Neural, Parallel, and Scientific Computations 29 (2021), No.4, 230-251

ON RANDOM POLYNOMIALS-II: A SURVEY

V. THANGARAJ AND M. SAMBANDHAM

Formerly at Ramanujan Institute for Advanced Study in Mathematics,
University of Madras, Chennai - 600005, India
thangarajv@yahoo.com
Department of Mathematics, Morehouse College, Atlanta, GA 30314, U.S.A.
msambandham@yahoo.com

ABSTRACT. Herein, we continue to summarise a set of fundamental results of various random polynomials including orthogonal polynomials. Also we outline how Kac-Rice formula is generalised in various dimensions. This paper contains the second part of survey of selected results on the real/complex zeros of random polynomials and asymptotic results of expected number of zeros of random polynomials in higher dimensions. Expected number of zeros of random orthogonal polynomials is methodologically presented to initiate further research.

Mathematics Subject Classification(2020): 60H25, 47B80, 47H40

Key Words and Phrases: Concentration of zeros, Kac-Rice Formula higher dimensions, Random Orthogonal Polynomials.

1 INTRODUCTION

The study of the roots of random polynomials is among the most important and popular topics in Mathematical Analysis and in some areas of Physics. For almost a century, a considerable amount of literature about this problem has been studied via fields such as probability, geometry, random algebraic geometry, algorithm complexity, quantum physics, etc. In spite of its rich history, it is still an extremely active field of research. There are several reasons that lead to consider random polynomials and several ways to randomize them (see Bharucha-Reid and Sambandham [4] and Farahmand [15]).

Received March 1, 2021
www.dynamicpublishers.org; www.dynamicpublishers.com;
https://doi.org/10.46719/npsc20212944.

2 CONCENTRATION OF ZEROS OF RANDOM POLYNOMIALS: WHERE? WHEN? HOW?

In this section, we collect results on the concentration of zeros of random polynomials. Our aim is to present the some cherries on the top of ice cream. We are not establishing any new results.

2.1 NEAR THE POINTS -1 AND +1

We recall that a random Kac polynomial is of the form $F(t) = \sum_{j=0}^{n} X_j t^j$ where the coefficients X_j are independent Gaussian random variables of mean zero and variance one. A classical result of Kac [21] asserts that the zeros of Kac random polynomials of large degree tend to accumulate around +1 and -1.

2.2 NEAR THE UNIT CIRCLE

In the case of random polynomials when the coefficients are complex standard Gaussian random variables in

$$F(z) = F_n(z) = \sum_{j=0}^{n-1} X_j z^j, z \in \mathbb{C},$$

Hammersley [19] has proved that the zeros lie on the unit circle $S^1 = \{|z| = 1\}$. This ensemble of random polynomials has been studied in detail in [20],[24], [30],[35], and references therein. Recently, Ibragimov and Zaporozhets [24] have proved that for independent and identically distributed (IID) real or complex random variables X_j,

$$\mathbb{E}[\log(1 + |X_j|)] < 1 \tag{2.1}$$

is a necessary and sufficient condition for zeros of random Kac polynomials with complex coefficients to accumulate near the unit circle.

Remark 2.1. We notice that the asymptotic distribution of zeros of Kac polynomials is independent of the choice of the probability law of random coefficients under condition (2.1). This phenomenon is referred to as *global universality* for zeros of random polynomials.

2.3 NEAR THE BOUNDARY OF A SIMPLY CONNECTED DOMAIN

Shiffman and Zelditch [36] have remarked that it is an implicit choice of an inner product that has produced the concentration of zeros of Kac polynomials with complex Gaussian coefficients around the unit circle S^1. More generally, for a simply connected domain $\Omega \Subset \mathbb{C}$ with real analytic boundary $\partial\Omega$ and a fixed orthonormal basis (ONB) $\left\{P_j\right\}_{j=1}^{n+1}$ induced by a measure $\rho(z)|\mathrm{d}z|$ where $\rho \in \mathcal{C}^{\omega}(\partial\Omega)$ and $|dz|$ denote arc-length, Shiffman and Zelditch have proved that zeros of random polynomials

$$F(z) = \sum_{j=1}^{n+1} X_j P_j(z) \text{ where } X_j \text{ IID}$$

concentrate near the boundary $\partial\Omega$ as $n \to \infty$.

2.4 NEAR THE SUPPORT OF EQUILIBRIUM MEASURE

Furthermore, the empirical measures δ_z of zeros

$$\frac{1}{n}\sum_{\{z:F(z)=0\}}\delta_z$$

converge weakly to the equilibrium measure $\mu_{\bar{\Omega}}$. Recall that for a non-polar compact set $K \subset \mathbb{C}$ the equilibrium measure μ_K is the unique minimizer of the logarithmic energy functional

$$\nu \to \iint \log\frac{1}{|z-w|}\,\mathrm{d}\nu(z)\,\mathrm{d}\nu(w)$$

over all probability measures supported on K. Later, Bloom [6] has observed that $\bar{\Omega}$ is replaced by a regular compact set $K \subset \mathbb{C}$, the inner product has been defined in terms of any Bernstein Markov measure (see also [9] for a generalization of this result to $\mathbb{C}^m$ for Gaussian random pluricomplex polynomials). More recently, Pritsker and Ramachandran [38] have observed that (2.1) is a necessary and sufficient condition for zeros of random linear combinations of Szegö, Bergman, or Faber polynomials (associated with Jordan domains bounded with analytic curves) to accumulate near the support of the corresponding equilibrium measure.

3 EXPECTED NUMBER OF ZEROS OF RANDOM POLYNOMIALS IN HIGHER DIMENSIONS

We trace the generalisation of the classical result of Kac and independently by Rice on the estimation of number of real zeros of random polynomials. The Kac-Rice formula derived by them has manifested in higher dimensions which are totally astonishing facts. Let us track its path in the dense forest of random polynomials. Let us now travel in a new cosmos.

3.1 ZEROS OF RANDOM POLYNOMIALS IN $\mathbb{R}$

The asymptotic and bound type results of Kac [21] are sketched here for reference.

Theorem 3.1 ([21]). *Let $F_n(z) = \sum_{j=0}^{n-1} X_j z^j, z \in \mathbb{R}$ be the Kac's random real algebraic polynomial where $(X_1, X_2, \ldots, X_{n-1}) \sim \mathcal{N}(0, \mathbb{I}_{n\times n-1})$.*

$$\mathbb{P}\left\{\bar{X} = (X_0, \ldots, X_{n-1}) \in A\right\} = \int_A \frac{e^{-\|x\|^2/2}}{(2\pi)^{n/2}}\mathrm{d}x.$$

Let $\mathcal{N}_n$ be the number of real zeroes of F. Then

$$\mathbb{E}\{\mathcal{N}_n\} = \frac{4}{\pi}\int_0^1 \frac{\{1-[nx^{n-1}(1-x^2)/(1-x^{2n})]^2\}^{1/2}}{1-x^2}\mathrm{d}x$$

For large n, the asymptotic value of

$$\mathbb{E}\{\mathcal{N}_n\} \sim \frac{2\log n}{\pi}.$$

Further for large n, an upper bound is

$$\mathbb{E}\{\mathcal{N}_n\} \leq \frac{2\log n}{\pi} + \frac{14}{\pi}.$$

In 1969 , Stevens [43] has obtained

$$\frac{2}{\pi}\log n - 0.6 \leq \mathbb{E}\mathcal{N}_{n-1} \leq \frac{2}{\pi}\log n + 1.4$$

which is an improvement of both lower and upper bounds for the expected number of real zeros of a Kac random polynomial.

In 1973, Wilkins [48] has further improved Kac's upper bound to

$$\mathbb{E}\mathcal{N}_n < \tfrac{2}{\pi}\log n + 1.116 \quad (n \text{ odd })$$
$$\mathbb{E}\mathcal{N}_n < \tfrac{2}{\pi}\log n + 1.113 \quad (n \text{ even })$$

3.2 KAC-RICE FORMULA IN TWO DIMENSIONS

A generalisation of Euler's observation of 1751 (in fact already noted by Descartes in 1639) is known as Euler characteristic that on "triangulating" a sphere into F regions, E edges, and V vertices, we have $V - E + F = 2$. If one triangulates any surface then $\chi = V - E + F$ is a number which does not depend on how the triangulation is done. In algebraic topology and polyhedral combinatorics, the Euler characteristic (or Euler number, or Euler-Poincaré characteristic) is a topological invariant, a number that describes a topological space's shape or structure regardless of the way it is twisted. In general it is denoted by χ (Greek lower-case letter chi). The Euler characteristic of a set $A \subset \mathbb{R}^2$, denoted $\chi(A)$ whenever it is well defined, is a topological invariant used in many circumstances viz., in the study of nonparametric spatial statistics, random media, topological index in astronomy, brain imagery, or oceanography. Geometric properties of level sets of multivariate random field have occupied the researchers for the past two decades. We invoke here some known definitions for our discussion. Here one may feel how the Kac-Rice formula has been generalised on the lines of one dimension case.

In general, let $m \geqslant 1, W \subset \mathbb{R}^m$ be measurable. Let f be a $\mathcal{C}^{1,1}$ function on W, i.e. continuously differentiable with Lipschitz gradient. Note $M(f) = \{x \in W : \nabla f(x) = 0\}$ is the *set of critical points,* and for $u \in \mathbb{R}, M(f,u) = M(f) \cap \{f \geqslant u\}$. Also note that $V(f) = f(M(f))$ is the *set of critical values.*

It is very well-known that the Sard's theorem, also known as Sard's lemma or the Morse–Sard theorem, is a result in mathematical analysis that asserts that the set of critical values (that is, the image of the set of critical points) of a smooth function f from one Euclidean space or manifold to another is a null set, i.e., it has Lebesgue measure 0. By Sard's Theorem, $V(f)$ has Lebesgue measure 0 has been used frequently in the proofs.

We state here the known definitions for our purpose.

Definition 3.2. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, the set $\{f \geqslant u\} = \left\{x \in \mathbb{R}^2 : f(x) \geqslant u\right\}$, for $u \in \mathbb{R}$ is called an *excursion set,* or *upper-level set* of f.

Definition 3.3. A $\mathcal{C}^2$ function $f : \mathbb{R}^2 \to \mathbb{R}$ is called a *Morse* function if all its critical points are non-degenerate, i.e. if for $x \in M(f)$, the Hessian matrix

$$H_f(x) = \begin{pmatrix} \partial^2_{11} f(x) & \partial^2_{12} f(x) \\ \partial^2_{21} f(x) & \partial^2_{22} f(x) \end{pmatrix}$$

of f at x is non-singular.

Definition 3.4. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *Morse above some value* $u \in \mathbb{R}$ if $H_f(x)$ is non-singular for $x \in M(f, u)$. In that case, the set of critical points $x \in M(f)$ with $f(x) \geqslant u$ is locally finite. Also, the number of positive eigenvalues of $H_f(x)$ is called the index of $x \in M(f)$.

Many results on the mean Euler characteristic of random excursions address Gaussian random fields, because their finite dimensional distributions are easier to handle, and more general results require the field to satisfy strong density requirements. But we note that the general variographic approach [28] to compute the mean value of a bidimensional weak version of the Euler characteristic which does not require density hypothesis.

Definition 3.5. Given a sufficiently regular function f on $\mathbb{R}^2$, the *Euler primitive* of f, to a smooth test function $h : \mathbb{R} \to \mathbb{R}$ is

$$\chi_f(h) = \int_{\mathbb{R}} h(u)\chi(\{f \geqslant u\})\mathrm{d}u.$$

It is known that the Euler primitive can be written as a proper Lebesgue integral over $\mathbb{R}^2$, involving the first and second order derivatives of f. Here $\mathbf{u}_1, \mathbf{u}_2$ are the canonical basis of $\mathbb{R}^2$, $\partial_i f$ are the partial derivative of f along $\mathbf{u}_i$, $i = 1, 2$, and $\partial^2_{ii} f$ is the second order partial derivative in direction $\mathbf{u}_i$.

Given a $\mathcal{C}^{1,1}$ function f over some bounded measurable $W \subset \mathbb{R}^2, f$ is twice differentiable a.e., and for any $\mathcal{C}^1$ function $h : \mathbb{R} \to \mathbb{R}$, introduce for $i \in \{1, 2\}$,

$$\gamma_i(x, f, h) = 1_{\{\nabla f(x) \in Q_i\}} \left[\partial_i f(x)^2 h'(f(x)) + \partial^2_{ii} f(x) h(f(x))\right], x \in W$$

$\gamma(x, f, h) = \gamma_1(x, f, h) + \gamma_2(x, f, h)$, and $I_f(h) = \int_W \gamma(x, f, h)\mathrm{d}x$. In addition, we might ask additional properties from the test function h, such as that to be twice continuously differentiable, or have compact support. For details, one may refer to [26].

Theorem 3.6 ([26]). *Let $h : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function with compact support. Let $f : W \subset \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function which is Morse above* $\min(\mathrm{supp}(h))$, *and such that* $\{f \geqslant \min(\mathrm{supp}(h))\}$ *is compact and contained in W 's interior. Then $\chi_f(h)$ is well defined and*

$$\chi_f(h) = I_f(h).$$

If f is Morse and has compact excursion sets, then Theorem 3.6 leads to

$$\chi_f(h) = -\sum_{i=1}^{2} \int_{\mathbb{R}^2} \left[h'(f(x))\partial_i f(x)^2 + h(f(x))\partial_{ii}^2 f(x)\right] 1_{\{\nabla f(x)\in Q_i\}} \mathrm{d}x \tag{3.1}$$

where Q_1 and Q_2 are two quarter planes defined as

$$Q_1 = \{(s,t) \in \mathbb{R}^2 : t < s < 0\}, Q_2 = \{(s,t) \in \mathbb{R}^2 : s < t < 0\}.$$

This formula (3.1) is the two-dimensional version of Kac-Rice formula for Morse function.

The classical literature gives the Euler characteristic of an excursion in function of the indexes of its critical points above the considered level. Namely, for a Morse function f and $u \in \mathbb{R}$ such that $\{f \geqslant u\}$ is compact and does not have critical points on its boundary, we have

$$\chi(\{f \geqslant u\}) = \sum_{k=0}^{2} (-1)^k \mu_k(f,u) \tag{3.2}$$

where $\mu_k(f,u) = \#\{x \in \{f \geqslant u\} : \nabla f(x) = 0$ and the Hessian matrix of f in x has exactly k positive eigenvalues $\}$.

Remark 3.7. In practice, use of this formula in dimension 2, see [13], is the counting measure on the right hand side that is captured through an integral over a neighbourhood of the critical points, see [1],

$$\chi(\{f \geqslant u\}) = \lim_{\varepsilon\to 0} \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^2} 1_{\{\|\nabla f(x)\|_\infty \leqslant \varepsilon\}} 1_{\{f(x)\geqslant u\}} \det\left(H_f(x)\right) \mathrm{d}x$$

Remark 3.8. Another known formula to which (3.1) can be compared is the one-dimensional version of the co-area formula. To introduce this formula in $\mathbb{R}^2$, call perimeter of a set $A \subset \mathbb{R}^2$, denoted by Per(A), the 1−dimensional Hausdorff measure of its topological boundary. In $\mathbb{R}^2$, the co-area formula expresses the perimeter of the level sets of a locally Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$, as a function of a differential operator applied to f. For $h : \mathbb{R} \to \mathbb{R}$, a bounded measurable function is given by

$$\int_{\mathbb{R}} h(u)\,\mathrm{Per}(\{f \geqslant u\})\mathrm{d}u = \int_{\mathbb{R}^2} h(f(x))\|\nabla f(x)\|\mathrm{d}x \tag{3.3}$$

Interesting Observation: We observe that (3.1) is an analogue of (3.2) for the Euler characteristic. The perimeter and the Euler characteristic form a remarkable pair of functionals as they are central in the theory of convex bodies. They are, with the volume function, the only homogeneous additive continuous functionals of poly-convex sets of $\mathbb{R}^2$. In both cases, thanks to (3.1) and (3.2), their integral against a test function can be computed in terms of a spatial integral involving f and its derivatives. This gives hope for similar formulae for all additive functionals in higher dimensions ($m \geq 1$). (A catch indeed!)

Remark 3.9. The formula in (3.1) can be seen as a two-dimensional analogue of the Kac-Rice formula. In dimension 1, the Euler characteristic, denoted by $\chi^{(1)}(\cdot)$, is the number of connected components. For the excursion set $\{f \geqslant u\}$ of a $\mathcal{C}^1$ function f with compact level sets, it corresponds to the number of up-crossings of level u, provided u is not a critical value. For a smooth function h with compact support, the integral version of Kac-Rice formula is given by

$$\int_{\mathbb{R}} h(u)\chi^{(1)}(\{f \geqslant u\})\mathrm{d}u = \int_{\mathbb{R}} h(f(x))\,|f'(x)|\,\mathrm{d}x \tag{3.4}$$

where the integrand of the left hand member is properly defined for almost all u.

By a Theorem from [27] states that for $u \notin V(f)$, the Euler characteristic can be expressed as the limit of some quantity $\delta_{u,\varepsilon} \in \mathbb{R}$ that is explicit in [27]

$$\chi(\{f \geqslant u\}) = \lim_{\varepsilon\to 0} \delta_{\varepsilon,u}(f).$$

For f like in Theorem 3.6 and $u \notin V(f)$, (3.2) yields that $\chi(\{f \geqslant u\})$ is constant on a neighbourhood of u. For ε sufficiently small and $\delta_\varepsilon : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^1$ with support in $[-\varepsilon, \varepsilon]$ such that $\int_{-\varepsilon}^{\varepsilon} h(v)\mathrm{d}v = 1$,

$$\chi(f \geqslant u) = \lim_{\varepsilon\to 0} \chi_f(\delta_\varepsilon) = \lim_{\varepsilon\to 0} I_f(\delta_\varepsilon)$$

This formula can be seen as a 2 -dimensional analogue of the celebrated Kac-Rice formula, obtained by taking $h = \delta_\varepsilon$ in (3.3).

3.3 EXTENSION TO NON-MORSE FUNCTIONS

We notice that in [28], the validity of the result about the Euler characteristic of excursions only requires $\mathcal{C}^{1,1}$ regularity, i.e. continuous differentiability with Lipschitz gradient. In contrast, Theorem 3.6 requires $\mathcal{C}^2$ regularity and Morse behaviour around the critical points. Still one may believe that the conclusion could be valid under $\mathcal{C}^{1,1}$ regularity (under such assumptions, the second order partial derivatives are well defined a.e.).

To support and motivate this claim, we record here that it holds if the function f is radial.

Theorem 3.10 ([26]). *Assume $f(x) = \psi\left(\left\|x^2\right\|\right), x \in \mathbb{R}^2$, for some $\mathcal{C}^{1,1}$ function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ that vanishes at ∞. Then for almost all (a. a.) $u > 0, \chi(\{f \geqslant u\})$ is well defined and bounded by 1, and for $h : \mathbb{R}_+ \to \mathbb{R}_+$ a $\mathcal{C}^1$ function with compact support in $(0, \infty)$, we have*

$$\chi_f(h) = I_f(h) = \int_0^{\psi(0)} h(u)\mathrm{d}u$$

Open Problem: What is a general result and its proof in dimension 2?

3.4 ZEROS OF RANDOM POLYNOMIALS IN $\mathbb{C}$

A classical result of Hammersley [19] (see also [35]) is that the zeros of a random complex polynomial

$$F_n(z) = \sum_{j=0}^{n-1} X_j z^j, z \in \mathbb{C}$$

mostly tend towards the unit circle $|z| = 1$ as the degree $n \to \infty$, when the coefficients X_j are independent complex Gaussian random variables of mean zero and variance one.

We invoke the explicit formula for the expectation of the number, $\nu_n(\Omega)$, of zeros of a random polynomial, obtained by Kac [21].

$$F_n(z) = \sum_{j=0}^{n-1} X_j z^j, \quad z \in \mathbb{C} \tag{3.5}$$

in *any measurable subset* Ω *of the reals.* Here, $X_0, \dots, X_{n-1}$ are independent standard normal random variables (where Kac's argument is the most natural choice for a "typical" polynomial since this distribution is invariant under orthogonal transformations). In fact, for each $n > 1$, he has obtained an explicit intensity function g_n for which

$$\mathbb{E}\nu_n(\Omega) = \int_\Omega g_n(x)\mathrm{d}x$$

Here, in [23] this result has been extended by deriving an explicit formula for the expected number of zeros in *any measurable subset* Ω *of the complex plane* $\mathbb{C}$. Namely, they have shown that

$$\mathbb{E}\nu_n(\Omega) = \int_\Omega h_n(x, y)\, \mathrm{d}x\mathrm{d}y + \int_{\Omega\cap\mathbb{R}} g_n(x)\mathrm{d}x$$

where h_n is an explicit intensity function. (Note that the usual identification between a point z of the complex plane and its real and imaginary parts, x and y are carried out carefully.)

The intensity function h_n is conveniently expressed in terms of the following three real-valued functions defined on $\mathbb{C}$

$$B_k(z) = \sum_{j=0}^{n-1} j^k |z|^{2j}, \quad z \in \mathbb{C}, \quad k = 0, 1, 2$$

and the following two complex-valued functions

$$A_k(z) = \sum_{j=0}^{n-1} j^k z^{2j}, \quad z \in \mathbb{C}, \quad k = 0, 1.$$

Finally, let

$$D_0(z) = \sqrt{B_0^2(z) - |A_0|^2 (z)} \tag{3.6}$$

The outstanding main result obtained in [23] is

Theorem 3.11 ([23]). *For each region $\Omega \in \mathbb{C}$,*

$$\mathbb{E}\nu_n(\Omega) = \int_\Omega h_n(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\cap\mathbb{R}} g_n(x)\mathrm{d}x \tag{3.7}$$

where

$$h_n = \frac{B_2D_0^2 - B_0\left(B_1^2 + |A_1|^2\right) + B_1\left(A_0\bar{A}_1 + \bar{A}_0A_1\right)}{\pi|z|^2D_0^3}$$

and

$$g_n = \frac{\left(B_0B_2 - B_1^2\right)^{1/2}}{\pi|z|B_0}$$

For detailed proof of the above results, one may consult [23].

In 1995, considering the case when $\left\{\eta_j\right\}$ are real-valued standard Gaussian, Shepp and Vanderbei [35] have obtained a formula for the expected number of zeros of P_n in

$$P_n(z) = \eta_n z^n + \eta_{n-1}z^{n-1} + \cdots + \eta_1 z + \eta_0 \tag{3.8}$$

off the real line. In their work they have obtained the limits

$$\lim_{n\to\infty}\rho_n^{\mathbb{C}}(z) = \frac{1}{\pi\left(1-|z|^2\right)^2}\sqrt{1-\left|\frac{1-|z|^2}{1-z^2}\right|^2} \tag{3.9}$$

and

$$\lim_{n\to\infty}\rho_n^{\mathbb{R}}(x) = \frac{1}{\pi}\frac{1}{\left|1-x^2\right|} \tag{3.10}$$

where $\rho_n^{\mathbb{C}}(z)$ is the intensity function for the number of purely complex zeros of the random polynomial.

Within the proof of computing the above limits, Shepp and Vanderbei [35] have shown that as $n \to \infty$, uniformly about $n-(2/\pi)\log n$ of zeros of P_n accumulate on the unit circle, and about $(2/\pi)\log n$ of real roots concentrate at ± 1. Ibragimov and Zeitouni[23] have generalized the work of Shepp and Vanderbei [35] by giving the limit of the expected value of a scaled version of the expected number of zeros of the random algebraic polynomial P_n in a disk of radius r when the random variables $\left\{\eta_j\right\}$ are IID with common distribution that belongs to the domain of attraction of an $\alpha-$stable law.

The formulae provided by Shepp and Vanderbei[35] for the intensity functions for the number of real and complex zeros of the random algebraic polynomial have been generalized by Feldheim[17] and independently by Vanderbei [50]. These general formulae give the intensity functions for random sums of the form

$$\sum_{k=0}^{n}\eta_k f_j(z)$$

where $\{\eta_k\}$ are IID real-valued standard Gaussian random variables, and $\{f_k\}$ are entire functions that are real-valued on the real line.

Let us first start with a motivating example. Consider the complex Kac polynomial

$$\mathcal{P} = P_n(z) = \sum_{k=0}^{n} \eta_k z^k$$

with $\eta_j = \alpha_j + i\beta_j$, $j = 0, 1, \ldots, n$, where $\left\{\alpha_j\right\}_{j=0}^{n}$ and $\left\{\beta_j\right\}_{j=0}^{n}$ are sequences of IID standard Gaussian random variables.

Define

$$A(s,t) := \{z \in \mathbb{C} : 0 \leq s < |z| < t\}.$$

Using a classical result by Hammersely[19] that gives a formula for the expected number of zeros of P_n, we have the following result.

Theorem 3.12 ([19]). *For the complex Kac polynomial $P_n(z)$ we have*

$$\mathbb{E}\left[N_n(A(s,t))\right] = \frac{1}{1-t^2} - \frac{n+1}{1-t^{2n+2}} - \left(\frac{1}{1-s^2} - \frac{n+1}{1-s^{2n+2}}\right)$$

provided the annulus $A(s,t)$ does not contain the unit circle.

3.5 ZEROS OF RANDOM POLYNOMIALS IN $\mathbb{C}^m$

This section contains some mind blowing results(see [9])! Let K be a compact set in $\mathbb{C}^m$ and let μ a Borel probability measure on K. Assume that K is non-pluripolar and let V_K be its pluricomplex Green function. Let $\mathcal{P} = P_n(z) = \sum_{k=0}^{n} \eta_k z^k$ be a holomorphic polynomial. Also assume that K is regular (i.e., $F = V_K^*$) and that μ satisfies the following Bernstein-Markov (BM) inequality.

The Bernstein-Markov Inequality: *Let μ be a finite positive Borel measure on K. The measure μ is said to satisty a Bernstein-Markov inequality, if, for each $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that*

$$\|(P)\|_K \leq C e^{\varepsilon \deg(\mathcal{P})} \|\mathcal{P}\|_{L^2(\mu)} \tag{3.11}$$

for all holomorphic polynomials $\mathcal{P}$. Essentially, the BM inequality says that the L^2 norms and the sup *norms of a sequence of holomorphic polynomials of increasing degrees are "asymptotically equivalent".*

The above inequality has an interesting historical background [18]. The chemist Mendeleev in 1887 invented the periodic table of the elements which lead to a study of the specific gravity of a solution as a function of the percentage of the dissolved substance [34]. From the real life situation, it is used in testing beer and wine for alcoholic content, and in testing the cooling system of an automobile for concentration of anti-freeze. Mendeleev's study paved a way to mathematical problems of great interest, some of which are inspiration to do research in Mathematics. He discussed this problem with Markov in 1889 who converted into a mathematical problem and subsequently generalised by Bernstein in 1926.

Gaussian Measure γ_n: Let

$$(f,g) = \int_K f\bar{g}\,\mathrm{d}\mu \tag{3.12}$$

be the Hermitian product. Consider the space P_n of holomorphic polynomials of degree $\leq n$ on $\mathbb{C}^m$ with the Gaussian probability measure γ_n that is induced by the Hermitian inner product (3.12). We write $F \equiv F_n = \sum_{j=1}^{d(n)} X_j p_j$ where $\left\{p_j\right\}$ is an orthonormal basis of $\mathcal{P}_n$ with respect to (3.12) and $d(n) = \dim \mathcal{P}_n = \begin{pmatrix} n+m \\ m \end{pmatrix}$. Identifying $F \in \mathcal{P}_n$ with $X = (X_1, \ldots, X_{d(n)}) \in \mathbb{C}^{d(n)}$, we have

$$\mathrm{d}\gamma_n(s) = \frac{1}{\pi^{d(n)}} e^{-|x|^2}\mathrm{d}x.$$

Notice that the measure γ_n is independent of the choice of orthonormal basis $\left\{p_j\right\}$. In other words, a random polynomial in the ensemble (P_n, γ_n) is a polynomial $F = \sum_j X_j p_j$, where the X_j are independent complex Gaussian random variables with mean 0 and variance 1.

Now we are ready to state a multivariate version on the expected distribution of simultaneous zeros of random polynomials orthonormalized on a compact set.

Theorem 3.13 ([9]). *Let μ be a Borel probability measure on a regular compact set $K \subset \mathbb{C}^m$, and suppose that (K, μ) satisfies the Bernstein-Markov inequality. Let $1 \leq k \leq m$, and let $\left(\mathcal{P}_n^k, \gamma_n^k\right)$ denote the ensemble of $k-$tuples of IID Gaussian random polynomials of degree $\leq n$ with the Gaussian measure* $\mathrm{d}\gamma_n$ *induced by $L^2(\mu)$. Then*

$$\frac{1}{n^k}\mathbf{E}_{\gamma_n^k}\left(Z_{F_1,\ldots,F_k}\right) \to \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^k \quad \textit{weak}\ ^*, \quad \textit{as } N \to \infty \tag{3.13}$$

where V_K is the pluricomplex Green function of K with pole at infinity.

Special Case 1:
Let K be the unit polydisk in $\mathbb{C}^m$. Then $V_K = \max_{j=1}^m \log^+ \left|z_j\right|$, the Silov boundary of K is the product of the circles $\left|z_j\right| = 1$, $(j = 1, \ldots, m)$ and $\mathrm{d}\mu_{eq} = \left(\frac{1}{2\pi}\right)^m \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m$ where $\mathrm{d}\theta_j$ is the angular measure on the circle $\left|z_j\right| = 1$. The monomials $z^J := z_1^{j_1} \cdots z_m^{j_m}$, for $|J| \leq N$, form an orthonormal basis for $\mathcal{P}_n$. A random polynomial in the ensemble is of the form

$$F(z) = \sum_{|J| \leq N} X_J z^J$$

where the X_J are independent complex Gaussian random variables of mean zero and variance one.

By Theorem 3.13, $\mathbf{E}_{\gamma_n^m}\left(Z_{F_1,\ldots,F_m}\right) \to \left(\frac{1}{2\pi}\right)^m \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m$ in weak *, as $n \to \infty$. In particular, the common zeros of m random polynomials tend to the product of the unit circles $\left|z_j\right| = 1$ for $j = 1, \ldots, m$

Special Case 2:
Let K be the unit ball $\{\|z\| \leq 1\}$ in $\mathbb{C}^m$. Then the Silov boundary of K is its topological

boundary $\{\|z\| = 1\}, V_K(z) = \log^+ \|z\|$, and μ_{eq} is the invariant hypersurface measure on $\|z\| = 1$ normalized to have total mass one.

Some established results on expected distributions of zeros are sketched here. The one-dimensional case of (3.13) is given in [5], which generalizes the results in [36] for the case where K is a real-analytic domain in $\mathbb{C}$ (or its boundary). Generalizations of (3.13) to weighted equilibrium measures are given in [6], and generalizations to equilibrium measures on pseudoconcave domains in compact Kähler manifolds are given by Berman [3]. It has also been shown in [6] that (3.13) holds for certain non-Gaussian random polynomials on $\mathbb{C}$. Results on the distribution of zeros of polynomials on $\mathbb{C}$ with random real coefficients are given by Shepp-Vanderbei [35], Ibragimov-Zeitouni [23], and others.

We may notice that the distributions of zeros for the measures on $\mathcal{P}_N$ considered here are quite different from those of the SU$(m + 1)$ ensembles (for example, in [44], [37], [7], [8], [12]). The Gaussian measure on the SU($m + 1$) polynomials is based on the inner product

$$\langle f, g\rangle_N = \int_{S^{2m+1}} F_N \overline{G_N} \mathrm{d}\mu$$

where $F_N, G_N \in \mathbb{C}[z_0, z_1, \dots, z_m]$ denote the degree N homogenizations of f and g respectively. It follows easily from the SU$(m + 1)$ -invariance of the inner product that the expected distribution of simultaneous zeros equals $\frac{N^m}{\pi^m}\omega^m$ (exactly), where ω is the Fubini-Study Kähler form (on $\mathbb{C}^m \subset \mathbb{CP}^m$). We note that, unlike (3.12), this inner product depends on N; indeed, $\left\|z^J\right\|_N^2 = \frac{m!(N-|J|)!j_1!\cdots j_m!}{(N+m)!}$ ([44],equation (30))

Theorem 3.14 ([9]). *Let $\left(\mathcal{P}_N^m, \gamma_N^m\right)$ denote the ensemble of m -tuples of IID standard Gaussian random polynomials of degree $\leq N$ with the Gaussian measure $d\gamma_N$ induced by $L^2\left(S^{2m-1}, \mu\right)$, where μ is the invariant measure on the unit sphere $S^{2m-1} \subset \mathbb{C}^m$ Then*

$$\mathbf{E}_{\gamma_N^m}\left(Z_{f_1,\dots,f_m}\right) = D_N\left(\log\|z\|^2\right)\left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^m$$

where

$$\frac{1}{N^{m+1}} D_N\left(\frac{u}{N}\right) = \frac{1}{\pi^m} F_m''(u) F_m'(u)^{m-1} + O\left(\frac{1}{N}\right) \text{ with}$$

$$F_m(u) = \log\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}u^{m-1}}\left(\frac{e^u - 1}{u}\right)\right]$$

An Open Problem:(Bloom-Shiffman[9]) Find scaling limits for more general sets in $\mathbb{C}^m$.

4 RANDOM ORTHOGONAL POLYNOMIALS

Let $\phi_0(x), \phi_1(x), \phi_2(x), \dots$ be a sequence of polynomials orthogonal with respect to a given positive-valued weight function $\omega(x)$ over the interval (a, b) where one or both of a and b may be infinite and let $\psi_n(x) = g_n^{-1/2}\phi_n(x)$ with

$$g_n = \int_a^b \omega(x)\phi_n^2(x)\mathrm{d}x. \tag{4.1}$$

Let $f(x)$ be defined by

$$f(x) \equiv f(\mathbf{c}; x) = \sum_{k=0}^{N} c_k \psi_k(x) \tag{4.2}$$

where the coefficients $c_0, c_1, c_2, \ldots$ form a sequence of IID standard Gaussian random variables. We take the ordered set $c_0, c_1, \ldots, c_n$ as the point c in an $(n+1)$−dimensional real vector space $\mathbb{R}^{n+1}$. The probability that the point c lies in an "infinitesimal rectangle" $\prod$ (c) with sides of lengths $\mathrm{d}c_0, \mathrm{d}c_1, \cdots, \mathrm{d}c_n$ is

$$\mathrm{d}P(c) = \prod_{k=0}^{n} \left\{ (2\pi)^{-1/2} \exp\left(-\frac{1}{2} c_k^2 \right) \mathrm{d}c_k \right\}.$$

Let $N(\mathbf{c}; \alpha, \beta)$ denote the number of zeros of the polynomial (4.2) in the interval $\alpha \leqq x \leqq \beta$. Das [10] has established the formula

$$\int_{R_{n+1}} N(c; \alpha, \beta) \mathrm{d}P(c) = \frac{1}{\pi} \int_{a}^{\beta} \left[\frac{S_n(x) + R_n(x)}{D_n(x)} - \frac{1}{4} \frac{Q_n^2(x)}{D_n^2(x)} \right]^{1/2} \mathrm{d}x \tag{4.3}$$

where

$$\begin{aligned} D_n(x) &= \phi'_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'_n(x) \\ Q_n(x) &= \phi''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi''_n(x) \\ R_n(x) &= \tfrac{1}{2} \left\{ \phi''_{n+1}(x)\phi'_n(x) - \phi'_{n+1}(x)\phi''_n(x) \right\} \end{aligned}$$

and

$$S_n(x) = \frac{1}{6} \left\{ \phi'''_{n+1}(x)\phi_n(x) - \phi_{n+1}(x)\phi'''_n(x) \right\}.$$

When n is large, Das has obtained an estimate of the integrand in the right-hand side of (4.3) in terms of n and x only in an easily integrable form, since only two functions $\phi_n(x)$ and $\phi_{n+1}(x)$ are now involved. He has obtained the following asymptotic estimate.

Theorem 4.1 ([10]). *Let $P_1^*(x)$ be the normalized Legendre polynomial $\left(k + \frac{1}{2}\right)^{1/2} P_k(x)$, where*

$$P_k(x) = \frac{1}{2^k} \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(x^2 - 1 \right)^k$$

the Legendre polynomial. Here $a = -1$ and $b = 1$ and $\omega(x) \equiv 1$ in (4.1). Further $\psi_k(x) = P_k^*(x)$ with $g_n = \left(n + \frac{1}{2}\right)^{1/2}$. The average number ν_n of zeros of

$$F_n(x) = c_0 P_0^*(x) + c_1 P_1^*(x) + \cdots + c_k P_k^*(x) + \cdots + c_n P_n^*(x) \tag{4.4}$$

in $(-1, 1)$ where c_i are IID standard Gaussian random variables, is asymptotically equal to $n/\sqrt{3}$ when n is sufficiently large.

In [10], he has shown that $\nu_n \sim 3^{-\frac{1}{2}} n$ for large n (In fact, his analysis indicates that $\nu_n = 3^{-\frac{1}{2}} n \left[1 + O\left\{ (\log n)^{-3} \right\} \right]$.) Wilkins [49] has obtained somewhat better result that

$$\nu_n = 3^{-\frac{1}{2}} n + o\left(n^5\right)$$

for any positive δ. The analysis in [49] is similar to that of [10], but it involves a more detailed treatment of the asymptotic expansion for $P_n(t)$ when n is large.

Let the coefficients c_j be dependent Gaussian with moment matrix with $\rho_{ii} = \sigma^2$ and $\rho_{ij} = \rho, 0 < \rho < 1, i \neq j$. Comparing the results of Farahmand [16] for the Legendre polynomials with the algebraic polynomials, in the cases of independent versus dependent, significant differences in the behavior are exhibited. Sambandham [41] has shown that $\mathbb{E}N_n(-\infty, \infty)$ for the algebraic case with dependent Gaussian coefficients is half that of the independent case. However, Farahmand[16] has shown that in the case of Legendre polynomials, the expected number of zeros is invariant for both dependent and independent Gaussian cases.

Theorem 4.2 ([16]). *If the coefficients of $F_n(x)$ in* (4.4) *are dependent Gaussian with the above covariance matrix and mean μ then, for all sufficiently large n, the expected number of real zeros of $P_n(x)$ is*

$$\mathbb{E}N_n(-1, 1) \sim \frac{n}{\sqrt{3}}$$

In another direction, let us define a real zero of $F_n(x, \omega)$ as $u-$sharp when it up-crosses the $x-$axis with slope greater than u or down-crosses it with slope smaller than $-u$. Let the number of $u-$sharp crossings of $F_n(x, \omega)$ in the interval (a, b) be $S_u(a, b)$. Farahmand's method indicates that in the case of independent coefficients, most of the crossings of random Legendre polynomials are $u-$sharp. That is, unlike algebraic cases, $\mathbb{E}S_u(-1, 1)$ is independent of u.

Theorem 4.3 ([16]). *If the coefficients of $F_n(x)$ in* (4.4) *are independent Gaussian with mean μ, then for all u such that $u/n^3 \to 0$ as $n \to \infty$, the expected number of $u-$sharp crossings is*

$$\mathbb{E}S_u(-1, 1) \sim \frac{n}{\sqrt{3}}$$

Let us present here the results obtained by Lubinsky et al.[33] on the random orthogonal polynomials. We state a result on the number of real zeros for the random linear combinations of rather general functions. It has its origin in the papers of Kac [21],[22], [25] who used the monomial basis, and was extended to trigonometric polynomials and other bases, see Farahmand [15] and Das [10], Das and Bhatt [11]. We are particularly interested in the bases of orthonormal polynomials, which is the case considered by Das [10]. For any set $E \subset \mathbb{C}$, we use the notation $N_n(E)$ for the number of zeros of random functions (4.5) (or random orthogonal polynomials of degree at most n) located in E. The expected number of zeros in E is denoted by $\mathbb{E}[N_n(E)]$, with $\mathbb{E}[N_n(a, b)\ ]$ being the expected number of zeros in $(a, b) \subset \mathbb{R}$.

Theorem 4.4 ([33]). *Let $[a, b] \subset \mathbb{R}$, and consider real valued functions $g_j(x) \in \mathcal{C}^1([a, b])$, $j = 0, \ldots, n$, with $g_0(x)$ being a nonzero constant. Define the random function*

$$G_n(x) = \sum_{j=0}^{n} c_j g_j(x) \qquad (4.5)$$

where the coefficients c_j are IID random variables with Gaussian distribution $\mathcal{N}\left(0,\sigma^2\right), \sigma > 0$. If there is $M \in \mathbb{N}$ such that $G_n'(x)$ has at most M zeros in (a,b) for all choices of coefficients, then the expected number of real zeros of $G_n(x)$ in the interval (a,b) is given by

$$\mathbb{E}\left[N_n(a,b)\right] = \frac{1}{\pi}\int^{b} \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)}\mathrm{d}x \tag{4.6}$$

where

$$A(x) = \sum_{j=0}^{n} g_j^2(x), \quad B(x) = \sum_{j=1}^{n} g_j(x)g_j'(x) \quad \textit{and} \quad C(x) = \sum_{j=1}^{n}\left[g_j'(x)\right]^2. \tag{4.7}$$

For random Jacobi polynomials, Das and Bhatt [11] have concluded that $\mathbb{E}\left[N_n(-1,1)\right]$ is asymptotically equal to $n/\sqrt{3}$ too. They have also provided estimates for the expected number of real zeros of random Hermite and Laguerre polynomials, but those arguments contain some significant gaps. Farahmand ([14], [15], [16]) has considered various generalizations of these results for the level crossings of random sums of Legendre polynomials with coefficients that may have different distributions.

For the orthonormal polynomials $\left\{p_j(x)\right\}_{j=0}^{\infty}$ associated with positive Borel measure μ, define the reproducing kernel by

$$K_n(x,y) = \sum_{j=0} p_j(x)p_j(y)$$

and the differentiated kernels by

$$K_n^{(k,l)}(x,y) = \sum_{j=0}^{n-1} p_j^{(k)}(x)p_j^{(l)}(y), \quad k,l \in \mathbb{N}\cup\{0\}.$$

The strategy is to apply Theorem 4.4 with $g_j = p_j$, so that

$$A(x) = K_{n+1}(x,x), \quad B(x) = K_{n+1}^{(0,1)}(x,x) \quad \text{and} \quad C(x) = K_{n+1}^{(1,1)}(x,x). \tag{4.8}$$

We use universality limits for the reproducing kernels of orthogonal polynomials (see Lubinsky ([31], [32]), Totik ([45], [46])), and asymptotic results on zeros of random polynomials (cf. Pritsker [39]) give asymptotics for the expected number of real zeros for a wider class of random orthogonal polynomials.

Theorem 4.5 ([33]). *Let $K \subset \mathbb{R}$ be a finite union of closed and bounded intervals, and let μ be a positive Borel measure supported on K such that $d\mu(x) = w(x)dx$ and $w > 0$ a.e. on K. If for every $\varepsilon > 0$ there is a closed set $S \subset K$ of Lebesgue measure $|S| < \varepsilon$, and a constant $C > 1$ such that $C^{-1} < w < C$ a.e. on $K\backslash S$, then the expected number of real zeros of random orthogonal polynomials* (4.5) *with Gaussian coefficients satisfy*

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[N_n(\mathbb{R})\right] = \frac{1}{\sqrt{3}}. \tag{4.9}$$

A simple example of the orthogonality measure μ satisfying the above conditions is given by the density w that is continuous on K except for finitely many points, and has finitely many zeros on K. More specifically, one may consider the generalized Jacobi with weight of the form $w(x) = v(x) \prod_{j=1}^{J} \left|x - x_j\right|^{\alpha_j}$, where $v(x) > 0, x \in K$, and $\alpha_j > -1, j = 1, \ldots, J$. Theorem 4.5 is a consequence of more precise and general local results given below. In order to state the result, we need the notion of the equilibrium measure ν_K of a compact set $K \subset \mathbb{C}$. This is the unique probability measure supported on K that minimizes the energy

$$I[\nu] = -\iint \log|z - t| \mathrm{d}\nu(t) \mathrm{d}\nu(z)$$

amongst all probability measures ν with support on K. The logarithmic capacity of K is

$$\operatorname{cap}(K) = \exp\left(-I\left[\nu_K\right]\right)$$

When we say that a compact set K is regular, this means regularity in the sense of Dirichlet problem (or potential theory). See Ransford [40] for further details.

We also need the notion of a measure μ regular in the sense of Stahl et al.[42](STU). If $K = \operatorname{supp} \mu$ and where γ_n is the

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\operatorname{cap}(K)}$$

where γ_n is the leading coefficient of p_n, then we say that μ is STU-regular. A sufficient condition for this is that K consists of finitely many intervals and $\mu' = w > 0$ a.e. in those intervals.

Theorem 4.6 ([33]). *Let μ be an* STU *regular measure with compact support $K \subset \mathbb{R}$, which is regular in the sense of potential theory. Let O be an open set in which μ is absolutely continuous, and such that for some $C > 1$*

$$C^{-1} \leq \mu' \leq C \text{ a.e., in } O \tag{4.10}$$

Then given any compact subinterval $[a, b]$ of O, we have

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[N_n([a, b])\right] = \frac{1}{\sqrt{3}} \nu_K([a, b]) \tag{4.11}$$

where ν_K is the equilibrium measure of K.

This is a special case of the following result, where μ need not to be STU regular. The asymptotic lower bound requires very little of μ.

Theorem 4.7 ([33]). *Let μ be a measure on the real line with compact support K.*
(a) Assume that $\mu' > 0$ a.e. in the interval $[a, b]$. Then

$$\liminf_{n\to\infty} \frac{1}{n} \mathbb{E}\left[N_n([a, b])\right] \geq \frac{1}{\sqrt{3}} \nu_K([a, b]) \tag{4.12}$$

(*b*) *Suppose in addition that* (4.10) *holds, and that* $[a,b] \subset O$. *Then*

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}\,[N_n([a,b])] \le \frac{1}{\sqrt{3}} \inf_L \nu_L([a,b]) \tag{4.13}$$

where the inf *is taken over all regular compact sets* $L \subset K$ *such that* $L \supset [a,b]$, *and the restriction* $\mu|_L$ *of* μ *to* L *is* STU *regular.*

Now we discuss the zeros of random sums of orthogonal polynomials, based on the work of Shiffman and Zelditch [36]. Consider a set $\{p_k(z)\}$ of orthogonal polynomials. Let $Z_0, Z_1, \ldots$ be a sequence of IID complex Gaussians with mean zero and variance one. Then, a random sum of orthogonal polynomials is a random polynomial of the form

$$P_n(z) = \sum_{k=0}^{n} Z_k p_k(z). \tag{4.14}$$

In order to correctly formulate the results of Shiffman and Zelditch [36], we need to give a few definitions. To start, let $\mathcal{P}_n$ be the space of polynomials defined on $\mathbb{C}$, with degree less than or equal to n. For Ω a simply connected bounded domain in $\mathbb{C}$ with real analytic boundary (which will henceforth be called a simply connected bounded $\mathcal{C}^\omega$ domain; see [2] for further references), we define the inner product on $\mathcal{P}_n$ by

$$\langle f, \bar{g}\rangle_{\partial\Omega,\rho} := \int_{\partial\Omega} f(z)\overline{g(z)}\rho(z)|\mathrm{d}z| \tag{4.15}$$

where ρ is a weight function, $\rho \in \mathcal{C}^\omega(\partial\Omega)$, the space of real analytic functions on a real analytic boundary $\partial\Omega$.

Now, given a compact set $K \in \mathbb{C}$, the equilibrium measure for this set is defined as the unique probability measure that minimizes the energy

$$I(\mu) = -\int_K\int_K \log|z-w|\mathrm{d}\mu(z)\mathrm{d}\mu(w)$$

(see [29], [47] for further reference). This measure will be denoted as μ_K. If $\{p_k(z)\}$ is an orthonormal basis of $\mathcal{P}_n$ orthogonalized over a domain Ω satisfying certain properties, Shiffman and Zelditch [36] have shown that the zeros of $P_n(z)$ are distributed themselves in the limit according to the equilibrium measure for $\bar{\Omega}$. By a slight abuse of notation, let μ_Ω represent this measure. In the case of the closed unit disk, S^1, this is simply Lebesgue measure on the circle, denoted by δ_{S^1}. This statement will be made more precise as follows.

If we let $\{p_k(z)\}$ be an orthonormal basis of $\mathcal{P}_n$ according to the inner product in (4.15), we can write any arbitrary $P_n \in \mathcal{P}_n$ in the form of (4.14). A Gaussian measure on $\mathcal{P}_n$ will then be defined by the condition that the Z_k 's are IID complex Gaussians with mean zero and unit variance. This measure will be denoted by $\gamma^n_{\Omega,\rho}$. An expectation with respect to $\left(\mathcal{P}_n, \gamma^n_{\Omega,\rho}\right)$ will be written as $\mathrm{E}^n_{\partial\Omega,\rho}$. Finally, we need to introduce the normalized distribution of zeros for P_n. This is defined as

$$\widetilde{Z}^n_{P_n} := \frac{1}{n}\sum_{P_n(z)=0} \delta_z$$

In a nutshell, it measures the zeros of P_n. We are now ready to state the main result of Shiffman and Zelditch [36].

Theorem 4.8 ([36]). *Suppose that Ω is a simply connected bounded $\mathcal{C}^\omega$ domain and that ρ is a positive $\mathcal{C}^\omega$ density on $\partial\Omega$. Then,*

$$\mathrm{E}^n_{\partial\Omega,\rho}\left[\widetilde{Z}^n_{P_n}\right] = \mu_\Omega + O\left(\frac{1}{n}\right) \tag{4.16}$$

where μ_Ω is the equilibrium measure of $\bar{\Omega}$. As a further note on notation, in this context $O(f(n))$ corresponds to a distribution $T_n \in \mathcal{D}'(\mathbb{C})$ such that

$$|\langle T_n, \phi\rangle| \le c_\phi f(n), \quad \forall\phi \in \mathcal{D}(\mathbb{C})$$

where c_ϕ does not depend on n.

This result has motivated the investigation of a similar problem, where the random sums of orthogonal polynomials are composed of the “classic” orthogonal polynomials. These would include the Chebyshev, Legendre, and Hermite polynomials. Since the aforementioned polynomials are all orthogonalized on the real line, or some subset thereof, the given theorem of Shiffman and Zelditch[36] would not apply. Thus, in what follows we will lay the groundwork for an investigation into the zeros of such random sums of orthogonal polynomials. We will also present some results pertaining to the specific case of Chebyshev polynomials of the first kind.

The discussion here will be closely based on the work of Shiffman and Zelditch in [36], where the necessary changes are made to handle the case when Ω is a subset of the real line, rather than a simply connected bounded $\mathcal{C}^\omega$ domain in $\mathbb{C}$.

At first, we formulate a specific case of orthonormal polynomials on the closed unit disk.

Theorem 4.9 ([36]). *Let $\mu = \delta_{S^1}$ denote Haar measure on S^1, and let $\rho \equiv 1$. Then*

$$\mathrm{E}^n_{S^1,\rho}\left[n\widetilde{Z}^n_{P_n}\right] = \frac{i}{2\pi}\left[\frac{1}{\left(|z|^2-1\right)^2} - \frac{(n+1)^2|z|^{2n}}{\left(|z|^{2n+2}-1\right)^2}\right]\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

Furthermore, $\mathrm{E}^n_{S^1,\rho}\left[n\widetilde{Z}^n_{P_n}\right] = n\mu + O(1)$; that is, for all test forms $\phi \in \mathcal{D}(\mathbb{C})$

$$\mathrm{E}^n_{S^1,\rho}\left[\sum_{\{z:P_n(z)=0\}} \phi(z)\right] = \frac{n}{2\pi}\int_0^{2\pi} \phi\left(e^{i\theta}\right)\mathrm{d}\theta + O(1)$$

In particular, $\mathrm{E}^n_{S^1,\rho}\left[\widetilde{Z}^n_{P_n}\right] \to \mu$ in $\mathcal{D}'(\mathbb{C})$

Using Theorem 4.9, the idea is to reduce all the other cases back to the unit disk. In order to accomplish this goal, we must introduce some additional notation.

Denoting the unit disk as U and letting $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, for a simply connected bounded domain Ω let

$$\Phi : \widehat{\mathbb{C}}\backslash\Omega \longrightarrow \widehat{\mathbb{C}}\backslash U$$

be a conformal mapping for which $\Phi(\infty) = \infty$ and $\Phi'(\infty) \in \mathbb{R}^+$. Letting $*$ denote the pullback, it is a known result that the equilibrium measure for Ω is then given by

$$\mu_\Omega = \Phi^* \delta_{S^1} \tag{4.17}$$

or equivalently,

$$\int_\Omega \phi(z) d\mu_\Omega(z) = \frac{1}{2\pi} \int_0^{2\pi} \phi \circ \Phi^{-1}\left(e^{i\theta}\right) \mathrm{d}\theta$$

We will now look more closely at our specific sequence of orthonormal polynomials. For the interval [-1,1] , consider the conformal mapping

$$\Phi(z) = z + \left(z^2 - 1\right)^{1/2} \tag{4.18}$$

which maps $\mathbb{C}\backslash[-1,1]$ to $\mathbb{C}\backslash U$. Additionally, $\Phi(\infty) = \infty, \Phi'(\infty) = 1$, and Φ takes the interval [-1,1] to the upper half of the boundary of U. Also, let the weight function ρ be given by

$$\rho(z) = (1-z)^\alpha (1-z)^\beta$$

where $\alpha > -1, \beta > -1$. The orthogonal polynomials generated by this weight function are called the Jacobi Polynomials. Let us take up the Chebyshev polynomials of the first kind, which arise when $\alpha = \beta = -\frac{1}{2}$. These are given by

$$\widetilde{T}_k(z) = \frac{1}{2}\left(\Phi^k(z) + \Phi^{-k}(z)\right). \tag{4.19}$$

Note that the $\widetilde{T}_k(z)$'s form an orthogonal set, but are not orthonormal. We will define the orthonormal set of Chebyshev polynomials of the first kind by

$$T_0(z) = \frac{1}{\sqrt{\pi}} \widetilde{T}_0(z) \tag{4.20}$$

$$T_k(z) = \sqrt{\frac{2}{\pi}} \widetilde{T}_k(z), \quad k > 0$$

We are now ready to state a result.

Theorem 4.10 ([36]). *Let $Z_1, Z_2, \ldots$ be a sequence of independent complex Gaussians, with mean zero and variance one. Consider the random sum of orthogonal polynomials given by*

$$P_n(z) = \sum_{k=0}^n Z_k T_k(z)$$

where $T_k(z)$ is the $k-$th orthonormal Chebyshev polynomial of the first kind defined above. Let ρ be the weight function given by $\rho(z) = (1-z)^{-1/2}(1+z)^{-1/2}$. Then, for $\Omega = [-1,1]$

$$\mathrm{E}^n_{\partial\Omega,\rho}\left(\widetilde{Z}^n_{P_n}\right) = \mu_\Omega + O\left(\frac{1}{n}\right)$$

The foundation laid in this section triggers to probe on some future work in this area. As the Chebyshev polynomials of the first kind, the zeros of P_n converge to the equilibrium distribution, we firmly believe that similar results should hold for the Legendre polynomials, as well as (i) the Jacobi polynomials and (ii) Hermite polynomials. Thus, a study the extension of Shiffman and Zelditch's work to other orthogonal polynomials has to be initiated.

5 CONCLUSION

In section 1, we recorded interesting results on the concentration of zeros of random polynomials that occur in different situations. Section 2 focused on the discussion on random polynomials in higher dimensions. Also Kac-Rice formula in higher dimensions was discussed with nice results. Computational details on the expected number of complex zeros of random polynomials were also outlined. In section 3, expected zeros of random orthogonal polynomials was methodologically presented to initiate further research. In the course of presentation of results, a cross section of research works were consulted. As this topic has influenced many researchers in the globe, some important contributions might have been left unintentionally. It is proposed to continue this task in the case of random polynomials in most general situations.

REFERENCES

[1] Adler, R.J. and Taylor, J.E. *Random Fields and Geometry.* Springer, 2007.

[2] Bell, S. R. *The Cauchy transform, potential theory, and conformal mapping. Studies in Advanced Mathematics.* CRC Press, Boca Raton, FL, 1992.

[3] Berman, R. Bergman kernels, random zeroes and equilibrium measures for polarized pseudoconcave domains, preprint, 2006.

[4] Bharucha-Reid, A.T. and Sambandham, M. *Random Polynomials.* Academic Press, Orlando, FL. 1986.

[5] Bloom, T. Random polynomials and Green functions, *Int. Math. Res. Not.* 1689–1708, 2005.

[6] Bloom, T. Random polynomials and (pluri)potential theory, *Ann. Polon. Math.* **91**(2–3), 131–141, 2007.

[7] Bleher, P., Shiffman, B., and Zelditch, S. Poincaré-Lelong approach to universality and scaling of correlations between zeros, *Comm. Math. Phys.* **208**, 771–785, 2000.

[8] Bleher, P., Shiffman, B., and Zelditch, S. Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.* **142**, 351–395, 2000.

[9] Bloom, T. and Shiffman, B. Zeros of random polynomials on $\mathbb{C}^m$, *Math. Res. Lett.* **14**(3), 469–479, 2007.

[10] Das, M. Real zeros of a random sum of orthogonal polynomials, *Proc. Amer. Math. Soc.* **27**(1), 147–153, 1971.

[11] Das, M. and Bhatt, S. S. Real roots of random harmonic equations, *Indian J. Pure Appl. Math.* **13**, 411–420, 1982.

[12] Dinh, T.-C. and Sibony, N. Distribution des valeurs de transformations méromorphes et applications, *Comment. Math. Helv.* **81**, 221–258, 2006.

[13] Estrade, A. and Leon, J.R. A central limit theorem for the Euler characteristic of a Gaussian excursion set. *Ann. Prob.* **44**(6), 3849–3878, 2016.

[14] Farahmand, K. Level crossings of a random orthogonal polynomial, *Analysis*, **16**, 245–253, 1996.

[15] Farahmand, M. *Topics in Random Polynomials,* Pitman Res. Notes Math. **393**, 1998.

[16] Farahmand, K. On random orthogonal polynomials, *J. Appl. Math. Stochastic Anal.* **14**, 265–274, 2001.

[17] Feldheim, N. Zeros of Gaussian analytic functions with translation-invariant distribution, *Israel J. Math.* **195**, 317–345, 2012.

[18] Govil, N. K. and Mohapatra, R.N. Markov and BernsteinType Inequalities for Polynomials, *J. of Inequal. & Appl.* **3**, 349–387, 1999.

[19] Hammersley, J. M. The zeros of a random polynomial, *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability,* 1954–1955, University of California Press, Berkeley and Los Angeles, Vol. **II**, 89–111, 1956.

[20] Hughes, C. P and Nikeghbali, A. The zeros of random polynomials cluster uniformly near the unit circle, *Compos. Math.* **144**(3), 734–746, 2008.

[21] Kac, M. On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.* **49**, 314–320, 1943.

[22] Kac, M. On the average number of real roots of a random algebraic equation. II, *Proc. London Math. Soc.* **50**, 390–408, 1948.

[23] Ibragimov, I. and Zeitouni, O. On roots of random polynomials, *Trans. Amer. Math. Soc.* **349**, 2427–2441, 1997.

[24] Ibragimov, I. and Zaporozhets, D. On distribution of zeros of random polynomials in complex plane, *Prokhorov and Contemporary Probability Theory,* Springer, Berlin. 303–323, 2013.

[25] Kac, M. *Nature of probability reasoning, Probability and related topics in Physical Sciences,* Proceedings of the Summer Seminar, Boulder, Colo., 1957, Vol. **I** Interscience Publishers, London-New York, 1959.

[26] Lachièze-Rey, R. Two-dimensional Kac-Rice formula: Application to shot noise processes excursions. Preprint arXiv:1607.05467v1, 2016.

[27] Lachièze-Rey, R. Bicovariograms and Euler characteristic of regular sets. Preprint arXiv 1510.00501v3, 2017.

[28] Lachièze-Rey, R. Bicovariograms and Euler characteristic of Random fields excursions. Preprint arXiv 1510.00502v4, 2018.

[29] Landkof, N.S. *Foundations of modern potential theory.* Springer-Verlag, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[30] Littlewood, J. E. and Offord, A.C. On the number of real roots of a random algebraic equation-III. *Rec. Math. [Mat. Sbornik]* N.S. **12**(54), 277–286, 1943.

[31] Lubinsky, D.S. A new approach to universality limits involving orthogonal polynomials' *Ann. Math.* **170**, 915–939, 2009.

[32] Lubinsky, D.S. Bulk universality holds in measure for compactly supported measures. *J. Anal. Math.* **116**, 219–253, 2012.

[33] Lubinsky, D.S., Pritsker, I.E., and Xie, X. Expected number of real zeros for random linear combinations of orthogonal polynomials. *Proc. Amer. Math.Soc.* **144**(4), 1631–1642, 2016.

[34] Mendeleev, D. *Investigation of Aqueous Solutions Based on Specific Gravity* (Russian), St. Petersburg, 1887.

[35] Shepp, L. A. and Vanderbei, R.J. The complex zeros of random polynomials. *Trans. Amer. Math. Soc.* **347**(11), 4365–4384, 1995.

[36] Shiffman, B. and Zelditch, S. Equilibrium distribution of zeros of random polynomials. *Int. Math. Res. Not.* **1**(25), 25–49, 2003.

[37] Shiffman, B. and Zelditch, S. Number variance of random zeros on complex manifolds. *Geom. Funct. Anal.* **18**, 1422–1475, 2008.

[38] Pritsker, I. and Ramachandran, K. Equidistribution of zeros of random polynomials. *J. Approx. Theory.* **215**, 106–117, 2017.

[39] Pritsker, I. E. Zero distribution of random polynomials. *Journal d'Analyse Mathématique.* **134**(2), 719–745, 2018.

[40] Ransford, T. *Potential Theory in the Complex Plane.* Cambridge Univ. Press, Cambridge, 1995.

[41] Sambandham, M. *Contributions to the study of random polynomials with dependent random coefficients.* Ph.D. Thesis, Annamalai University, Annamalai Nagar, India, 1976.

[42] Stahl, H. and Totik, V. *General Orthogonal Polynomials.* Cambridge Univ. Press, Cambridge, 1992.

[43] Stevens, D.C. The average number of real zeros of a random polynomial. Communications on Pure and Applied Mathematics. **22**(4), 457–477, 1969.

[44] Shiffman, B. and Zelditch, S. Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Commun. Math. Phys.* **200**, 661–683, 1999.

[45] Totik, V. Asymptotics for Christoffel Functions for general measures on the real line. *J. Anal. Math.* **81**, 283–303, 2000.

[46] Totik, V. Asymptotics of Christoffel functions on arcs and curves. *Adv. Math.* **252**, 114–149, 2014.

[47] Tsuji, M. *Potential theory in modern function theory.* Maruzen Co. Ltd., Tokyo, 1959.

[48] Wilkins, Jr. J.E. An upper bound for the expected number of real zeros of a random polynomial. *J. Math. Anal. Appl.* **42**, 569–577, 1973.

[49] Wilkins, Jr. J.E. The expected value of the number of real zeros of a random sum of Legendre polynomials. *Proc. Amer. Math. Soc.* **125**(5), 1531–1538, 1997.

[50] Vanderbei, R. The complex zeros of random sums. *arXiv:* 1508.05162v1 Aug. 21, 2015.