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STUDY OF TWO SYSTEM OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS VIA LAPLACE TRANSFORM METHOD

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ABSTRACT. In this work, we will provide an analytical method to compute the solution of the linear coupled system of Caputo fractional differential equations with initial conditions. The standard method adopted for the system of ordinary differential equations using the exponential of a matrix will not be useful, since the Mittag-Leffler function does not have the nice property of the exponential function. In addition, the variation of parameter cannot be adopted for fractional differential equations. Here we have used the Laplace transform method to solve the system of Caputo fractional differential equations when the order of the derivative is q and $0 < q < 1$. The method yields the integer results as a special case. Our method also works for scalar sequential Caputo fractional differential equations of order nq, since it can be reduced to n systems of q^{th} order Caputo fractional differential equations with initial conditions.

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Key Words: Mittag-Leffler Function, Caputo Fractional Derivative, Sequential.

1. INTRODUCTION

The concept of fractional-order derivatives, also known as non-integer order derivatives, dates back to the 17th century. By the end of the nineteenth century, the requisite mathematics of fractional-order derivatives and arbitrary-order integrals had nearly been finished. Later, it was discovered that arbitrary order derivatives provide an ideal foundation for modeling real-world issues in a range of fields, and their importance has grown as a result of their numerous applications in diverse branches of science and engineering. See [1, 3, 4, 7, 8, 12, 17, 13, 21, 26, 27, 30, 31] and the references therein for some analysis and applications of fractional differential equations. For numerical work in fractional differential equation, see [2, 9, 23]. In the study of Caputo differential equation with initial conditions, we will reduce to integer differential equation with initial conditions if q is the order of the fractional derivative tends to an integer. In addition, if we can compute the solution of the Caputo fractional differential equation, then, we can demonstrate that we can choose the value of q as a parameter, to enhance the mathematical model as in [18]. In this work, we provide a methodology to compute the solution of the two system of Caputo fractional differential equations of order q and $q < 1$, with initial conditions. However, we cannot use the methods of

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the integer order since the Mittag-Leffler function needed to solve the Caputo fractional differential equations, do not enjoy the nice property of the exponential function of the corresponding integer order. In addition, we cannot use the variation of the parameter method since the nice form of the product rule is not available for fractional derivatives. In order to solve the linear, homogeneous or non-homogeneous system of equations with integer order, we could use the fundamental matrix solution method by computing the matrix e^{At}. This method can be used to solve both homogeneous and non-homogeneous linear system with initial conditions. The use of the inverse matrix of the fundamental matrix e^{At}, which is essentially e^{-At}, is required for this procedure. But, we can not extend this approach to the linear Caputo fractional differential system. Similarly to e^{At}, $E_{q,1}^{At}$ is the fundamental matrix solution of the linear Caputo fractional differential system

$$^{c}D_{0+}^{q}(u) = Au,$$

where A is any $N \times N$, constant matrix.
The Mittag-Leffler function, which is a generalization of the exponential function, is represented by $E_{q,1}^{At}$. However, the inverse of $E_{q,1}^{At}$ is not $E_{q,1}^{-At}$. Thus, we can not even solve the homogeneous linear system of Caputo fractional differential equations with initial conditions using the fundamental matrix solution method. Also, we can not use variation of parameter method to solve the non-homogeneous linear scalar Caputo fractional differential equation since there is no product rule for Caputo derivative. In this work, we provide a method to solve the linear non-homogeneous Caputo fractional differential system by using the Laplace transform method. However, in order to perform the inverse Laplace transform, we must first determine the eigenvalues of the matrix A. The solutions will depend on the roots of the determinant of $A - \lambda I$, being real and distinct, real and coincident and complex roots. We will obtain the solution form in all the three cases. In the integer case of the pure complex root case, the eigenfunctions are standard trigonometric functions of the form $sin\lambda t$ and $cos\lambda t$. In the corresponding situation, the eigenfunctions of the linear Caputo fractional differential system are the fractional trigonometric functions $sin_{q,1}(\lambda t^q)$ and $cos_{q,1}(\lambda tq)$. The fractional trigonometric functions $sin_{q,1}(\lambda t^q)$ and $cos_{q,1}(\lambda tq)$ are generalizations of the integer trigonometric functions of the form $sin\lambda t$ and $cos\lambda t$. In this work, we will show that the linear Caputo sequential fractional differential equation of order nq can be reduced to n system of linear Caputo fractional differential equations with initial conditions. Numerically, we will show that the fractional trigonometric functions $\sin_{q,1} t^q$ and $\cos_{q,1} t^q$ have damping behavior without a damping term in the sequential differential equation of order $2q$. This behavior of the fractional trigonometric function is useful in establishing the asymptotic stability of the equilibrium solution when we use the Caputo fractional derivative of order $q < 1$ in the model instead of the integer model. If the eigenvalues are of the form $\lambda \pm i\mu$, then the Generalized fractional trigonometric functions $G\sin_{q,1}(\lambda + i\mu)t^q$ and $G\cos_{q,1}(\lambda + i\mu)t^q$ are required. We will provide the Laplace transform table which will be useful in solving the linear non-homogeneous system of Caputo fractional differential equations with initial conditions.

2. PRELIMINARY RESULTS

In this section, we will recall some definitions and known results which play a key role in our main results.

Definition 2.1. The Riemann-Liouville fractional integral of order q defined by

$$D_{0+}^{-q}u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds, \tag{2.1}$$

where $0 < q \leq 1$ and $\Gamma(q)$ is the Gamma function.

Definition 2.2. The Riemann-Liouville (left-sided) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:

$$D_{0+}^{q}u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}u(s)ds,\ t > 0. \tag{2.2}$$

The Riemann-Liouville integral of order q for any function is same as the Caputo integral of order q.

Definition 2.3. The Caputo (left-sided)fractional derivative of $u(t)$ of order $q,\ n-1 \leq nq < n$, is given by the equation:

$$^{c}D_{0+}^{nq}u(t) = \frac{1}{\Gamma(n-nq)}\int_0^t (t-s)^{n-nq-1}u^{n}(s)ds,\ t \in [0,\infty),\ t > t_0, \tag{2.3}$$

where $u^n(t) = \frac{d^n(u)}{dt^n}$.

In particular, if q is an integer, then both Caputo derivative and integer derivative are same. Note that the Caputo integral of order q for any function is same as the Riemann-Liouville integral of order q. See [10, 12, 20] for more details in Caputo and Riemann-Liouville fractional derivative.

Definition 2.4. The Caputo (left) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:

$$^{c}D_{0+}^{q}u(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}u'(s)\mathrm{d}s. \tag{2.4}$$

We are just replacing n by 1 in above definition.

Next, we define the two parameter Mittag-Leffler function which will be useful in solving the systems of linear Caputo fractional differential equations using the Laplace Transform. See [15, 16, 22] for more in fractional differential equations with applications.

Definition 2.5. The two parameter Mittag-Leffler function is defined as

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+r)}, \tag{2.5}$$

where $q,\ r > 0$, and λ is a constant. Furthermore, for $r = q$, (2.5) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+q)}. \tag{2.6}$$

If $q = 1$ and $r = 1$ in (2.5), then we have,

$$E_{1,1}(\lambda t) = \sum_{k=0}^{\infty}\frac{(\lambda t)^k}{\Gamma(k+1)} = e^{\lambda t}, \tag{2.7}$$

where $e^{\lambda t}$ is the usual exponential function.

See [10, 12, 14, 20] for more details on Mittag-Leffler function.
Here, below we have defined fractional trigonometric functions and generalized fractional trigonometric functions of order q which will be required in our main results.

Definition 2.6. The fractional trigonometric functions $\sin_{q,1}(\lambda t^q)$ and $\cos_{q,1}(\lambda t^q)$, are given by

$$\sin_{q,1}(\lambda t^q) = \frac{1}{2i}[E_{q,1}(i\lambda t^q) - E_{q,1}(-i\lambda t^q)] \tag{2.8}$$

and

$$\cos_{q,1}(\lambda t^q) = \frac{1}{2}[E_{q,1}(i\lambda t^q) + E_{q,1}(-i\lambda t^q)], \tag{2.9}$$

respectively.

We can also define $\sin_{q,q}(\lambda t^q)$ and $\cos_{q,q}(\lambda t^q)$ in a similar way using $E_{q,q}(\lambda t^q)$ in place of $E_{q,1}(\lambda t^q)$.

Definition 2.7. The Generalized fractional trigonometric functions $Gsin_{q,1}((\lambda+i\mu)t^q)$ and $Gcos_{q,1}((\lambda+i\mu)t^q)$, are given by

$$Gsin_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2i}[E_{q,1}((\lambda + i\mu)t^q) - E_{q,1}((\lambda - i\mu)t^q)] \tag{2.10}$$

and

$$Gcos_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2}[E_{q,1}((\lambda + i\mu)t^q) + E_{q,1}((\lambda - i\mu)t^q)], \tag{2.11}$$

respectively.

We can also define $Gsin_{q,q}((\lambda + i\mu)t^q)$ and $Gcos_{q,q}((\lambda + i\mu)t^q)$ in a similar way.
Note that the generalized fractional trigonometric function can not be expressed in simple form as the integer trigonometric function, since the Mittag-Leffler function does not enjoy the properties of an exponential function. When $q = 1$, then (2.8), (2.9), (2.10) and (2.11) will give $\sin \lambda t$, $\cos \lambda t$, $e^{\lambda t} \sin \mu t$ and $e^{\lambda t} \cos \mu t$ respectively.

We turn our attention now to transform method, which will provide not just a tool for obtaining solutions, but a framework for understanding the structure of linear Caputo fractional differential equations.

Definition 2.8. The Laplace transform F(s) of a function f(t) is

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt,$$

defined for all s such that the integral converges.

Since ${}^cD^q_{0+}f(t)$ is in the convolution integral form, the Laplace transform of ${}^cD^q_{0+}f(t)$ such that $0 < q \le 1$ is given by

$$\mathcal{L}[{}^cD^q_{0+}f(t)] = s^q F(s) - s^{q-1}f(0),$$

where $F(s) = \mathcal{L}(f(t))$.
See [19] for the initial work on Laplace transform for fractional differential equations.

Definition 2.9. The Caputo fractional derivative of $u(t)$ of order nq for $n - 1 < nq < n$, is said to be sequential Caputo fractional derivative of order q, if the relation

$${}^cD^{nq}_{0+}u(t) = {}^cD^q_{0+}({}^cD^{(n-1)q}_{0+})u(t), \tag{2.12}$$

holds for $n = 2, 3 \ldots.$

Note that the equation (2.12) can also be written as

$$^{c}D^{kq}_{0+}u(t) = {^{c}D^{q}_{0+}}(^{c}D^{q}_{0+})(^{c}D^{q}_{0+})\ldots k\ times \ldots (^{c}D^{q}_{0+})u(t),$$

for $k = 2, 3, 4 \ldots$.
See [5, 24, 25, 28] for some more work on sequential fractional differential equations.

Next we recall a lemma which is useful in taking the Laplace transform of a sequential Caputo fractional derivative of order nq.

Lemma 2.10. *The Laplace transform of a sequential Caputo fractional derivative of $u(t)$ of order q such that $n-1 < nq < n$, is given by*

$$\mathcal{L}(^{c}D^{nq}_{0+}u(t)) = s^{nq}U(s) - s^{nq-1}u(0) - s^{(n-1)q-1}(^{c}D^{q}_{0+}u(0)) - s^{(n-2)q-1}(^{c}D^{2q}_{0+}u(0))$$
$$\cdots - s^{q-1}(^{c}D^{(n-1)q}_{0+}u(0)),$$

where $U(s) = \mathcal{L}(u(t))$.

For details of the proof, see [29].
Below, we have developed a Laplace transform table for certain basic functions which are useful for our main results.

Laplace transform Table 1			
S.N	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}(f(t))$	
1.	t^q	$\frac{\Gamma(q+1)}{s^{q+1}}$	$s > 0, q > -1$
2.	$E_{q,1}(\pm\lambda t^q)$	$\frac{s^{q-1}}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
3.	$t^{q-1}E_{q,q}(\pm\lambda t^q)$	$\frac{1}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
4.	$\frac{t^q}{q}E_{q,q}(\pm\lambda t^q)$	$\frac{s^{q-1}}{(s^q \mp \lambda)^2}$	$s^q > \lambda, q > -1$
5.	$sin_{q,1}(\lambda t^q)$	$\frac{\lambda s^{q-1}}{s^{2q} + \lambda^2}$	$s > 0$
6.	$cos_{q,1}(\lambda t^q)$	$\frac{s^{2q-1}}{s^{2q} + \lambda^2}$	$s > 0$
7.	$t^{q-1}sin_{q,q}(\lambda t^q)$	$\frac{\lambda}{s^{2q} + \lambda^2}$	$s > 0$
8.	$t^{q-1}cos_{q,q}(\lambda t^q)$	$\frac{s^q}{s^{2q} + \lambda^2}$	$s > 0$
9.	$E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q}E_{q,q}(\lambda t^q)$	$\frac{s^{2q-1}}{(s^q - \lambda)^2}$	

Laplace transform Table 2			
S.N	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}(f(t))$	
10.	$t^{q-1}\sum_{k=0}^{\infty}\dfrac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+q)}$	$\dfrac{s^q}{(s^q-\lambda)^2}$	
11.	$t^{2q-1}\sum_{k=0}^{\infty}\dfrac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+2q)}$	$\dfrac{1}{(s^q-\lambda)^2}$	
12.	$Gcos_{q,1}\{(\lambda+i\mu)t^q\}$	$\dfrac{s^{q-1}(s^q-\lambda)}{(s^q-\lambda)^2+\mu^2}$	
13.	$Gsin_{q,1}\{(\lambda+i\mu)t^q\}$	$\dfrac{\mu s^{q-1}}{(s^q-\lambda)^2+\mu^2}$	
14.	$t^{q-1}Gcos_{q,q}\{(\lambda+i\mu)t^q\}$	$\dfrac{s^q-\lambda}{(s^q-\lambda)^2+\mu^2}$	
15.	$t^{q-1}Gsin_{q,q}\{(\lambda+i\mu)t^q\}$	$\dfrac{\mu}{(s^q-\lambda)^2+\mu^2}$	

3. MAIN RESULTS

We divided the main findings into three sections. In the first part, we will show how to reduce linear sequential Caputo fractional differential equations of order nq to n systems of qth order Caputo fractional differential equations with initial conditions. In the second section, we will devise a method for solving two system of q^{th} order Caputo fractional differential equations. Finally, we will discuss some numerical results.

We developed a Laplace transform method in this work to solve the linear sequential Caputo fractional differential equations of order nq that are sequential of order q such that $n-1 < nq < n$. All our results yield the integer results as a special case when q tends to 1.

3.1. **Solution of linear sequential Caputo fractional differential equations of order nq with initial conditions.** Although our aim is to develop a method to compute the solutions of linear system of Caputo fractional differential equations with initial conditions, initially we would like to show that nq order linear sequential Caputo fractional differential equations can be reduced to n systems of q^{th} order linear Caputo fractional differential equations.

Consider the linear sequential Caputo fractional differential equations of order nq with initial conditions of the form:

$$ {}^cD_{0+}^{nq}u + a_1^c D_{0+}^{(n-1)q}u + a_2^c D_{0+}^{(n-2)q}u + \cdots + a_n u = f(t), \tag{3.1}$$

with ${}^cD_{0+}^{kq}u(0) = b_k$ for $k = 0, 1, 2, \ldots n-1$.

To transform nq order linear sequential Caputo fractional differential equations to n system of q^{th} order linear Caputo fractional differential equations, we have label

$$u = u_1.$$

Then the resulting q^{th} order n linear system of Caputo fractional differential equations is

$$\begin{cases} {}^cD^q_{0+}u_k = u_{k+1} \quad \text{for } k = 1, 2, \dots n-1, \\ {}^cD^q_{0+}u_n = -a_n u_1 - \dots - a_2 u_{n-1} - a_1 u_n + f(t). \end{cases} \tag{3.2}$$

Now the above system can be written as

$${}^cD^q_{0+}u = Au + F,$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix},$$

and $u_k(0) = {}^cD^{kq}_{0+}u(0) = b_k$ for $k = 0, 1, 2, \dots n-1$.

Remark 3.1. In general, we can reduce the linear nq order sequential Caputo fractional differential equations to n systems of linear Caputo fractional differential equation of order q.

3.2. **Solution of linear Caputo fractional order with two system.** In this work, we will develop a method to solve a two system of q^{th} order linear Caputo fractional differential equations. For that purpose, consider the two system of linear Caputo fractional differential equation of order q of the following form:

$${}^cD^q_{0+}u(t) = Au(t) + f(t), \quad u(0) = u_0, \quad 0 < q \le 1, \tag{3.3}$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$

Say $u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$

If $q = 1$, we can solve it using the fundamental matrix solution and then use the variation of parameter method to solve the non-homogeneous part. However, we can't use the fundamental matrix solution method for $q < 1$ for two reasons. The first reason is that we can not use a

variation of the parameter method. It will not work since there is no product rule for the fractional derivative. The second reason is that the Mittag-Leffler function does not enjoy the nice property that the exponential function does. As a result, the Laplace transform method is the best approach for solving two systems of linear Caputo fractional differential equations.
Now taking the Laplace transform of (3.3) , we get

$$s^q U(s) - s^{q-1}u_0 = AU(s) + F(s),$$

where $U(s) = \mathcal{L}(u(t))$ and $F(s) = \mathcal{L}(f(t))$.
Now solving for $U(s)$, we get

$$\begin{aligned} U(s) &= (s^q I - A)^{-1}\{s^{q-1}u_0 + F(s)\} \\ &= \begin{bmatrix} \dfrac{s^{2q-1}x_0 + s^{q-1}(a_{12}y_0 - a_{22}x_0)}{|s^q I - A|} \\ \\ \dfrac{s^{2q-1}y_0 + s^{q-1}(a_{21}x_0 - a_{11}y_0)}{|s^q I - A|} \end{bmatrix} + \begin{bmatrix} \dfrac{(s^q - a_{22})F_1(s) + a_{12}F_2(s)}{|s^q I - A|} \\ \\ \dfrac{a_{21}F_1(s) + (s^q - a_{11})F_2(s)}{|s^q I - A|} \end{bmatrix}. \end{aligned}$$

Let $|s^q I - A| = P(2)$, where $P(2)$ is second degree polynomial in s^q. Now solving for $U(s)$, we get

$$U(s) = \begin{bmatrix} \dfrac{s^{2q-1}x_0 + s^{q-1}(a_{12}y_0 - a_{22}x_0) + (s^q - a_{22})F_1(s) + a_{12}F_2(s)}{P(2)} \\ \\ \dfrac{s^{2q-1}y_0 + s^{q-1}(a_{21}x_0 - a_{11}y_0) + (s^q - a_{11})F_2(s) + a_{21}F_1(s)}{P(2)} \end{bmatrix}. \tag{3.4}$$

In order to obtain $u(t)$, we have to take the inverse Laplace transform of (3.4) on both sides. The inverse Laplace transform of right hand side depends on the roots of the polynomial $P(2)$. This leads to several cases.

Case 1: If the roots of $P(2)$ are real and distinct, say λ_1 and λ_2.
Then, $P(2) = (s^q - \lambda_1)(s^q - \lambda_2)$.
Now in order to get inverse of Laplace transform of above expression, let's get the inverse separately.

(a) For this, we are going to use partial fraction.
Let

$$\frac{s^q}{(s^q - \lambda_1)(s^q - \lambda_2)} = \frac{A_1}{(s^q - \lambda_1)} + \frac{B_1}{(s^q - \lambda_2)}.$$

Then using formulas 2 and 3 from Laplace transform table, we get

$$\mathcal{L}^{-1}[\frac{s^{2q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)}] = A_1 E_{q,1}(\lambda_1 t^q) + B_1 E_{q,1}(\lambda_2 t^q)$$

and

$$\mathcal{L}^{-1}[\frac{s^q F_1(s)}{(s^q - \lambda_1)(s^q - \lambda_2)}] = \int_0^t \{A_1 E_{q,q}(\lambda_1(t-s)^q) + B_1 E_{q,q}(\lambda_2(t-s)^q)\}(t-s)^{q-1} f_1(s)ds,$$

where $A_1 = \dfrac{\lambda_1}{\lambda_1 - \lambda_2}$ and $B_1 = \dfrac{\lambda_2}{\lambda_2 - \lambda_1}$.

(b) Let

$$\frac{1}{(s^q - \lambda_1)(s^q - \lambda_2)} = \frac{A_2}{(s^q - \lambda_1)} + \frac{B_2}{(s^q - \lambda_2)}.$$

Then using formulas 2 and 3 from Laplace transform table, we get

$$\mathcal{L}^{-1}[\frac{s^{q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)}] = A_2 E_{q,1}(\lambda_1 t^q) + B_2 E_{q,1}(\lambda_2 t^q)$$

and

$$\mathcal{L}^{-1}[\frac{a_{12}F_2(s)-a_{22}F_1(s)}{(s^q-\lambda_1)(s^q-\lambda_2)}]$$
$$=\int_0^t\{A_2E_{q,q}(\lambda_1(t-s)^q)+B_2E_{q,q}(\lambda_2(t-s)^q)\}(t-s)^{q-1}(a_{12}f_2(s)-a_{22}f_1(s))ds,$$

where $A_2=\dfrac{1}{\lambda_1-\lambda_2}$ and $B_2=\dfrac{1}{\lambda_2-\lambda_1}$.

Now taking the inverse Laplace transform of $U(s)$ in (3.4), we will get

$$\begin{aligned}x(t)&=\{A_1E_{q,1}(\lambda_1t^q)+B_1E_{q,1}(\lambda_2t^q)\}x_0\\&+\{A_2E_{q,1}(\lambda_1t^q)+B_2E_{q,1}(\lambda_2t^q)\}(a_{12}y_0-a_{22}x_0)\\&+\int_0^t\{A_1E_{q,q}(\lambda_1(t-s)^q)+B_1E_{q,q}(\lambda_2(t-s)^q)\}(t-s)^{q-1}f_1(s)ds\\&+\int_0^t\{A_2E_{q,q}(\lambda_1(t-s)^q)+B_2E_{q,q}(\lambda_2(t-s)^q)\}(t-s)^{q-1}(a_{12}f_2(s)-a_{22}f_1(s))ds\end{aligned}$$

and

$$\begin{aligned}y(t)&=\{A_1E_{q,1}(\lambda_1t^q)+B_1E_{q,1}(\lambda_2t^q)\}y_0\\&+\{A_2E_{q,1}(\lambda_1t^q)+B_2E_{q,1}(\lambda_2t^q)\}(a_{21}x_0-a_{11}y_0)\\&+\int_0^t\{A_1E_{q,q}(\lambda_1(t-s)^q)+B_1E_{q,q}(\lambda_2(t-s)^q)\}(t-s)^{q-1}f_2(s)ds\\&+\int_0^t\{A_2E_{q,q}(\lambda_1(t-s)^q)+B_2E_{q,q}(\lambda_2(t-s)^q)\}(t-s)^{q-1}(a_{21}f_1(s)-a_{11}f_2(s))ds.\end{aligned}$$

Case 2: If the roots of $P(2)$ are real and equal, (say λ), then $P(2)=(s^q-\lambda)^2$. Similarly, using formulas 9, 4, 10 and 11 from Laplace transform table, we get

$$\begin{aligned}x(t)&=\{E_{q,1}(\lambda t^q)+\frac{\lambda t^q}{q}E_{q,q}(\lambda t^q)\}x_0+\frac{t^q}{q}E_{q,q}(\lambda t^q)(a_{12}y_0-a_{22}x_0)\\&+\int_0^t\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k(t-s)^{qk+q-1}}{\Gamma(qk+q)}f_2(s)\\&+\int_0^t\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k(t-s)^{qk+2q-1}}{\Gamma(qk+2q)}(a_{21}f_1(s)-a_{11}f_2(s))ds\end{aligned}$$

and

$$\begin{aligned}y(t)&=\{E_{q,1}(\lambda t^q)+\frac{\lambda t^q}{q}E_{q,q}(\lambda t^q)\}y_0+\frac{t^q}{q}E_{q,q}(\lambda t^q)(a_{21}x_0-a_{11}y_0)\\&+\int_0^t\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k(t-s)^{qk+q-1}}{\Gamma(qk+q)}f_1(s)\\&+\int_0^t\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k(t-s)^{qk+2q-1}}{\Gamma(qk+2q)}(a_{12}f_2(s)-a_{22}f_1(s))ds.\end{aligned}$$

Case 3: If the roots of $P(2)$ are purely imaginary, (say $\pm\lambda i$).
Then, $P(2)=(s^q-i\lambda)(s^q+i\lambda)=s^{2q}+\lambda^2$.
Similarly, using formulas 6, 5, 8 and 7 from Laplace transform table, we get

$$x(t) = cos_{q,1}(\lambda t^q)x_0 + \frac{1}{\lambda}sin_{q,1}(\lambda t^q)(a_{12}y_0 - a_{22}x_0) + \int_0^t (t-s)^{q-1}cos_{q,q}(\lambda(t-s)^q)f_1(s)ds$$
$$+ \int_0^t \frac{(t-s)^{q-1}}{\lambda}sin_{q,q}(\lambda(t-s)^q)(a_{12}f_2(s) - a_{22}f_1(s))ds$$

and

$$y(t) = cos_{q,1}(\lambda t^q)y_0 + \frac{1}{\lambda}sin_{q,1}(\lambda t^q)(a_{21}x_0 - a_{11}y_0) + \int_0^t (t-s)^{q-1}cos_{q,q}(\lambda(t-s)^q)f_2(s)ds$$
$$+ \int_0^t \frac{(t-s)^{q-1}}{\lambda}sin_{q,q}(\lambda(t-s)^q)(a_{21}f_1(s) - a_{11}f_2(s))ds.$$

Case 4: If the roots of $P(2)$ are complex, (say $\lambda + \mu i$ and $\lambda - \mu i$).
Then, $P(2) = (s^q - \lambda - \mu i)(s^q - \lambda + \mu i) = (s^q - \lambda)^2 + \mu^2$.
Now in order to get inverse of Laplace Transform, we are going to use partial fraction.
Let

$$\frac{s^q}{(s^q-\lambda)^2+\mu^2} = \frac{s^q-\lambda}{(s^q-\lambda)^2+\mu^2} + \frac{\lambda}{(s^q-\lambda)^2+\mu^2}.$$

Then using formulas 12 and 13 from Laplace transform table, we get

$$\mathcal{L}^{-1}[\frac{s^{2q-1}}{(s^q-\lambda)^2+\mu^2}] = Gcos_{q,1}\{(\lambda+i\mu)t^q\} + \frac{\lambda}{\mu}Gsin_{q,1}\{(\lambda+i\mu)t^q\}.$$

Similarly, using formulas 12, 13, 14 and 15 from Laplace transform table, we will get

$$x(t) = \{Gcos_{q,1}\{(\lambda+i\mu)t^q\} + \frac{\lambda}{\mu}Gsin_{q,1}\{(\lambda+i\mu)t^q\}\}x_0 + \frac{1}{\mu}Gsin_{q,1}\{(\lambda+i\mu)t^q\}(a_{12}y_0 - a_{22}x_0)$$
$$+ \int_0^t (t-s)^{q-1}\{Gcos_{q,q}\{(\lambda+i\mu)(t-s)^q\} + \frac{\lambda}{\mu}Gsin_{q,q}\{(\lambda+i\mu)(t-s)^q\}f_1(s)ds$$
$$+ \int_0^t \frac{(t-s)^{q-1}}{\mu}Gsin_{q,q}\{(\lambda+i\mu)(t-s)^q\}(a_{12}f_2(s) - a_{22}f_1(s))ds$$

and

$$y(t) = \{Gcos_{q,1}\{(\lambda+i\mu)t^q\} + \frac{\lambda}{\mu}Gsin_{q,1}\{(\lambda+i\mu)t^q\}\}y_0 + \frac{1}{\mu}Gsin_{q,1}\{(\lambda+i\mu)t^q\}(a_{21}x_0 - a_{11}y_0)$$
$$+ \int_0^t (t-s)^{q-1}\{Gcos_{q,q}\{(\lambda+i\mu)(t-s)^q\} + \frac{\lambda}{\mu}Gsin_{q,q}\{(\lambda+i\mu)(t-s)^q\}f_2(s)ds$$
$$+ \int_0^t \frac{(t-s)^{q-1}}{\mu}Gsin_{q,q}\{(\lambda+i\mu)(t-s)^q\}(a_{21}f_1(s) - a_{11}f_2(s))ds.$$

3.3. **Numerical Results.**

In this section, we provide some stability results of the equilibrium solution of prey and predator model using analysis and numerical approach. Let u_1 and u_2 be the prey and predator densities respectively. Now consider the prey and predator model

$${}^cD^q_{0+}u_1(t) = u_1(4 - u_1 - 3u_2)$$

and

$${}^cD^q_{0+}u_2(t) = u_2(-2 + u_1 + u_2),$$

where $0 < q < 1$. The only positive equilibrium solution of above system is $(1, 1)$ and we reduce the above problem to the corresponding linear system in the neighborhood of equilibrium. The linear system will be in the form

$${}^cD^q_{0+}v(t) = Av(t), \quad u(0) = u_0, \quad 0 < q < 1,$$

where $A = \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$ and $v(t) = \begin{bmatrix} u_1(t) - 1 \\ u_2(t) - 1 \end{bmatrix}$.

Now the eigen values of the matrix A are $\pm i\sqrt{2}$ and the corresponding eigenvectors are $sin_{q,1}(\sqrt{2}t^q)$ and $cos_{q,1}(\sqrt{2}t^q)$. For $q = 1$, we have only stability but for $q < 1$, we will have asymptotic stability because the graphs of $sin_{q,1}(\sqrt{2}t^q)$ and $cos_{q,1}(\sqrt{2}t^q)$ oscillates and exhibit damping behavior. Hence, by proper choice of $q < 1$, we will have asymptotic stability. See more in [11]. Here, below we draw the graphs of the solution near the equilibrium point $(1, 1)$.

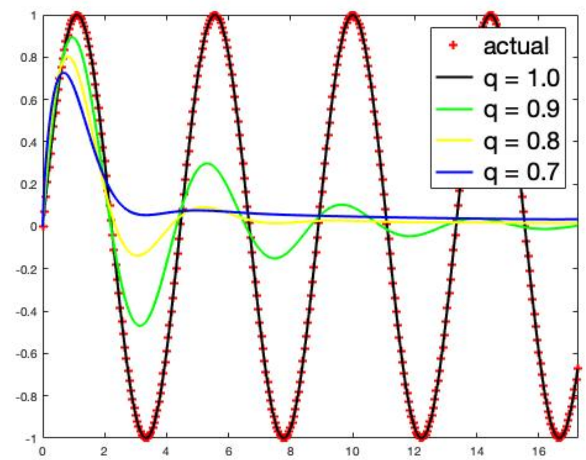

FIGURE 1. $\sin_{q,1}(\sqrt{2}t^q)$ graph.

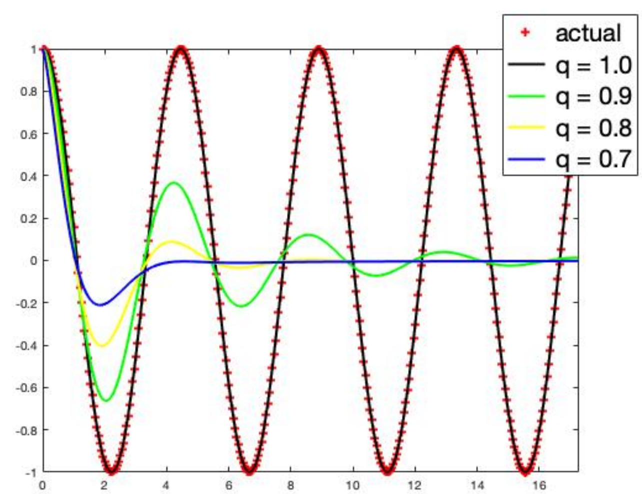

FIGURE 2. $\cos_{q,1}(\sqrt{2}t^q)$ graph.

Remark 3.3.1. Observe that when $q = 1$, the graphs exactly match the usual trigonometric functions. However, when $q < 1$, we can see that there is a damping that ensures the asymptotic stability of the equilibrium solution.

Similarly, if we consider the following prey and predator model

$$^cD^q_{0+}u_1(t) = u_1(1 - \frac{1}{2}u_1 - \frac{1}{2}u_2)$$

and

$$^cD^q_{0+}u_2(t) = u_2(\frac{-3}{2} + 2u_1 - \frac{1}{2}u_2),$$

where $0 < q \leq 1$.

Then the linear system will be in the form

$$^cD^q_{0+}v(t) = Bv(t), \quad u(0) = u_0, \quad 0 < q < 1,$$

where $B = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 2 & -\frac{1}{2} \end{bmatrix}$ and $v(t) = \begin{bmatrix} u_1(t) - 1 \\ u_2(t) - 1 \end{bmatrix}$.

Then the eigenvalues of matrix B are $-0.5 \pm i$ and we will have following figure of $Gsin_{q,1}(-0.5 + i)t^q$ and $Gcos_{q,1}(-0.5 + i)t^q$.

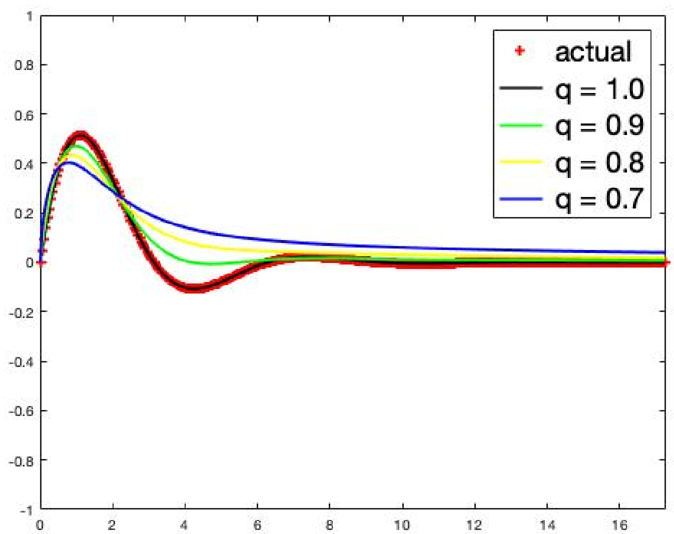

FIGURE 3. $Gsin_{q,1}(-0.5 + i)t^q$ graph.

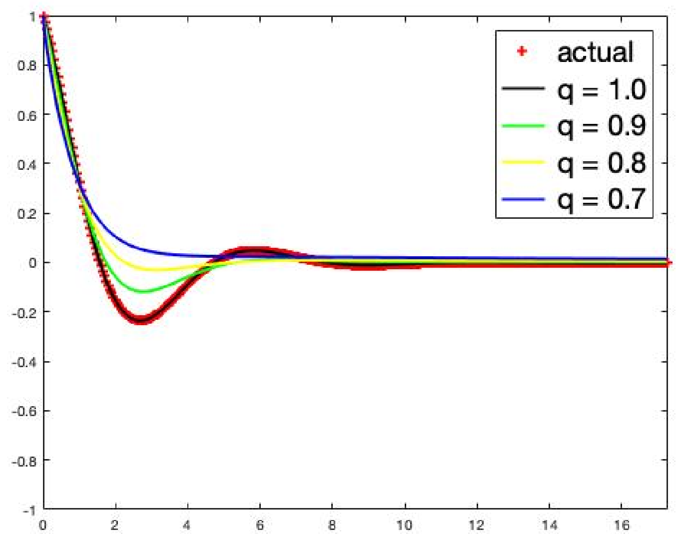

FIGURE 4. $Gcos_{q,1}(-0.5+i)t^q$ graph.

Finally, we present an example of a system of 2q order sequential Caputo differential equations which exhibits a damping nature without a damping term.

Example: Consider $2q$ order sequential Caputo fractional differential equation

$$^cD^{2q}u(t) + \lambda^2 u = 0, \quad 1 < 2q \leq 2,$$

subject to initial conditions

$$u(0) = 1, \quad {}^cD^q u(0) = 0,$$

where λ is a real number.
The roots of the characteristic equation are then $\pm\lambda i$, and the general solution of the above equation is

$$u(t) = Acos_{q,1}(\lambda t^q) + Bsin_{q,1}(\lambda t^q).$$

Using the initial conditions, the solution which satisfy above equation is given by

$$u(t) = cos_{q,1}(\lambda t^q). \tag{3.5}$$

If the initial conditions are changed to

$$u(0) = 0, \quad {}^cD^q u(0) = 1,$$

then the general solution will be

$$u(t) = \frac{1}{\lambda} sin_{q,1}(\lambda t^q).$$

Remark 3.3.2. Observe that the graphs of $sin_{q,1}(\lambda t^q)$ and $cos_{q,1}(\lambda t^q)$ have damping behavior similar to the graphs of figure 1 and figure 2. However, if $q = 1$, we will have usual trigonometric functions $sin\lambda t$ and $cos\lambda t$ which has no damping behavior. See more in [6, 9].

CONCLUDING REMARKS

In this work, we have developed a method to solve the two linear system of Caputo fractional differential equations with initial conditions using the Laplace transform method. In general, there is no easy method to solve a general linear Caputo fractional differential equations of order nq, where $n-1 < nq < n$. In particular, if $q = 1$, then we have an n^{th} order linear differential equation, which can be theoretically solved by reducing it to n system of linear differential equations. This is possible for a fractional differential equation of order nq if the fractional derivative of order nq is sequential of order q. Further, the initial conditions should involve all fractional derivatives of lower order kq, with $k = 1, 2, \ldots (n-1)$. All our methods developed here yield the corresponding integer results as a special case. The methods developed here will help us to discuss the stability of the equilibrium solution of linearized system of the Caputo fractional differential equations with initial conditions. An important observation is that the equilibrium solution of the linear Caputo fractional differential system may be asymptotically stable even when the corresponding solution of the equilibrium solution of integer system is only locally stable. In addition, the solution of the Caputo fractional differential system may agree closely with the available data compared with the solution of the integer model. We plan to extend our method to third and higher order systems so that we can study biological models such as COVID-19 SIR and SEIR models.

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