STUDY OF THREE SYSTEMS OF NON-LINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS AND APPLICATIONS

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ABSTRACT. We shall provide an analytical method for solving three linear coupled systems of Caputo fractional differential equations with fractional initial conditions. Because the Mittag-Leffler function doesn't satisfy all the properties of the exponential function, we cannot use the integer order methods. Here we have used an efficient and convenient method, called the Laplace transform method, to solve the three systems of linear Caputo fractional differential equations with fractional initial conditions when the order of the fractional derivative is q and 0 < q < 1. In addition, the Laplace-Adomian decomposition method allows us to obtain an approximation of the non-linear SIR epidemic model of fractional order q. All the methods we have adopted here yield integer results as a special case. Our method also works for scalar linear sequential Caputo fractional differential equations of order nq, since it can be reduced to n systems of qth order linear Caputo fractional differential equations with initial conditions.

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Key Words: Mittag-Leffler Function, Caputo Fractional Derivative, Laplace-Adomian method.

1. INTRODUCTION

The widely studied subject of fractional differential equations has remarkably gained prominence and popularity in the last few decades due to its proved applications in a wide range of science, economics, and engineering fields. It has encompassed a wide range of topics, such as initial value problems, boundary value problems, and the stability of fractional equations. Fractional differential equations of arbitrary order are the generalization of ordinary differential equations of integer order. Many scientists and mathematicians are now attracted to the field of differential systems of fractional order due to its numerous applications. The fractional derivative has a global nature and the integer derivative has a local nature. From a modeling point of view, appropriate order fractional differential systems are more suitable than the integer order. As a result, mathematicians were able to improve the model using real world data by utilizing the order q of the fractional derivative as a parameter. See [3, 4, 7, 8, 11, 16, 12, 20, 25, 26, 27, 30, 31] and the references therein for some analysis and applications of fractional differential equations. For numerical work in fractional differential equation, see [2, 9, 23].

Received November 1, 2021 1061-5369 \$15.00 ©Dynamic Publishers, Inc. www.dynamicpublishers.org; www.dynamicpublishers.com; https://doi.org/10.46719/npsc20212941. In this work, we present an analytic method for solving the three systems of Caputo fractional differential equations of order q with $0 < q \leq 1$, with initial conditions. The Mittag-Leffler function, which is required to solve the Caputo fractional differential equations, does not have the nice properties of the exponential function of the appropriate integer order, so we cannot utilize the integer order methods. Furthermore, because the product rule is not available for fractional derivatives, we cannot use the variation of the parameter technique. $E_{q,1}(At^q)$ represents the Mittag-Leffler function, which is the generalization of the exponential function, but the inverse is not $E_{q,1}(-At^q)$, because of the fact that $E_{q,1}(At^q) * E_{q,1}(-At^q) \neq 1$. This holds true even when A is a constant. Hence, we cannot solve it using the fundamental matrix solution method, and therefore an effective and convenient method for solving Caputo fractional differential equations is needed. Various methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations, and dynamic systems with fractional derivatives, including the Laplace-Adomian decomposition method. Since the Caputo fractional derivative is in the convolution integral form, it is the ideal candidate to use the Laplace transform.

In this work, we initially provide a method to solve the three linear non-homogeneous Caputo fractional differential systems by using the Laplace transform method. Podlubny [19], proposed a method based on the Laplace transform approach which is used to solve numerous fractional differential equations. The Laplace transform method has played an important role in solving basic problems of differential equations. It is one of several valuable tools for solving fractional-order differential equations. The Laplace transform method has proved to be the most efficient and useful in the analysis and applications of fractional-order systems, from which some findings can be obtained quickly. When using Laplace transform method, in order to get an inverse Laplace transform of linear non-homogenous Caputo fractional differential equations with fractional initial conditions, we have developed necessary Laplace transform tables in our work.

It is easy to observe that the linear constant coefficient sequential differential equations of order nq with fractional initial conditions, can be reduced to n systems of qth order linear Caputo fractional differential equations with initial conditions. Thus, our study of systems will include the study of nq order sequential differential equations as a special case. See, [18] for more details on sequential. Finally, we have found a theoretical approximate solution of a non-linear SIR epidemic model of the fractional order q by using the Laplace-Adomian decomposition method. Using the above theoretical results, we have also provided numerical methods for a specific epidemic SIR model for different values of the fractional order q including q = 1. The purpose of this computation is to choose the value of the order q as a parameter to improve our model to fit the data. See, [1, 21] for more details.

2. PRELIMINARIES RESULTS

In this section, we will recall some definitions and known results that play a key role in our main results.

Definition 2.1. The Riemann-Liouville fractional integral of order q defined by

(2.1)
$$D_{0+}^{-q}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds,$$

where $0 < q \leq 1$ and $\Gamma(q)$ is the Gamma function.,

Definition 2.2. The Riemann-Liouville (left-sided) fractional derivative of u(t) of order q, when 0 < q < 1, is defined as:

(2.2)
$$D_{0+}^{q}u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{0}^{t}(t-s)^{-q}u(s)ds, \ t > 0.$$

The Riemann-Liouville integral of order q for any function is same as the Caputo integral of order q.

Definition 2.3. The Caputo (left-sided)fractional derivative of u(t) of order nq, n-1 < nq < n, is given by the equation:

(2.3)
$${}^{c}D_{0+}^{nq}u(t) = \frac{1}{\Gamma(n-nq)} \int_{0}^{t} (t-s)^{n-nq-1} u^{n}(s) ds, t \in [0,\infty), \ t > t_{0},$$

where $u^n(t) = \frac{d^n(u)}{dt^n}$.

In particular, if q is an integer, then both the Caputo derivative and the integer derivative are one and the same. Note that the Caputo integral of order q for any function is the same as the Riemann-Liouville integral of order q. See [10, 11, 19] for more details on Caputo and Riemann-Liouville fractional derivatives.

Definition 2.4. The Caputo (left) fractional derivative of u(t) of order q for n = 1, when 0 < q < 1, is defined as:

(2.4)
$${}^{c}D_{0+}^{q}u(t) = \frac{1}{\Gamma(1-q)} \int_{0}^{t} (t-s)^{-q} u'(s) \mathrm{d}s.$$

Next, we define the two parameter Mittag-Leffler functions, which will be useful in solving the three systems of linear Caputo fractional differential equations using the Laplace transform. See [14, 15, 22], for more on fractional differential equations with applications.

Definition 2.5. The two parameter Mittag-Leffler function is defined as

(2.5)
$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+r)}$$

where q, r > 0, and λ is a constant. Furthermore, for r = q, (2.5) reduces to

(2.6)
$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}$$

If q = 1 and r = 1 in (2.5), then we have, $E_{1,1}(\lambda t) = e^{\lambda t}$, the usual exponential function.

See [10, 11, 13, 19], for more details on Mittag-Leffler functions.

Here, we have defined fractional trigonometric functions and generalized fractional trigonometric functions of order q which will be required in our main results.

Definition 2.6. The fractional trigonometric functions $sin_{q,1}(\lambda t^q)$ and $cos_{q,1}(\lambda t^q)$, are given by

(2.7)
$$sin_{q,1}(\lambda t^q) = \frac{1}{2i} [E_{q,1}(i\lambda t^q) - E_{q,1}(-i\lambda t^q)]$$

 and

(2.8)
$$\cos_{q,1}(\lambda t^q) = \frac{1}{2} [E_{q,1}(i\lambda t^q) + E_{q,1}(-i\lambda t^q)],$$

respectively.

We can also define $sin_{q,q}(\lambda t^q)$ and $cos_{q,q}(\lambda t^q)$ in a similar way using $E_{q,q}(\lambda t^q)$ in place of $E_{q,1}(\lambda t^q)$.

Definition 2.7. The generalized fractional trigonometric functions $Gsin_{q,1}((\lambda + i\mu)t^q)$ and $Gcos_{q,1}((\lambda + i\mu)t^q)$, are given by

(2.9)
$$Gsin_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2i} [E_{q,1}((\lambda + i\mu)t^q) - E_{q,1}((\lambda - i\mu)t^q)]$$

 and

(2.10)
$$G\cos_{q,1}((\lambda+i\mu)t^q) = \frac{1}{2}[E_{q,1}((\lambda+i\mu)t^q) + E_{q,1}((\lambda-i\mu)t^q)],$$

respectively.

We can also define $Gsin_{q,q}((\lambda + i\mu)t^q)$ and $Gcos_{q,q}((\lambda + i\mu)t^q)$ in a similar way.

Definition 2.8. The Laplace transform F(s) of a function f(t) is

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt,$$

defined for all s such that the integral converges.

Since ${}^{c}D_{0+}^{q}f(t)$ is in the convolution integral form, the Laplace transform of ${}^{c}D_{0+}^{q}f(t)$ is

$$\mathcal{L}[{}^{c}D_{0+}^{q}f(t)] = s^{q}F(s) - s^{q-1}f(0), \qquad 0 < q \le 1,$$

where $F(s) = \mathcal{L}(f(t))$.

See [17], for the initial work on the Laplace transform for fractional differential equations.

We have created a Laplace transform table for a few fundamental functions that will come in handy in our main results. For some of the functions which are not present in this table, see table from [18].

Laplace transform Table			
S.N	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}(f(t))$	
1.	t^q	$\frac{\Gamma(q+1)}{s^{q+1}}$	
2.	$E_{q,1}(\pm\lambda t^q)$	$\frac{s^{q-1}}{s^q \mp \lambda}$	
3.	$t^{q-1}E_{q,q}(\pm\lambda t^q)$	$\frac{1}{s^q \mp \lambda}$	
4.	$rac{t^q}{q}E_{q,q}(\pm\lambda t^q)$	$\frac{s^{q-1}}{(s^q \mp \lambda)^2}$	
5.	$E_{q,1}(\lambda t^q) + rac{\lambda t^q}{q} E_{q,q}(\lambda t^q)$	$\frac{s^{2q-1}}{(s^q-\lambda)^2}$	
6.	$t^{2q-1} \sum_{k=0}^{\infty} \frac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+2q)}$	$rac{1}{(s^q-\lambda)^2}$	
7.	$\sum_{k=0}^{\infty} \frac{k(k+1)}{2} \frac{(\lambda t^q)^{k-1}}{\Gamma(q(k-1)+1)}$	$\frac{s^{3q-1}}{(s^q-\lambda)^3}$	
8.	$t^q \sum_{k=0}^{\infty} \frac{k(k+1)}{2} \frac{(\lambda t^q)^{k-1}}{\Gamma(qk+1)}$	$\frac{s^{2q-1}}{(s^q-\lambda)^3}$	

9.	$t^{2q} \sum_{k=0}^{\infty} \frac{k(k+1)}{2} \frac{(\lambda t^q)^{k-1}}{\Gamma(q(k+1)+1)}$	$\frac{s^{q-1}}{(s^q-\lambda)^3}$
10.	$t^{q-1}E_{q,q}(\lambda t^{q}) + \sum_{k=0}^{\infty} \frac{(k+1)(k+4)}{2} \frac{\lambda^{k+1}t^{q(k+2)-1}}{\Gamma(qk+2q)}$	$\frac{s^{2q}}{(s^q-\lambda)^3}$
11.	$\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} \frac{\lambda^k t^{q(k+2)-1}}{\Gamma(qk+2q)}$	$\frac{s^q}{(s^q-\lambda)^3}$
12.	$\sum_{k=0}^{\infty} \frac{k(k+1)}{2} \frac{\lambda^{k-1} t^{q(k+2)-1}}{\Gamma(qk+2q)}$	$\frac{1}{(s^q - \lambda)^3}$
13.	$Gcos_{q,1}\{(\lambda+i\mu)t^q\}$	$\frac{s^{q-1}(s^q - \lambda)}{(s^q - \lambda)^2 + \mu^2}$
14.	$Gsin_{q,1}\{(\lambda+i\mu)t^q\}$	$\frac{\mu s^{q-1}}{(s^q - \lambda)^2 + \mu^2}$
15.	$t^{q-1}Gcos_{q,q}\{(\lambda+i\mu)t^q\}$	$\frac{s^q - \lambda}{(s^q - \lambda)^2 + \mu^2}$
16.	$t^{q-1}Gsin_{q,q}\{(\lambda+i\mu)t^q\}$	$\frac{\mu}{(s^q-\lambda)^2+\mu^2}$

3. MAIN RESULTS

We divided the main findings into two sections. In the first section, we will develop a method for solving three or more linear systems of qth order Caputo fractional differential equations. All our results yield integer results as a special case when q tends to 1. See, [18] for more details. The study of linear fractional differential systems is also useful in the stability results of the compartmental SIR model of epidemic diseases. In the second section, we will discuss the non-linear fractional system of the compartmental SIR model. Using Laplace-Adomian decomposition method, we have developed both computational and numerical results for measles SIR Model for different fractional order q with $0 < q \leq 1$. The spread of an epidemic SIR model can be seen here, [21].

3.1. Solution of linear Caputo fractional order with three systems. In this work, we will develop a method to solve three systems of qth order linear Caputo fractional differential equations. For that purpose, consider the three systems of linear Caputo fractional differential equations of order q, where $0 < q \leq 1$, of the following form:

$${}^{c}D_{0+}^{q}u(t) = Au(t) + f(t), \qquad u(0) = u_{0}, \qquad 0 < q \le 1,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_{1}(t) \\ f_{2}(t) \\ f_{3}(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}.$
Say $u_{0} = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} x_{0} \\ y_{0} \\ z_{0} \end{bmatrix}.$

Taking the Laplace Transform of reference, we get

$$s^{q}U(s) - s^{q-1}u_{0} = AU(s) + F(s)$$

$$U(s) = (s^{q}I - A)^{-1} \{ s^{q-1}u_0 + F(s) \},\$$

where

$$|(s^{q}I - A)| = P(3) = s^{3q} - s^{2q}(a_{11} + a_{22} + a_{33}) + s^{q}(a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}) + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}.$$

Hence,

$$(s^{q}I - A)^{-1} = \begin{bmatrix} \frac{s^{2q} - s^{q}(a_{22} + a_{33}) + d_{1}}{P(3)} & \frac{s^{q}a_{12} + e_{1}}{P(3)} & \frac{s^{q}a_{13} + f_{1}}{P(3)} \\ \frac{s^{q}a_{21} + d_{2}}{P(3)} & \frac{s^{2q} - s^{q}(a_{11} + a_{33}) + e_{2}}{P(3)} & \frac{s^{q}a_{23} + f_{2}}{P(3)} \\ \frac{s^{q}a_{31} + d_{3}}{P(3)} & \frac{s^{q}a_{32} + e_{3}}{P(3)} & \frac{s^{2q} - s^{q}(a_{11} + a_{22}) + f_{3}}{P(3)} \end{bmatrix}$$

where

 $\begin{array}{ll} a_{22}a_{33}-a_{23}a_{32}=d_1, & a_{23}a_{31}-a_{21}a_{33}=d_2, & a_{21}a_{32}-a_{22}a_{31}=d_3, \\ a_{13}a_{32}-a_{12}a_{33}=e_1, & a_{11}a_{33}-a_{13}a_{31}=e_2, & a_{12}a_{31}-a_{11}a_{32}=e_3, \\ a_{12}a_{23}-a_{13}a_{22}=f_1, & a_{13}a_{21}-a_{11}a_{23}=f_2, & a_{11}a_{22}-a_{12}a_{21}=f_3. \\ \text{Thus,} \end{array}$

$$U(s) = (s^{q}I - A)^{-1} \{ s^{q-1}u_0 + F(s) \},\$$

becomes

$$U(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}.$$

For simplicity, say

$$\begin{split} &z_0a_{13}+y_0a_{12}-x_0(a_{22}+a_{33})=b_1,\ z_0a_{23}+x_0a_{21}-y_0(a_{11}+a_{33})=b_2,\\ &x_0a_{31}+y_0a_{32}-z_0(a_{11}+a_{22})=b_3,\\ &x_0(a_{22}a_{33}-a_{23}a_{32})+y_0(a_{13}a_{32}-a_{12}a_{33})+z_0(a_{12}a_{23}-a_{13}a_{22})=c_1,\\ &x_0(a_{23}a_{31}-a_{21}a_{33})+y_0(a_{11}a_{33}-a_{13}a_{31})+z_0(a_{13}a_{21}-a_{11}a_{23})=c_2,\\ &x_0(a_{21}a_{32}-a_{22}a_{31})+y_0(a_{12}a_{31}-a_{11}a_{32})+z_0(a_{11}a_{22}-a_{12}a_{21})=c_3.\\ &\text{Then, we have} \end{split}$$

$$\begin{split} U_1(s) &= \frac{s^{3q-1}}{P(3)} x_0 + \frac{s^{2q-1}}{P(3)} b_1 + \frac{s^{q-1}}{P(3)} c_1 + \frac{s^{2q} F_1(s)}{P(3)} \\ &+ \frac{s^q}{P(3)} \{F_3(s) a_{13} + F_2(s) a_{12} - F_1(s) (a_{22} + a_{33})\} + \frac{F_1(s)}{P(3)} d_1 + \frac{F_2(s)}{P(3)} e_1 + \frac{F_3(s)}{P(3)} f_1, \\ U_2(s) &= \frac{s^{3q-1}}{P(3)} y_0 + \frac{s^{2q-1}}{P(3)} b_2 + \frac{s^{q-1}}{P(3)} c_2 + \frac{s^{2q} F_2(s)}{P(3)} \\ &+ \frac{s^q}{P(3)} \{F_1(s) a_{21} + F_3(s) a_{23} - F_2(s) (a_{11} + a_{33})\} + \frac{F_1(s)}{P(3)} d_2) + \frac{F_2(s)}{P(3)} e_2 + \frac{F_3(s)}{P(3)} f_2, \\ U_3(s) &= \frac{s^{3q-1}}{P(3)} z_0 + \frac{s^{2q-1}}{P(3)} b_3 + \frac{s^{q-1}}{P(3)} c_3 + \frac{s^{2q} F_3(s)}{P(3)} \\ &+ \frac{s^q}{P(3)} \{F_1(s) a_{31} + F_2(s) a_{32} - F_3(s) (a_{11} + a_{22})\} + \frac{F_1(s)}{P(3)} d_3 + \frac{F_2(s)}{P(3)} e_3 + \frac{F_3(s)}{P(3)} f_3, \end{split}$$

where P(3) is the three degree polynomial and has three roots.

Case 1: If the roots of P(3) are real and distinct, say λ_1 , λ_2 and λ_3 .

Then, $P(3) = (s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3).$

Now, in order to get the inverse of the Laplace transform of the above expression, let's get the inverse separately.

(a)
$$\mathcal{L}^{-1}\left[\frac{s^{3q-1}}{(s^q-\lambda_1)(s^q-\lambda_2)(s^q-\lambda_3)}\right]$$

Using formula 2 from the Laplace transform table after the partial fraction, we get

$$\mathcal{L}^{-1}\left[\frac{s^{3q-1}}{(s^q-\lambda_1)(s^q-\lambda_2)(s^q-\lambda_3)}\right] = A_1 E_{q,1}(\lambda_1 t^q) + B_1 E_{q,1}(\lambda_2 t^q) + C_1 E_{q,1}(\lambda_3 t^q),$$

where

$$A_1 = \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \ B_1 = \frac{\lambda_2^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \ C_1 = \frac{\lambda_3^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},$$

(b)
$$\mathcal{L}^{-1}\left[\frac{s^{2q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right]:$$

Similarly, we will get

$$\mathcal{L}^{-1}\left[\frac{s^{2q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right] = A_2 E_{q,1}(\lambda_1 t^q) + B_2 E_{q,1}(\lambda_2 t^q) + C_2 E_{q,1}(\lambda_3 t^q),$$

where

$$A_2 = \frac{\lambda_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad B_2 = \frac{\lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \quad C_2 = \frac{\lambda_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.$$
(c) $\mathcal{L}^{-1}\left[\frac{s^{q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right]$:

Similarly, we will get

$$\mathcal{L}^{-1}\left[\frac{s^{q-1}}{(s^q-\lambda_1)(s^q-\lambda_2)(s^q-\lambda_3)}\right] = A_3 E_{q,1}(\lambda_1 t^q) + B_3 E_{q,1}(\lambda_2 t^q) + C_3 E_{q,1}(\lambda_3 t^q),$$

where

$$A_{3} = \frac{1}{(\lambda_{1} - \lambda_{2})(\lambda_{1} - \lambda_{3})}, B_{3} = \frac{1}{(\lambda_{2} - \lambda_{1})(\lambda_{2} - \lambda_{3})}, C_{3} = \frac{1}{(\lambda_{3} - \lambda_{1})(\lambda_{3} - \lambda_{2})}.$$

(d) $\mathcal{L}^{-1}[\frac{s^{2q}F_{1}(s)}{(s^{q} - \lambda_{1})(s^{q} - \lambda_{2})(s^{q} - \lambda_{3})}]:$

Using formula 3 from the Laplace transform table after the partial fraction, we get

$$\mathcal{L}^{-1}\left[\frac{s^{2q}}{(s^{q}-\lambda_{1})(s^{q}-\lambda_{2})(s^{q}-\lambda_{3})}\right] = A_{1}t^{q-1}E_{q,q}(\lambda_{1}t^{q}) + B_{1}t^{q-1}E_{q,q}(\lambda_{2}t^{q}) + C_{1}t^{q-1}E_{q,q}(\lambda_{3}t^{q})$$

Now using the convolution, we get $2a \nabla (\cdot)$

$$\mathcal{L}^{-1}\left[\frac{s^{2q}F_{1}(s)}{(s^{q}-\lambda_{1})(s^{q}-\lambda_{2})(s^{q}-\lambda_{3})}\right]$$

$$=\int_{0}^{t}(t-s)^{q-1}\left\{A_{1}E_{q,q}(\lambda_{1}(t-s)^{q})+B_{1}E_{q,q}(\lambda_{2}(t-s)^{q})+C_{1}E_{q,q}(\lambda_{3}(t-s)^{q})\right\}f_{1}(s)ds.$$
(e) $\mathcal{L}^{-1}\left[\frac{s^{q}F_{3}(s)}{(s^{q}-\lambda_{1})(s^{q}-\lambda_{2})(s^{q}-\lambda_{3})}\right]$:

Similarly, using formula 3 from the Laplace transform table

$$\mathcal{L}^{-1}\left[\frac{s^{q}}{(s^{q}-\lambda_{1})(s^{q}-\lambda_{2})(s^{q}-\lambda_{3})}\right] = A_{2}t^{q-1}E_{q,q}(\lambda_{1}t^{q}) + B_{2}t^{q-1}E_{q,q}(\lambda_{2}t^{q}) + C_{2}t^{q-1}E_{q,q}(\lambda_{3}t^{q}).$$

Now using the convolution, we get

$$\begin{split} \mathcal{L}^{-1}[\frac{s^q F_3(s)}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}] \\ &= \int_0^t (t-s)^{q-1} \{A_3 E_{q,q}(\lambda_1(t-s)^q) + B_3 E_{q,q}(\lambda_2(t-s)^q) + C_3 E_{q,q}(\lambda_3(t-s)^q)) \} f_3(s) ds, \\ \text{and so on for } F_1 \text{ and } F_2. \\ (f) \text{ Similarly, } \mathcal{L}^{-1}[\frac{1 * F_1(s)}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}] \\ &= \int_0^t (t-s)^{q-1} \{A_2 E_{q,q}(\lambda_1(t-s)^q) + B_2 E_{q,q}(\lambda_2(t-s)^q) + C_2 E_{q,q}(\lambda_3(t-s)^q)) \} f_1(s) ds. \\ \text{Now, taking the inverse Laplace transform of } U_1(s), we get \\ x(t) &= \{A_1 E_{q,1}(\lambda_1 t^q) + B_1 E_{q,1}(\lambda_2 t^q) + C_1 E_{q,1}(\lambda_3 t^q) \} x_0 \\ &+ \{A_2 E_{q,1}(\lambda_1 t^q) + B_2 E_{q,1}(\lambda_2 t^q) + C_2 E_{q,1}(\lambda_3 t^q) \} b_1 \\ &+ \{A_3 E_{q,1}(\lambda_1 t^q) + B_3 E_{q,1}(\lambda_2 t^q) + C_3 E_{q,1}(\lambda_3 t^q) \} c_1 \\ &+ \int_0^t (t-s)^{q-1} \{A_1 E_{q,q}(\lambda_1(t-s)^q) + B_1 E_{q,q}(\lambda_2(t-s)^q) + C_1 E_{q,q}(\lambda_3(t-s)^q)) \} f_1(s) ds \\ &+ \int_0^t (t-s)^{q-1} \{A_2 E_{q,q}(\lambda_1(t-s)^q) + B_3 E_{q,q}(\lambda_2(t-s)^q) + C_2 E_{q,q}(\lambda_3(t-s)^q)) \} f_3(s) a_{13} \\ &+ f_2(s) a_{12} - f_1(s)(a_{22} + a_{33}) \} ds + \int_0^t (t-s)^{q-1} \{A_3 E_{q,q}(\lambda_1(t-s)^q) \\ &+ B_3 E_{q,q}(\lambda_2(t-s)^q) + C_3 E_{q,q}(\lambda_3(t-s)^q) \} \{f_1(s) d_1 + f_2(s) e_1 + f_3(s) f_1 \} ds. \end{split}$$

Similarly, taking the inverse Laplace transform of $U_2(s)$ and $U_3(s)$, we get

$$\begin{split} y(t) &= \{A_1 E_{q,1}(\lambda_1 t^q) + B_1 E_{q,1}(\lambda_2 t^q) + C_1 E_{q,1}(\lambda_3 t^q)\} y_0 \\ &+ \{A_2 E_{q,1}(\lambda_1 t^q) + B_2 E_{q,1}(\lambda_2 t^q) + C_2 E_{q,1}(\lambda_3 t^q)\} b_2 \\ &+ \{A_3 E_{q,1}(\lambda_1 t^q) + B_3 E_{q,1}(\lambda_2 t^q) + C_3 E_{q,1}(\lambda_3 t^q)\} c_2 \\ &+ \int_0^t (t-s)^{q-1} \{A_1 E_{q,q}(\lambda_1 (t-s)^q) + B_1 E_{q,q}(\lambda_2 (t-s)^q) + C_1 E_{q,q}(\lambda_3 (t-s)^q)\} f_2(s) ds \\ &+ \int_0^t (t-s)^{q-1} \{A_2 E_{q,q}(\lambda_1 (t-s)^q) + B_2 E_{q,q}(\lambda_2 (t-s)^q) + C_2 E_{q,q}(\lambda_3 (t-s)^q)\} \{f_1(s) a_{21} \\ &+ f_3(s) a_{23} - f_2(s)(a_{11} + a_{33}\} ds + \int_0^t (t-s)^{q-1} \{A_3 E_{q,q}(\lambda_1 (t-s)^q) \\ &+ B_3 E_{q,q}(\lambda_2 (t-s)^q) + C_3 E_{q,q}(\lambda_3 (t-s)^q)\} \{f_1(s) d_2 + f_2(s) e_2 + f_3(s) f_2\} ds \end{split}$$

 and

$$\begin{split} z(t) &= \{A_1 E_{q,1}(\lambda_1 t^q) + B_1 E_{q,1}(\lambda_2 t^q) + C_1 E_{q,1}(\lambda_3 t^q)\} z_0 \\ &+ \{A_2 E_{q,1}(\lambda_1 t^q) + B_2 E_{q,1}(\lambda_2 t^q) + C_2 E_{q,1}(\lambda_3 t^q)\} b_3 \\ &+ \{A_3 E_{q,1}(\lambda_1 t^q) + B_3 E_{q,1}(\lambda_2 t^q) + C_3 E_{q,1}(\lambda_3 t^q)\} c_3 \\ &+ \int_0^t (t-s)^{q-1} \{A_1 E_{q,q}(\lambda_1 (t-s)^q) + B_1 E_{q,q}(\lambda_2 (t-s)^q) + C_1 E_{q,q}(\lambda_3 (t-s)^q))\} f_3(s) ds \\ &+ \int_0^t (t-s)^{q-1} \{A_2 E_{q,q}(\lambda_1 (t-s)^q) + B_2 E_{q,q}(\lambda_2 (t-s)^q) + C_2 E_{q,q}(\lambda_3 (t-s)^q))\} \{f_1(s) a_{31} \\ &+ f_2(s) a_{32} - f_3(s)(a_{11} + a_{22}) ds + \int_0^t (t-s)^{q-1} \{A_3 E_{q,q}(\lambda_1 (t-s)^q) \\ &+ B_3 E_{q,q}(\lambda_2 (t-s)^q) + C_3 E_{q,q}(\lambda_3 (t-s)^q)\} \{f_1(s) d_3 + f_2(s) e_3 + f_3(s) f_3\} ds, \end{split}$$

which is the solution of our systems.

Case 2: If the roots of P(3) are real and equal, say λ . Then, $P(3) = (s^q - \lambda)^3$.

Now, in order to get the inverse of the Laplace transform of the above expression, let's get the inverse separately.

(a)
$$\mathcal{L}^{-1}[\frac{s^{3q-1}}{(s^q-\lambda)^3}]$$
:

Using the partial fraction,

$$\frac{s^{3q-1}}{(s^q - \lambda)^3} = \frac{s^{2q-1}}{(s^q - \lambda)^2} + \frac{s^{2q-1}}{(s^q - \lambda)^2} * \frac{\lambda}{(s^q - \lambda)^2}$$

Using formulas 3 and 5 from the Laplace transform table , we get

$$\mathcal{L}^{-1}\left[\frac{s^{3q-1}}{(s^q - \lambda)^3}\right] = E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q} E_{q,q}(\lambda t^q) \\ + \int_0^t \{E_{q,1}(\lambda (t-s)^q) + \frac{\lambda (t-s)^q}{q} E_{q,q}(\lambda (t-s)^q)\} \lambda s^{q-1} E_{q,q}(\lambda s^q) ds.$$
(b) $\mathcal{L}^{-1}\left[\frac{s^{2q-1}}{(s^q - \lambda)^3}\right]$:

Using the partial fraction,

$$\frac{s^{2q-1}}{(s^q - \lambda)^3} = \frac{s^{2q-1}}{(s^q - \lambda)^2} * \frac{1}{(s^q - \lambda)}$$

Using formulas 3 and 5 from the Laplace transform table , we get

$$\mathcal{L}^{-1}\left[\frac{s^{2q-1}}{(s^q - \lambda)^3}\right] = \int_0^t \{E_{q,1}(\lambda(t-s)^q) + \frac{\lambda(t-s)^q}{q}E_{q,q}(\lambda(t-s)^q)\}s^{q-1}E_{q,q}(\lambda s^q)ds.$$
(c) $\mathcal{L}^{-1}\left[\frac{s^{q-1}}{(s^q - \lambda)^3}\right]$:

Using the partial fraction,

$$\frac{s^{q-1}}{(s^q - \lambda)^3} = \frac{s^{q-1}}{(s^q - \lambda)^2} * \frac{1}{(s^q - \lambda)}.$$

Using the formulas 3 and 4 from the Laplace transform table , we get

$$\mathcal{L}^{-1}\left[\frac{s^{q-1}}{(s^q-\lambda)^3}\right] = \int_0^t \left\{\frac{(t-s)^q}{q} E_{q,q}(\lambda(t-s)^q)\right\} s^{q-1} E_{q,q}(\lambda s^q) ds.$$

$$\frac{{}^q F_1(s)}{(s^q-\lambda)^3} = \frac{1}{2} \sum_{k=1}^{t} \frac{1}{2}$$

(d) $\mathcal{L}^{-1}[\frac{s^{2q}F_1(s)}{(s^q - \lambda)^3}]$:

Using the partial fraction,

$$\frac{s^{2q}}{(s^q - \lambda)^3} = \frac{1}{(s^q - \lambda)} + \frac{2\lambda}{(s^q - \lambda)^2} + \frac{\lambda^2}{(s^q - \lambda)^3}.$$

Using formulas 3, 6 and 12 from the Laplace transform table , we get

$$\mathcal{L}^{-1}\left[\frac{s^{2q}}{(s^{q}-\lambda)^{3}}\right] = t^{q-1}E_{q,q}(\lambda t^{q}) + \sum_{k=0}^{\infty}\frac{(k+1)(k+4)}{2}\frac{\lambda^{k+1}t^{q(k+2)-1}}{\Gamma(qk+2q)}.$$

Now, using the convolution, we get

$$\mathcal{L}^{-1}\left[\frac{s^{2q}F_1(s)}{(s^q-\lambda)^3}\right] = \int_0^t \{(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q) + \sum_{k=0}^\infty \frac{(k+1)(k+4)}{2} \frac{\lambda^{k+1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)}\} f_1(s).$$

(e)
$$\mathcal{L}^{-1}\left[\frac{s^q \{F_3(s)a_{13} + F_2(s)a_{12} - F_1(s)(a_{22} + a_{33})\}}{(s^q - \lambda)^3}\right]$$
:

Using the partial fraction,

$$\frac{s^q}{(s^q - \lambda)^3} = \frac{1}{(s^q - \lambda)^2} + \frac{\lambda}{(s^q - \lambda)^3}$$

Using formulas 6 and 12 from the Laplace transform table , we get

$$\mathcal{L}^{-1}\left[\frac{s^{q}}{(s^{q}-\lambda)^{3}}\right] = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} \frac{\lambda^{k} t^{q(k+2)-1}}{\Gamma(qk+2q)}.$$

Using the convolution, we get

$$\mathcal{L}^{-1}\left[\frac{s^q \{F_3(s)a_{13} + F_2(s)a_{12} - F_1(s)(a_{22} + a_{33})\}}{(s^q - \lambda)^3}\right] = \int_0^t \sum_{k=0}^\infty \frac{(k+1)(k+2)}{2} \frac{\lambda^k (t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \\ * \{f_3(s)a_{13} + f_2(s)a_{12} - f_1(s)(a_{22} + a_{33})\} ds.$$

(f) Using the formula 12 from the Laplace table and using the convolution,

$$\mathcal{L}^{-1}\left[\frac{F_1(s)d_1 + F_2(s)e_1 + F_3(s)f_1}{(s^q - \lambda)^3}\right] = \int_0^t \sum_{k=0}^\infty \frac{k(k+1)}{2} \frac{\lambda^{k-1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_1(s)d_1 + f_2(s)e_1 + f_3(s)f_1\} ds.$$

Now, taking the inverse Laplace transform of $U_1(s)$, we get

$$\begin{split} x(t) &= \{E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q} E_{q,q}(\lambda t^q) + \int_0^t \{E_{q,1}(\lambda (t-s)^q) + \frac{\lambda (t-s)^q}{q} E_{q,q}(\lambda (t-s)^q)\} \\ &\quad * \lambda s^{q-1} E_{q,q}(\lambda s^q) ds \} x_0 + b_1 \int_0^t \{E_{q,1}(\lambda (t-s)^q) + \frac{\lambda (t-s)^q}{q} E_{q,q}(\lambda (t-s)^q)\} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\quad + c_1 \int_0^t \{\frac{(t-s)^q}{q} E_{q,q}(\lambda (t-s)^q)\} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\quad + \int_0^t \{(t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) + \sum_{k=0}^\infty \frac{(k+1)(k+4)}{2} \frac{\lambda^{k+1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \} f_1(s) ds \\ &\quad + \int_0^t \sum_{k=0}^\infty \frac{(k+1)(k+2)}{2} \frac{\lambda^k (t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_3(s)a_{13} + f_2(s)a_{12} - f_1(s)(a_{22} + a_{33})\} ds \\ &\quad + \int_0^t \sum_{k=0}^\infty \frac{k(k+1)}{2} \frac{\lambda^{k-1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_1(s)d_1 + f_2(s)e_1 + f_3(s)f_1\} ds. \end{split}$$

Similarly, taking the inverse Laplace transform of $U_2(s)$ and $U_3(s)$, we get,

$$\begin{split} y(t) &= \{E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q} E_{q,q}(\lambda t^q) + \int_0^t \{E_{q,1}(\lambda (t-s)^q) + \frac{\lambda (t-s)^q}{q} E_{q,q}(\lambda (t-s)^q)\} \\ &\quad * \lambda s^{q-1} E_{q,q}(\lambda s^q) ds \} y_0 + b_2 \int_0^t \{E_{q,1}(\lambda (t-s)^q) + \frac{\lambda (t-s)^q}{q} E_{q,q}(\lambda (t-s)^q) \} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\quad + c_2 \int_0^t \{\frac{(t-s)^q}{q} E_{q,q}(\lambda (t-s)^q) \} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\quad + \int_0^t \{(t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) + \sum_{k=0}^\infty \frac{(k+1)(k+4)}{2} \frac{\lambda^{k+1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \} f_2(s) ds \\ &\quad + \int_0^t \sum_{k=0}^\infty \frac{(k+1)(k+2)}{2} \frac{\lambda^k (t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_1(s)a_{21} + f_3(s)a_{23} - f_2(s)(a_{11} + a_{33})\} ds \\ &\quad + \int_0^t \sum_{k=0}^\infty \frac{k(k+1)}{2} \frac{\lambda^{k-1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_1(s)d_2 + f_2(s)e_2 + f_3(s)f_2\} ds \end{split}$$

and

$$\begin{split} z(t) &= \{E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q} E_{q,q}(\lambda t^q) + \int_0^t \{E_{q,1}(\lambda(t-s)^q) + \frac{\lambda(t-s)^q}{q} E_{q,q}(\lambda(t-s)^q)\} \\ &\quad * \lambda s^{q-1} E_{q,q}(\lambda s^q) ds \} z_0 + b_3 \int_0^t \{E_{q,1}(\lambda(t-s)^q) + \frac{\lambda(t-s)^q}{q} E_{q,q}(\lambda(t-s)^q)\} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\quad + c_3 \int_0^t \{\frac{(t-s)^q}{q} E_{q,q}(\lambda(t-s)^q)\} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\quad + \int_0^t \{(t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) + \sum_{k=0}^\infty \frac{(k+1)(k+4)}{2} \frac{\lambda^{k+1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \} f_3(s) ds \\ &\quad + \int_0^t \sum_{k=0}^\infty \frac{(k+1)(k+2)}{2} \frac{\lambda^k(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_1(s)a_{31} + f_2(s)a_{32} - f_3(s)(a_{11} + a_{22})\} ds \\ &\quad + \int_0^t \sum_{k=0}^\infty \frac{k(k+1)}{2} \frac{\lambda^{k-1}(t-s)^{q(k+2)-1}}{\Gamma(qk+2q)} \{f_1(s)d_3 + f_2(s)e_3 + f_3(s)f_3\} ds. \end{split}$$

Case 3: If the roots of P(3) are one real root, say λ_1 and two complex roots, say $\lambda_2 = \lambda + i\mu$ and $\lambda_3 = \lambda - i\mu$. Then, $P(3) = (s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)$. Now, let's get the inverse Laplace transform term by term.

(a)
$$\mathcal{L}^{-1}\left[\frac{s^{3q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right]$$
: Using the partial fraction,
$$\frac{s^{2q}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)} = \frac{A_1}{s^q - \lambda_1} + \frac{B_1}{s^q - \lambda_2} + \frac{C_1}{s^q - \lambda_3},$$
where

where

$$A_1 = \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \ B_1 = \frac{\lambda_2^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \ C_1 = \frac{\lambda_3^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.$$

Then, using formula 2 from the Laplace transform table, we get

(3.1)
$$\mathcal{L}^{-1}\left[\frac{s^{3q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right] = A_1 E_{q,1}(\lambda_1 t^q) + B_1 E_{q,1}((\lambda + i\mu)t^q) + C_1 E_{q,1}((\lambda - i\mu)t^q).$$

From equations (2.9) and (2.10), we have

$$Gcos_{q,1}((\lambda + i\mu)t^q) \pm i \ Gsin_{q,1}((\lambda + i\mu)t^q) = E_{q,1}((\lambda \pm i\mu)t^q).$$

Then (3.1) becomes,

$$\mathcal{L}^{-1} \frac{s^{3q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)} = A_1 E_{q,1}(\lambda_1 t^q) + (B_1 + C_1) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_1 - C_1) i \ G \sin_{q,1}((\lambda + i\mu)t^q).$$

(b) Similarly, using formula 2 from the Laplace transform table, we get,

$$\mathcal{L}^{-1}\left[\frac{s^{2q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right] = A_2 E_{q,1}(\lambda_1 t^q) + (B_2 + C_2) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_2 - C_2) i \ G \sin_{q,1}((\lambda + i\mu)t^q),$$

where

$$A_2 = \frac{\lambda_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad B_2 = \frac{\lambda_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \quad C_2 = \frac{\lambda_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.$$

(c) Similarly,

$$\mathcal{L}^{-1}\left[\frac{s^{q-1}}{(s^q - \lambda_1)(s^q - \lambda_2)(s^q - \lambda_3)}\right] = A_3 E_{q,1}(\lambda_1 t^q) + (B_3 + C_3)Gcos_{q,1}((\lambda + i\mu)t^q) + (B_3 - C_3)i\ Gsin_{q,1}((\lambda + i\mu)t^q),$$

where

$$A_{3} = \frac{1}{(\lambda_{1} - \lambda_{2})(\lambda_{1} - \lambda_{3})}, \ B_{3} = \frac{1}{(\lambda_{2} - \lambda_{1})(\lambda_{2} - \lambda_{3})}, \ C_{3} = \frac{1}{(\lambda_{3} - \lambda_{1})(\lambda_{3} - \lambda_{2})}$$

(d) Using formula 3 from the Laplace transform table after partial fraction, we get

$$\mathcal{L}^{-1}\left[\frac{s^{2q}}{(s^{q}-\lambda_{1})(s^{q}-\lambda_{2})(s^{q}-\lambda_{3})}\right] = A_{1}t^{q-1}E_{q,q}(\lambda_{1}t^{q}) + t^{q-1}\{(B_{1}+C_{1})G\cos_{q,q}((\lambda+i\mu)t^{q}) + (B_{1}-C_{1})i\ Gsin_{q,q}((\lambda+i\mu)t^{q})\}.$$

(e) Using formula 3 from the Laplace transform table after partial fraction, we get

$$\mathcal{L}^{-1} \frac{s^{q}}{(s^{q} - \lambda_{1})(s^{q} - \lambda_{2})(s^{q} - \lambda_{3})} = A_{2}t^{q-1}E_{q,q}(\lambda_{1}t^{q}) + t^{q-1}\{(B_{2} + C_{2})Gcos_{q,q}((\lambda + i\mu)t^{q}) + (B_{2} - C_{2})i\ Gsin_{q,q}((\lambda + i\mu)t^{q})\}.$$

(f) Similarly,

$$\mathcal{L}^{-1}\left[\frac{1}{(s^{q}-\lambda_{1})(s^{q}-\lambda_{2})(s^{q}-\lambda_{3})}\right] = A_{3}t^{q-1}E_{q,q}(\lambda_{1}t^{q}) + t^{q-1}\{(B_{3}+C_{3})Gcos_{q,q}((\lambda+i\mu)t^{q}) + (B_{3}-C_{3})i\ Gsin_{q,q}((\lambda+i\mu)t^{q})\}.$$

Now, taking the inverse Laplace transform of $U_1(s)$, we get

$$\begin{split} x(t) &= \{A_1 E_{q,1}(\lambda_1 t^q) + (B_1 + C_1) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_1 - C_1)i \ G \sin_{q,1}((\lambda + i\mu)t^q)\} x_0 \\ &+ \{A_2 E_{q,1}(\lambda_1 t^q) + (B_2 + C_2) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_2 - C_2)i \ G \sin_{q,1}((\lambda + i\mu)t^q)\} b_1 \\ &+ \{A_3 E_{q,1}(\lambda_1 t^q) + (B_3 + C_3) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_3 - C_3)i \ G \sin_{q,1}((\lambda + i\mu)t^q))\} c_1 \\ &+ \int_0^t [A_1 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{(B_1 + C_1) G \cos_{q,q}((\lambda + i\mu)(t - s)^q) + (B_1 - C_1) \\ &* i \ G \sin_{q,q}((\lambda + i\mu)(t - s)^q)\}] f_1(s) ds + \int_0^t [A_2 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{(B_2 + C_2) \\ &* G \cos_{q,q}((\lambda + i\mu)(t - s)^q) + (B_2 - C_2)i \ G \sin_{q,q}((\lambda + i\mu)(t - s)^q)\}] * \{f_3(s)a_{13} + f_2(s)a_{12} \\ &- f_1(s)(a_{22} + a_{33})\} ds + \int_0^t [A_3 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{(B_3 + C_3) G \cos_{q,q}((\lambda + i\mu)(x + i\mu)$$

Similarly, taking the inverse Laplace transform of $U_2(s)$ and $U_3(s)$, we get

$$\begin{split} y(t) &= \{A_1 E_{q,1}(\lambda_1 t^q) + (B_1 + C_1) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_1 - C_1)i \; G \sin_{q,1}((\lambda + i\mu)t^q)\} y_0 \\ &+ \{A_2 E_{q,1}(\lambda_1 t^q) + (B_2 + C_2) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_2 - C_2)i \; G \sin_{q,1}((\lambda + i\mu)t^q)\} b_2 \\ &+ \{A_3 E_{q,1}(\lambda_1 t^q) + (B_3 + C_3) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_3 - C_3)i \; G \sin_{q,1}((\lambda + i\mu)t^q))\} c_2 \\ &+ \int_0^t [A_1 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{(B_1 + C_1) G \cos_{q,q}((\lambda + i\mu)(t - s)^q) + (B_1 - C_1) \\ &* i \; G \sin_{q,q}((\lambda + i\mu)(t - s)^q)\}] f_2(s) ds + \int_0^t [A_2 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{(B_2 + C_2) \\ &* G \cos_{q,q}((\lambda + i\mu)(t - s)^q) + (B_2 - C_2)i \; G \sin_{q,q}((\lambda + i\mu)(t - s)^q)\}] * \{f_1(s)a_{21} + f_3(s)a_{23} \\ &- f_2(s)(a_{11} + a_{33})\} ds + \int_0^t [A_3 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{(B_3 + C_3) * G \cos_{q,q}((\lambda + i\mu)(x + i$$

and

$$\begin{split} z(t) &= \{A_1 E_{q,1}(\lambda_1 t^q) + (B_1 + C_1) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_1 - C_1)i \; G \sin_{q,1}((\lambda + i\mu)t^q) \} z_0 \\ &+ \{A_2 E_{q,1}(\lambda_1 t^q) + (B_2 + C_2) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_2 - C_2)i \; G \sin_{q,1}((\lambda + i\mu)t^q) \} b_3 \\ &+ \{A_3 E_{q,1}(\lambda_1 t^q) + (B_3 + C_3) G \cos_{q,1}((\lambda + i\mu)t^q) + (B_3 - C_3)i \; G \sin_{q,1}((\lambda + i\mu)t^q)) \} c_3 \\ &+ \int_0^t [A_1 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{ (B_1 + C_1) G \cos_{q,q}((\lambda + i\mu)(t - s)^q) + (B_1 - C_1) \\ &* i \; G \sin_{q,q}((\lambda + i\mu)(t - s)^q) \}] f_3(s) ds + \int_0^t [A_2 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{ (B_2 + C_2) \\ &* G \cos_{q,q}((\lambda + i\mu)(t - s)^q) + (B_2 - C_2)i \; G \sin_{q,q}((\lambda + i\mu)(t - s)^q) \}] * \{f_1(s)a_{31} + f_2(s)a_{32} \\ &- f_3(s)(a_{11} + a_{22}) \} ds + \int_0^t [A_3 E_{q,q}(\lambda_1 (t - s)^q) + (t - s)^{q-1} \{ (B_3 + C_3) * G \cos_{q,q}((\lambda + i\mu)) \\ &* (t - s)^q) + (B_3 - C_3)i \; G \sin_{q,q}((\lambda + i\mu)(t - s)^q) \}] \{f_1(s)d_3 + f_2(s)e_3 + f_3(s)f_3\} ds. \end{split}$$

EXAMPLE

Consider the 3q order linear sequential Caputo fractional differential equation

(3.2)
$${}^{c}D_{0+}^{3q}u(t) + a {}^{c}D_{0+}^{2q}u(t) + b {}^{c}D_{0+}^{q}u(t) = \lambda u, \qquad 2 < 3q \le 3,$$

subject to initial conditions

$$u(0) = a_0, \quad {}^c D^q_{0+} u(0) = a_1, \quad {}^c D^{2q}_{0+} u(0) = a_2$$

where λ is a real number. To convert (3.2) into three systems of linear Caputo fractional differential equations of order q, we assume

$$u_{1}(t) = u(t),$$

$$u_{2}(t) = {}^{c}D_{0+}^{q}u(t),$$

$$u_{3}(t) = {}^{c}D_{0+}^{2q}u(t).$$

Then, we have

$$\begin{cases} {}^{c}D_{0+}^{q}u_{1}(t) = u_{2}(t), & u_{1}(0) = a_{0} \\ {}^{c}D_{0+}^{q}u_{2}(t) = u_{3}(t), & u_{2}(0) = a_{1} \\ {}^{c}D_{0+}^{q}u_{3}(t) = \lambda u_{1}(t) - bu_{2}(t) - au_{3}(t), & u_{3}(0) = a_{2}. \end{cases}$$

Hence, the above system can be written as the three systems of linear Caputo fractional differential equations of order q in the following matrix form:

$${}^{c}D_{0+}^{q}u(t) = Au(t), \qquad u(0) = u_{0}, \qquad 0 < q \le 1,$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -b & -a \end{bmatrix}, u(t) = \begin{bmatrix} u_{1}(t) \\ u_{2}(t) \\ u_{3}(t) \end{bmatrix} \text{ and } u_{0} = \begin{bmatrix} a_{0} \\ a_{1} \\ a_{2} \end{bmatrix}.$

Now, (3.2) has been reduced to 3 systems of linear Caputo fractional differential equations, which can be solved by the method we have already developed.

3.2. Solving Fractional Non-linear SIR Model using Laplace-Adomian decomposition method. In this section, we are going to consider the simple epidemic model, which is compartmental SIR model with a fractional derivative in place of the usual integer derivative. The following SIR model consists of three non-linear systems of Caputo fractional differential equations, which are in the form:

(3.3)
$${}^{c}D_{0+}^{q}S(t) = -\beta S(t)I(t)$$
$${}^{c}D_{0+}^{q}I(t) = \beta S(t)I(t) - \gamma I(t)$$
$${}^{c}D_{0+}^{q}R(t) = \gamma I(t),$$

with the initial conditions

$$S(0) = N_1, \qquad I(0) = N_2, \qquad R(0) = N_3$$

where β is the contact rate and γ is the recovery rate. The total number of population N is divided into three sections at time t, where S(t) is the number of susceptible population, I(t) is the number of infected population and R(t) is the number of recovered population from the disease. In this model, we are assuming that the birth and death rates are the same during the small period of epidemic, there is no immigration, and recovered individuals are immune to disease, so that the population is constant during that time. So we have, N(t) = S(t) + I(t) + R(t) for any time t. Susceptible individuals are all of the individuals that are capable of becoming sick from an infection. Then those who are infected with the disease leave the susceptible category and are transmitted into the infected category. After some time, infected individuals moved to recovered individuals.

Initially, we will develop a theoretical method of computing the solution by the Laplace-Adomian decomposition method and plot some numerical results. Now, taking the Laplace transform of first equation of (3.3), we have

$$s^{q}\mathcal{L}(S(t)) - s^{q-1}S(0) = -\beta\mathcal{L}(S(t)I(t)),$$

and hence

$$\mathcal{L}(S(t)) = \frac{N_1}{s} - \frac{\beta}{s^q} \mathcal{L}(S(t)I(t)).$$

Similarly, using the initial conditions on all three equations of (3.3), we will have

$$\begin{cases} \mathcal{L}(S(t)) &= \frac{N_1}{s} - \frac{\beta}{s^q} \mathcal{L}(S(t)I(t)) \\ \mathcal{L}(I(t)) &= \frac{N_2}{s} + \frac{\beta}{s^q} \mathcal{L}(S(t)I(t)) - \frac{\gamma}{s^q} \mathcal{L}(I(t)) \\ \mathcal{L}(R(t)) &= \frac{N_3}{s} - \frac{\gamma}{s^q} \mathcal{L}(I(t)). \end{cases}$$

Now, suppose that the solutions S(t), I(t) and R(t) are in the form of infinite series as

$$S(t) = \sum_{k=0}^{\infty} S_k(t), \qquad I(t) = \sum_{k=0}^{\infty} I_k(t), \qquad R(t) = \sum_{k=0}^{\infty} R_k(t),$$

and the non-linear term S(t)I(t) is decomposed in the form of

$$S(t)I(t) = \sum_{k=0}^{\infty} A_k(t),$$

where A_k is given by

$$A_{k} = \frac{1}{\Gamma(k+1)} \frac{d^{k}}{d\lambda^{k}} \left[\sum_{j=0}^{k} \lambda^{k} S_{k}(t) \cdot \sum_{j=0}^{k} \lambda^{k} I_{k}(t) \right] \Big|_{\lambda=0}.$$

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Then, we will have

$$A_{0} = S_{0}I_{0}$$

$$A_{1} = S_{1}I_{0} + S_{0}I_{1}$$

$$A_{2} = S_{2}I_{0} + S_{1}I_{1} + S_{0}I_{2}$$

$$A_{3} = S_{3}I_{0} + S_{2}I_{1} + S_{1}I_{2} + S_{0}I_{3},$$

and so on.

Now, substituting all the series in the above systems yields

$$\mathcal{L}(\sum_{k=0}^{\infty} S_k(t)) = \frac{N_1}{s} - \frac{\beta}{s^q} \mathcal{L}(\sum_{k=0}^{\infty} A_k(t))$$
$$\mathcal{L}(\sum_{k=0}^{\infty} I_k(t)) = \frac{N_2}{s} + \frac{\beta}{s^q} \mathcal{L}(\sum_{k=0}^{\infty} A_k(t)) - \frac{\gamma}{s^q} \mathcal{L}(\sum_{k=0}^{\infty} I_k(t))$$
$$\mathcal{L}(\sum_{k=0}^{\infty} R_k(t)) = \frac{N_3}{s} - \frac{\gamma}{s^q} \mathcal{L}(\sum_{k=0}^{\infty} I_k(t)).$$

Comparing both sides of each individual system, we will get the following kind of recurrence relation:

$$\mathcal{L}(S_0(t)) = \frac{N_1}{s} \quad and \quad \mathcal{L}(S_k(t)) = -\frac{\beta}{s^q} \mathcal{L}(A_{k-1})$$

$$\mathcal{L}(I_0(t)) = \frac{N_2}{s} \quad and \quad \mathcal{L}(I_k(t)) = \frac{\beta}{s^q} \mathcal{L}(A_{k-1}) - \frac{\gamma}{s^q} \mathcal{L}(I_{k-1})$$

$$\mathcal{L}(R_0(t)) = \frac{N_3}{s} \quad and \quad \mathcal{L}(R_k(t)) = -\frac{\gamma}{s^q} \mathcal{L}(I_{k-1}),$$

for k = 1, 2, ...

To find S(t), I(t) and R(t), we have initially given S_0, I_0, R_0 and can determine S_1, I_1, R_1 as

$$S_{0}(t) = N_{1} \\ I_{0}(t) = N_{2} \\ R_{0}(t) = N_{3} \\ \end{cases} \quad and \quad \begin{cases} S_{1}(t) = \frac{-\beta N_{1} N_{2}}{\Gamma(q+1)} t^{q} \\ I_{1}(t) = \frac{(\beta N_{1} N_{2} - \gamma N_{2})}{\Gamma(q+1)} t^{q} \\ R_{1}(t) = \frac{\gamma N_{2}}{\Gamma(q+1)} t^{q}. \end{cases}$$

Similarly, using the above values

$$\int S_{2}(t) = \frac{\beta N_{1} N_{2} \{\beta (N_{2} - N_{1}) + \gamma\}}{\Gamma(2q + 1)} t^{2q}$$

$$I_{2}(t) = \frac{\{(\beta N_{1} - \gamma)^{2} N_{2} - \beta^{2} N_{1} N_{2}^{2}\}}{\Gamma(2q + 1)} t^{2q}$$

$$R_{2}(t) = \frac{\gamma (\beta N_{1} N_{2} - \gamma N_{2})}{\Gamma(2q + 1)} t^{2q}$$

and

$$S_{3}(t) = \left[-N_{1}N_{2}\{(\beta N_{1} - \gamma)^{2} + \beta N_{2}(\beta N_{2} + \gamma - 2\beta N_{1})\} - \frac{(\beta \gamma N_{1}N_{2}^{2} - \beta^{2}N_{1}^{2}N_{2}^{2})\Gamma(2q + 1)}{\Gamma(q + 1)^{2}} \right] \frac{\beta t^{3q}}{\Gamma(3q + 1)}$$

$$I_{3}(t) = \left[\beta N_{1}N_{2}\{(\beta N_{1} - \gamma)^{2} - 2\beta^{2}N_{1}N_{2} + \beta^{2}N_{2}^{2} + \beta\gamma N_{2}\} + \frac{\beta(\beta \gamma N_{1}N_{2}^{2} - \beta^{2}N_{1}^{2}N_{2}^{2})\Gamma(2q + 1)}{\Gamma(q + 1)^{2}} - \gamma\{(\beta N_{1} - \gamma)^{2}N_{2} - \beta^{2}N_{1}N_{2}^{2}\} \right] \frac{t^{3q}}{\Gamma(3q + 1)}$$

$$R_{3}(t) = \gamma\{(\beta N_{1} - \gamma)^{2}N_{2} - \beta^{2}N_{1}N_{2}^{2}\} \frac{t^{3q}}{\Gamma(3q + 1)}.$$

Similarly, we can find the other terms of the series recursively, and hence the solution in the form of a series is

$$S(t) = S_0 + S_1 + S_2 + \dots, \qquad I(t) = I_0 + I_1 + I_2 + \dots, \qquad R(t) = R_0 + R_1 + R_2 + \dots$$

Here, we have computed the first four terms of the series as a solution and will check the behavior of the solution for different orders q of the fractional derivative. Here, we have considered the parameters $\beta = 0.01$, $\gamma = 0.02$, $N_1 = 20$, $N_2 = 10$ and $N_3 = 5$.

Here, we have plotted the figure of Susceptible Population S(t), Infected Population I(t) and Recovered Population R(t) with the given parameters for different fractional orders q.



FIGURE 1. S(t)







FIGURE 3. R(t)

4. Concluding Remarks

In this work, we have developed a method to solve the three linear non-homogeneous systems of Caputo fractional differential equations of order q, where $0 < q \leq 1$, with initial conditions using the Laplace transform method. All our methods developed here yield the corresponding integer results as a special case.

In addition, our study also yields the study of nq order linear sequential Caputo fractional differential equations with fractional initial conditions as a special case. Our solution method yields the stability results of the equilibrium solutions of any non-linear Caputo differential systems with initial conditions. An important observation is that the equilibrium solution of the linear Caputo fractional differential system may be asymptotically stable even when the corresponding solution of the equilibrium solution of an integer system is locally stable. We have considered a non-linear systems of compartmental SIR model, and developed some theoretical and numerical results for the corresponding fractional SIR model using the Laplace-Adomian decomposition method. Our aim here is to choose the value of q, the order of the derivative appropriately such that our solution is closer to the data compared with the solution of the corresponding integer derivative. Thus, the value of q can be used as a parameter to enhance the mathematical model. In our future work, we plan to study non-linear systems related to infectious disease models, specifically COVID-19 SIR and SEIR models.

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