$\square$

Set $H\left(x\right)=\frac{e^{-kx}}{h(x)\psi(\Gamma(\alpha-\beta))}\psi\left(\int_A^x\sigma'\left(t\right)\left(\sigma\left(x\right)-\sigma\left(t\right)\right)^{\alpha-\beta-1}\delta\left(t\right)e^{kt}dt\right)$ and consider the condition

$$\text{(3.3)}\qquad\left\{\begin{array}{l}\text{there exists } i\in\{1,2...N\}\text{ such that}\\ \Lambda_r\left(p_i\right)\leq r^{\frac{p_i-2}{p_i-1}}\left(\sup\limits_{x\geq0}\left\{e^{-kx}\int_0^x\frac{dt}{h(t)}\right\}\right)^{-1},\end{array}\right.$$

where $r>0$ is the constant given in (1.2).

Theorem 3.5. *Assume that Hypothesis (1.2), (1.3) and (3.3) hold true. Then ivp (1.1) admits at least one positive solution. Moreover, if the function a is bounded, then the solutions of ivp (1.1) grow exponentially.*

Proof. Let $u\in K\cap\bar{B}\left(0,r\right)$, for $x>0$

$$\begin{aligned}
e^{-kx}Tu(x) &= \frac{e^{-kx}}{a\left(x\right)}\int_0^x\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\delta.u\right)+I_{0^+}^{\alpha,\sigma}\left(p(.,u)+q.f(.,u)\right)\right)(t)dt\\
&\leq e^{-kx}\int_0^x\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\delta.u\right)+I_{0^+}^{\alpha,\sigma}\left(p(.,u)+q.f(.,u)\right)\right)(t)dt\\
&\leq \psi_{p_i}\left(r\right)\Lambda_r\left(p_i\right).\sup_{x\geq0}\left\{e^{-kx}\int_0^x\frac{dt}{h(t)}\right\}\leq r
\end{aligned}$$

then

$$\|Tu\| \leq \|u\| .$$

We have that the compact operator T maps the closed bounded convex set $K \cap \bar{B}(0,r)$ into itself. So, Schauder's fixed point theorem guarantees existence of a fixed point u of T, which is a positive solution of ivp (1.1). Moreover, u is nontrivial since $f(.,0)$ does not vanish identically on any subinterval of $\mathbb{R}^+$.

Now, we have to prove that the solution u grows exponentially at ∞. As

$$\lim_{x\to+\infty} e^{-kx}u(x) = 0,$$

there exists $A, \theta > 0$, such that the function $e^{-kx}u(x)$ is nonincreasing on $[A, +\infty)$ and such that for $x > A, u(x) \geq \theta$. Then

$$\begin{aligned}
(a.u)'(x) &= (a.Tu)'(x) = \frac{1}{h(x)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(x) \\
&\geq \frac{1}{h(x)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)\right)(x) \\
&\geq \frac{1}{h(x)\psi(\Gamma(\alpha-\beta))}\psi\left(\int_A^x \sigma'(t)(\sigma(x)-\sigma(t))^{\alpha-\beta-1}\delta(t)u(t)dt\right) \\
&\geq \theta.H(x)u(x) \geq \frac{\theta.H(x)}{a(x)}(a.u)(x)
\end{aligned}$$

where

$$H(x) = \frac{e^{-kx}}{h(x)\psi(\Gamma(\alpha-\beta))}\psi\left(\int_A^x \sigma'(t)(\sigma(x)-\sigma(t))^{\alpha-\beta-1}\delta(t)e^{kt}dt\right)$$

It follows that there exists $c > 0$ such that for all $x \geq A$,

$$(a.u)(x) \geq c\exp\left(\int_A^x \frac{\theta.H(t)}{a(t)}dt\right)$$

leading to

$$u(x) \geq c_0 \exp\left(\int_A^x \theta_0.H(t)\,dt\right),$$

where

$$c_0 = \frac{c}{\sup\limits_{x\geq 0}\{a(x)\}} \text{ and } \theta_0 = \frac{\theta}{\sup\limits_{x\geq 0}\{a(x)\}}$$

which means that u is an exponentially growing solution. □ □

We consider the following hypothesis

(3.4)

$$\begin{cases}
\text{There exist } g_0 : \mathbb{R}^+ \to \mathbb{R}^+,\ r > 0 \text{ and } i \in \{1, 2...N\} \text{ such that for all } x \in (0,r] \text{ and all } t > 0 \\
p(t, e^{kt}x) + q.f(t, e^{kt}x) \leq g_0(t).x \text{ and there exists } i \in \{1, 2...N\} \text{ such that} \\
(r)^{\frac{2-p_i}{p_i-1}} \sup\limits_{x\geq 0}\left\{e^{-kx}\int_0^x \frac{1}{h(t)}\psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + \nu(t)\, I_{0^+}^{\alpha-\beta,\sigma}(g_0)\right)(t)dt\right\} \leq 1.
\end{cases}$$

Theorem 3.6. *If Hypothesis (1.2), (1.3) and (3.4) hold true, then ivp (1.1) admits at least one positive solution.*

Proof. For $u \in K \cap \bar{B}(0,r)$

$$\begin{aligned} Tu(x) &\leq \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + \nu(t)\, I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \end{aligned}$$

and then

$$\begin{aligned} e^{-kx}Tu(x) &\leq e^{-kx} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + \nu(t)\, I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq e^{-kx} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{u}_k\right) + \nu(t)\, I_{0^+}^{\alpha-\beta,\sigma}(g_0.\tilde{u}_k)\right)(t)dt \\ &\leq \psi_{p_i}(r)\, e^{-kx} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + \nu(t)\, I_{0^+}^{\alpha-\beta,\sigma}(g_0)\right)(t)dt \\ &\leq (r)^{\frac{1}{p_i - 1}} \sup_{x\geq 0}\left\{e^{-kx} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + \nu(t)\, I_{0^+}^{\alpha-\beta,\sigma}(g_0)\right)(t)dt\right\} \leq r \end{aligned}$$

So, Schauder's fixed point theorem guarantees existence of a fixed point u of T, which is a positive solution of ivp (1.1). □ □

Now, we consider the following hypothesis

(3.5)
$$\begin{cases} \text{There exist a function } \rho : \mathbb{R}^+ \to \mathbb{R}^+, \text{ such that for all } t > 0, \text{ if } x, y \in [0,r] \text{ then} \\ \left|f(t, e^{kt}x) - f(t, e^{kt}y)\right| \leq \rho(t) \, . \, |x - y| \text{ and such that} \\ \lim\limits_{x\to+\infty} \frac{1}{a(x)} \int_0^x \frac{A_r(t)}{h(t)} \left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k} + q.\rho\right)\right)(t)ds = 0, \end{cases}$$

where r is the constant given in hypothesis (1.2),

$$A_r(t) = \frac{1}{\sum_{i=1}^{i=N} (p_i - 1)\left(\psi\left(N_r^{(i)}(t)\right)\right)^{(p_i-2)}},$$

with

$$N_r^{(i)}(t) = \begin{cases} I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.r\right)(t) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} \bar{\eta}_{n,k}.r^n + (\lambda.r + \epsilon).q\right)(t), & \text{if } 1 < p_i \leq 2 \\ I_{0^+}^{\alpha,\sigma}(q) \, . \min\left\{f(t,x), (t,x) \in \mathbb{R}^+ \times \left[0, e^{kt}r\right]\right\} & \text{if } \quad p_i > 2 \end{cases}$$

Theorem 3.7. *Assume that Hypothesis (1.2), (1.3) and (3.5)hold true and one of the conditions (3.3) or (3.4) is satisfied.*
Then the positive solutions of problem (1.1) are locally asymptotically stable in K.

Proof. We have from theorems 3.5 and 3.6 that T admits a fixed point in $K\cap\bar{B}(0,r)$, which is a solution of ivp (1.1) in $\bar{B}(0,r)$.

Now, we show that the solutions are locally asymptotically stable in K. We assume that $u,v\in K\cap B(0,r)$ are solutions of ivp (1.1) such that $u\neq v$. For $x>0$, let

$$w(x)=u(x)-v(x).$$

We have

$$w(x)=u(x)-v(x)=Tu(x)-Tv(x)=\frac{1}{a(x)}\int_0^x\frac{1}{h(t)}\left(\psi(Bu)-\psi(Bv)\right)(t)\,dt$$

where

$$Bu(t)=I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.(u)^n+q.f(t,u)\right).$$

Moreover, there exists a function $\chi\in[\min(Bu,Bv),\max(Bu,Bv)]$ such that

$$\begin{aligned}|\psi(Bu)-\psi(Bv)|(x) &= A(\chi(t))\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.(u-v))\right)(t)\\ &\quad+A(\chi(t))\left(I_{0^+}^{\alpha,\sigma}(p(.,u)-p(.,v))\right)(t)\\ &\quad+A(\chi(t))\,I_{0^+}^{\alpha,\sigma}(q.[f(.,u)-f(.,v)])(t)\end{aligned}$$

where

$$A(\chi(t))=\frac{1}{\sum_{i=1}^{i=N}(p_i-1)(\psi(\chi(t)))^{(p_i-2)}}.$$

For $w\in\{u,v\}$ and $t\in[0,x]$

$$\begin{aligned}Bw(t) &= I_{0^+}^{\alpha-\beta,\sigma}(\delta.w)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.(w)^n+q.f(t,w)\right)\\ &\le I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{w}\right)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.(\tilde{w})^n+\lambda q.\tilde{w}+q\epsilon\right)\\ &\le I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.r\right)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.(r)^n+q(\lambda.r+\epsilon)\right)\end{aligned}$$

and

$$\begin{aligned}Bw(t) &\ge I_{0^+}^{\alpha,\sigma}(q.f(t,w))\\ &\ge I_{0^+}^{\alpha,\sigma}(q).\min\left\{f(t,x),(t,x)\in\mathbb{R}^+\times\left[0,e^{kt}.r\right]\right\},\end{aligned}$$

where $\tilde{w}(s) = e^{-ks}w(s) \in [0, r]$.
Then inequality of hypothesis (3.5) gives for $t > 0$

$$\begin{aligned}
|\psi(Bu) - \psi(Bv)|(t) &\leq A(\chi(t))\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.|u-v|) + I_{0^+}^{\alpha,\sigma}(|p(.,u) - p(.,u)|)\right)(t) \\
&\quad +A(\chi(t))\, I_{0^+}^{\alpha,\sigma}(q.|f(.,u) - f(.,v)|)(t) \\
&\leq A_r(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.|\tilde{u}-\tilde{v}|\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k}(r)^{n-1}|\tilde{u}-\tilde{v}|\right)\right)(t) \\
&\quad +A_r(t)\, I_{0^+}^{\alpha,\sigma}(q.\rho.|\tilde{u}-\tilde{v}|)(t) \\
&\leq rA_r(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k}.r^{n-1} + q.\rho\right)\right)(t) \\
&\leq \max\{1, r^m\}.A_r(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k} + q.\rho\right)\right)(t)
\end{aligned}$$

where

$$A_r(t) = \frac{1}{\sum_{i=1}^{i=N}(p_i - 1)\left(\psi\left(N_r^{(i)}(t)\right)\right)^{(p_i-2)}}.$$

Therefore, for $x > 0$ we have

$$|w(x)| \leq \frac{\max\{1, r^m\}}{a(x)}.\int_0^x \frac{1}{h(t)} A_1(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k} + q.\rho\right)\right)(x)$$

and from (3.5) we obtain $\lim_{x\to+\infty} w(x) = 0$ and we conclude that $\lim_{x\to\infty}|(u-v)(x)| = 0$. □ □

Assume that Hypothesis (1.2), (1.3) hold true and one of the conditions (3.3) or (3.4) is satisfied.
If
(3.6)

$$\left\{\begin{array}{c}
\text{There exist a bounded function } \rho_0 : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that for all } t > 0, \text{ if } x, y \in [0, r] \text{ then} \\
\left|f(t, e^{kt}x) - f(t, e^{kt}y)\right| \leq \rho_0(t).|x-y| \text{ and such that} \\
\lim_{x\to+\infty} \frac{1}{a(x)} \int_0^x \frac{A_r(t)}{h(t)} dt = 0,
\end{array}\right.$$

then the positive solutions of problem (1.1) are locally asymptotically stable in K.

Proof. The boundness of ρ_0 makes that $(q.\rho_0) \in L_\sigma^\alpha(\mathbb{R}^+, \mathbb{R}^+)$. As $\bar{\eta}_{n,k} \in L_\sigma^\alpha(\mathbb{R}^+, \mathbb{R}^+)$ and $\hat{\delta}_k \in L_\sigma^{\alpha-\beta}(\mathbb{R}^+, \mathbb{R}^+)$, we have

$$M = \sup_{t\geq 0}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right)(t) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k} + q.\rho_0\right)(t)\right) < \infty.$$

Then

$$\frac{1}{a(x)}\int_0^x \frac{1}{h(t)} A_r(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\bar{\eta}_{n,k} + q.\rho_0\right)\right)(t)dt \leq \theta(x)$$

where

$$\theta(x) = M.\frac{1}{a(x)}\int_0^x \frac{1}{h(t)} A_r(t)\, dt.$$

The condition (3.6) gives

$$\lim_{x\to+\infty} \theta(x) = 0$$

which means that the condition (3.5) holds. We deduce from theorem (3.7) that the positive solutions of problem (1.1) are locally asymptotically stable in K. □

Remark 3.8. If we assume that the conditions (1.2), (1.3) and (3.5) hold for all $r > 0$ and one of the conditions (3.3) or (3.4) is satisfied then ivp (1.1) is asymptotically stable in K.

Example 3.9. We consider the initial value problem

(3.7)

$$\begin{cases} -D_{0^+}^{\frac{1}{2},\omega,\sigma}\left(\phi_{p_1}\left(e^t.u'\right) + \phi_{p_2}\left(e^t.u'\right)\right)(t) + D_{0^+}^{\frac{1}{3},\omega,\sigma}\left(\delta.u\right)(t) + \eta(t).u(t) + q(t) f(t,u(t)) = 0, \ t > 0, \\ u(0) = 0. \end{cases}$$

We have $a(t) = 1$, $p(t,x) = \eta(t).x$ $(m = 1)$, $h(t) = e^t$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, with $\sigma(t) = 1 - e^{-x}$, $p_1, p_2 > 1$,

$$\delta(t) = \frac{1}{4}e^{-2t}\Gamma(\alpha - \beta + 1), \eta(t) = \frac{1}{4}e^{-2t}\Gamma(\alpha + 1), \ q = \Gamma(\alpha + 1),$$

and the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$f(t,x) = \frac{e^{-2t}}{4}x + \frac{1}{4}.$$

The conditions (1.2) and (1.3) are satisfied for $k = 2$ and $r = 1$ because $f(t, e^{2t}x) = \frac{1}{4}x + \frac{1}{4}$ $\left(\lambda = \epsilon = \frac{1}{4}\right)$, and

$$\begin{aligned} I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right)(x) &= \frac{1}{4}\Gamma(\alpha - \beta + 1) I_{0^+}^{\alpha-\beta,\sigma}(1)(x) = \frac{1}{4}(\sigma(x))^{\alpha-\beta} \le \frac{1}{4} < \infty \\ I_{0^+}^{\alpha,\sigma}(\bar{\eta}_k)(x) &= \frac{1}{4}\Gamma(\alpha + 1) I_{0^+}^{\alpha,\sigma}(1)(x) = \frac{1}{4}(\sigma(x))^{\alpha} \le \frac{1}{4} < \infty \\ I_{0^+}^{\alpha,\sigma}(q)(x) &= \Gamma(\alpha + 1) I_{0^+}^{\alpha,\sigma}(1)(x) = (\sigma(x))^{\alpha} \le 1 < \infty \end{aligned}$$

and

$$\frac{1}{h(x)} = e^{-x} \in L^1\left(\mathbb{R}^+\right) \subset L^1_{loc}\left(\mathbb{R}^+\right), \text{ and} \lim_{x\to+\infty} e^{-2x}\int_0^x \frac{dt}{h(t)} = 0.$$

Moreover, we have

$$\sup_{x\ge0} e^{-2x}\int_0^x \frac{dt}{h(t)} = \sup_{x\ge0} e^{-2x}\left(1 - e^{-x}\right) = \frac{4}{27} < 1$$

for $p \in \{p_1, p_2\}$

$$\begin{aligned} \Lambda_{r=1}(p) &= \sup_{x\ge0}\psi_p\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\bar{\eta}_k + \left(\lambda + \frac{\epsilon}{r}\right).q\right)\right) \\ &\le \psi_p(1) = 1 \le \left(\sup_{x\ge0}\left\{e^{-2x}\int_0^x \frac{dt}{h(t)}\right\}\right)^{-1}. \end{aligned}$$

This means that the condition (3.3) holds. Then we deduce from theorem (3.5) that ivp (3.7) admits at least one exponentially growing solution.

Example 3.10. We consider the following initial value problem (p(j))
(p(j))

$$\begin{cases} -D_{0^+}^{\frac{1}{2},\omega,t}\left(\phi_{p_1}\left(e^t.\left(a_j.u\right)'\right)+\phi_{p_2}\left(e^t.u'\right)\right)(t)+D_{0^+}^{\frac{1}{3},\omega,t}\left(\delta.u\right)(t)+\eta\left(t\right).u\left(t\right)+q\left(t\right)f(t,u(t))=0,\ t>0, \\ u(0)=0. \end{cases}$$

We have $\sigma(t) = t,\ a_j(t) = (t+1)^j,\ j > 1,\ p(t,x) = \eta(t).x\ (m=1),\ h(t) = (x+1)^{-\frac{1}{4}},\ \alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, with $p_1 = \frac{3}{2} < 2 < p_2 = 3$,

$$\delta(t) = \frac{\mu}{4\theta_1(1+t)^2}e^{-2t}\Gamma(\alpha-\beta+1), \eta(t) = \frac{\mu}{4\theta_2(1+t)^2}e^{-2t}\Gamma(\alpha+1),\ q = \mu.\frac{\Gamma(\alpha+1)}{\theta_2(1+t)^2},$$

where

$$\theta_1 = \sup\left\{\int_0^x \frac{(x-t)^{\alpha-\beta-1}}{(1+t)^2}dt, x\geq 0\right\},\ \theta_2 = \sup\left\{\int_0^x \frac{(x-t)^{\alpha-1}}{(1+t)^2}dt, x\geq 0\right\} \text{ and}$$

$$\mu = \phi_p\left(\sup_{x\geq 0}\left\{e^{-2x}\int_0^x (x+1)^{\frac{1}{4}}\,dt\right\}\right)^{\frac{-1}{2}}.$$

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$f(t,x) = \frac{e^{-2t}}{4}x + \frac{1}{4}.$$

As in the proof of the first example, we can easily show that (1.2), (1.3) and (3.3) are satisfied for $k = 2$, $p = p_1 = \frac{3}{2}$ and $r = 1$. Now, we show that the condition (3.6) of the corollary (3) holds.
For $x, y \in [0,1]$

$$\left|f\left(t,e^{2t}x\right) - f\left(t,e^{2t}y\right)\right| = \rho_0(t)\,|x-y|$$

where

$$\rho_0(t) = \frac{1}{4}.$$

And

$$\begin{aligned} A_{r=1}(t) &= \frac{1}{(p_1-1)\left(\psi\left(N_1^{(1)}(t)\right)\right)^{(p_1-2)} + (p_2-1)\left(\psi\left(N_1^{(2)}(t)\right)\right)^{(p_2-2)}} \\ &= \frac{1}{\frac{1}{2}\left(\psi\left(N_1^{(1)}(t)\right)\right)^{-\frac{1}{2}} + 2\left(\psi\left(N_1^{(2)}(t)\right)\right)} \\ &\leq \frac{1}{2}\left(\psi\left(N_1^{(2)}(t)\right)\right)^{-1}, \end{aligned}$$

where

$$
\begin{aligned}
N_1^{(1)}(t) &= I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right)(t) + I_{0^+}^{\alpha,\sigma}\left(\bar{\eta}_k + (\lambda+\epsilon).q\right)(t) \\
&= \frac{1}{4}(\sigma(t))^{\alpha-\beta} + \frac{3}{4}(\sigma(t))^{\alpha} \\
N_1^{(2)}(t) &= I_{0^+}^{\alpha,\sigma}(q).\min\left\{f(t,x),(t,x)\in\mathbb{R}^+\times\left[0,e^{kt}r\right]\right\} = \frac{1}{4}I_{0^+}^{\alpha,\sigma}(q)(t) \\
&= \frac{1}{4}(\sigma(t))^{\alpha} = \frac{t^\alpha}{4}
\end{aligned}
$$

and for $x > 16$

$$
\begin{aligned}
\int_0^x \frac{A_r(t)}{h(t)}dt &\leq \int_0^x \frac{1}{2h(t)\psi\left(N_1^{(2)}(t)\right)}dt = \int_0^{16} \frac{1}{2h(t)\psi\left(N_1^{(2)}(t)\right)}dt + \int_{16}^x \frac{1}{2h(t)\psi\left(N_1^{(2)}(t)\right)}dt \\
&\leq \int_0^{16} \frac{1}{2h(t)\psi\left(N_1^{(2)}(t)\right)}dt + \int_{16}^x \frac{1}{2h(t)\left(\frac{t^\alpha}{8}\right)^{\frac{1}{2}}}dt \\
&\leq \int_0^{16} \frac{(t+1)^{\frac{1}{4}}}{2\psi\left(N_1^{(2)}(t)\right)}dt + \sqrt{2}\int_{16}^x \frac{(t+1)^{\frac{1}{4}}}{t^{\frac{1}{4}}}dt
\end{aligned}
$$

Thus

$$
\lim_{x\to+\infty}\frac{1}{a_j(x)}\int_0^x \frac{A_r(t)}{h(t)}dt = 0.
$$

Consequently, for all $j > 1$, the ivp (p(j)) is locally asymptotically stable.

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