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A STUDY OF A LANGEVIN TYPE FRACTIONAL NONLOCAL NONLINEAR BOUNDARY VALUE PROBLEM WITH FINITELY MANY NONLINEARITIES

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ABSTRACT. This article deals with the investigation of a Langevin type fractional differential equation involving finitely many nonlinearities and equipped with nonlocal nonlinear fractional boundary conditions. The main results for the given problem are obtained by applying the modern tools of the fixed point theory. Examples are presented for the illustration of the obtained results.

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1. INTRODUCTION

Consider a Langevin type fractional differential equation involving finitely many nonlinearities and supplemented with nonlocal nonlinear fractional boundary conditions given by

$$(1.1)\qquad {}^cD^{\alpha}({}^cD^{\rho}+\mu)y(t)=\sum_{i=1}^{m}a_i f_i(t,y(t)),\quad 0<\alpha\le 1,\quad 1<\rho\le 2,\ 0<t<T$$

$$(1.2)\quad y'(0)=\sigma\ {}^cD^{\delta}y(T),\ y(\eta)=0,\ y(0)+\lambda y(T)=\omega g(y),\ 0<\delta<1,\ 0<\eta<T,$$

where ${}^cD^{\kappa}$ denote the Caputo fractional differential operators of order $\kappa\in\{\alpha,\rho,\delta\}$, $f_i:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $g:C([0,T],\mathbb{R})\to\mathbb{R}$ is a continuous function and $\mu,\sigma,\lambda,\omega\in\mathbb{R}$. It is imperative to note that the present configuration of the problem enables one to consider a variety of nonlinearities and develop the theory accordingly (for details, see the last section of the paper).

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The Langevin equation plays an active role in describing time evolution of the velocity of the Brownian motion [1, 2], gait variability [3], financial aspects [4], anomalous diffusion [5], diffusion with inertial effects [6], harmonization of a many-body problem [7], etc. Several interesting theoretical results for fractional Langevin equation equipped with a variety of boundary conditions can be found in the related literature, for instance, see [8]- [17].

The objective of this paper is to enrich the literature on the Langevin equation by considering it with finitely many nonlinearities and nonlocal nonlinear boundary conditions involving ordinary and fractional derivatives. We emphasize that the proposed study is new and significant in the given configuration. First of all we convert the given problem into a fixed point problem with the help of a lemma proved in Section 3. We present our first main result dealing with the existence of a unique solution for the given problem in Section 4. In Section 5, we present two existence results for the problem at hand, which are proved under different criteria. Some interesting facts about the work established in this paper are presented in the last section.

2. PRELIMINARIES

Let us begin this section with some fundamental concepts of fractional calculus.

Definition 2.1. ([18, 19]). The Riemann–Liouville fractional integral $I_a^{\upsilon}u$ of order $\upsilon > 0$ for a function $u \in L_1[a,b], -\infty < a < b < +\infty$, existing almost everywhere on $[a,b]$, is defined by

$$I_a^{\upsilon}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\upsilon-1} u(s)ds,$$

where Γ denotes the Euler Gamma function.

Definition 2.2. [18, 19]. For $u, u^{(m)} \in L_1[a,b]$, the Riemann–Liouville fractional derivative $D_a^{\upsilon}u$ of order $\upsilon \in (m-1, m], m \in \mathbb{N}$, existing almost everywhere on $[a,b]$, is defined as

$$D_a^{\upsilon}u(t) = \frac{d^m}{dt^m} I_a^{m-\upsilon}u(t) = \frac{1}{\Gamma(m-\upsilon)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\upsilon} u(s)ds.$$

In terms of Riemann–Liouville fractional derivative, we can express the Caputo fractional derivative ${}^cD_a^{\alpha}u$ as

$${}^cD_a^{\upsilon}u(t) = D_a^{\upsilon}\left[u(t) - u(a) - u'(a)\frac{(t-a)}{1!} - \ldots - u^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!}\right].$$

Remark 2.3. [18]. The Caputo fractional derivative ${}^cD_a^{\upsilon}u$ of order $\upsilon \in (m-1, m], m \in \mathbb{N}$ for $u \in AC^m[a,b]$ can also be defined as

$$ {}^cD_a^{\upsilon}u(t) = I_a^{m-\upsilon}u^{(m)}(t) = \frac{1}{\Gamma(m-\upsilon)} \int_a^t (t-s)^{m-1-\upsilon} u^{(m)}(s) ds. $$

Proposition 2.4. ([18]) For $\kappa > 0$ and $\alpha > 0$ with $n-1 < \alpha \leq n$, and $u \in L_1[a,b]$, we have the following properties:

(i) $I_a^{\upsilon} I_a^{\kappa} u(t) = I_a^{\kappa} I_a^{\upsilon} u(t) = I_a^{\upsilon+\kappa} u(t)$;

(ii) $I_a^{\upsilon}(t-a)^{\eta} = \dfrac{\Gamma(\eta+1)}{\Gamma(\upsilon+\eta+1)}(t-a)^{\upsilon+\eta}, \ \eta > -1$;

(iii) ${}^cD_a^{\upsilon}\left[I_a^{\upsilon}u(t)\right] = u(t)$;

(iv) $I_a^{\upsilon}\left[{}^cD_a^{\upsilon}u(t)\right] = u(t) - \displaystyle\sum_{p=0}^{n-1} \frac{u^{(p)}(a)(t-a)^p}{p!}, \ u \in C^n[a,b]$.

In the sequel, we write I^{σ} and ${}^cD^{\sigma}$ instead of I_a^{σ} and ${}^cD_a^{\sigma}$ respectively.

In the following section, we solve a linear variant (in terms of Langevin equation) of the problem (1.1).

3. LINEAR LANGEVIN EQUATION CASE

Lemma 3.1. *For a given $\chi \in C([0,1], \mathbb{R})$, the unique solution of the boundary value problem*

$$ {}^cD^{\alpha}({}^cD^{\rho} + \mu)y(t) = \chi(t) \qquad 0 < \alpha \leq 1, \qquad 1 < \rho \leq 2 \tag{3.1} $$

$$ y'(0) = \sigma\ {}^cD^{\delta}y(T), \ y(\eta) = 0, \ y(0) + \lambda y(T) = \omega g(y), \ 0 < \delta < 1, \ 0 < \eta < T, \tag{3.2} $$

is given by

$$ \begin{aligned} y(t) &= I^{\rho+\alpha}\chi(t) - \mu I^{\rho}y(t) + \sigma\varpi_1(t)\Big\{I^{\rho+\alpha-\delta}\chi(T) - \mu I^{\rho-\delta}y(T)\Big\} \\ &\quad +\varpi_2(t)\Big\{I^{\rho+\alpha}\chi(\eta) - \mu I^{\rho}y(\eta)\Big\} \\ &\quad +\varpi_3(t)\Big\{\lambda I^{\rho+\alpha}\chi(T) - \lambda\mu I^{\rho}y(T) - \omega g(y)\Big\}, \end{aligned} \tag{3.3} $$

where

$$ \begin{aligned} \varpi_1(t) &= \frac{1}{\Delta}\Big\{(\eta A_4 - \lambda T)\frac{t^{\rho}}{\Gamma(\rho+1)} + (A_5 - A_3A_4)t + (T\lambda A_3 - \eta A_5)\Big\}, \\ \varpi_2(t) &= \frac{1}{\Delta}\Big\{(-A_1A_4)\frac{t^{\rho}}{\Gamma(\rho+1)} + (A_2A_4)t + (A_1A_5 - \lambda T A_2)\Big\}, \\ \varpi_3(t) &= \frac{1}{\Delta}\Big\{(A_1)\frac{t^{\rho}}{\Gamma(\rho+1)} - (A_2)t + (\eta A_2 - A_1A_3)\Big\}, \end{aligned} \tag{3.4} $$

$$ A_1 = \frac{\sigma T^{1-\delta}}{\Gamma(2-\delta)} - 1, \ A_2 = \frac{\sigma T^{\rho-\delta}}{\Gamma(\rho-\delta+1)}, \ A_3 = \frac{\eta^{\rho}}{\Gamma(\rho+1)}, $$

$$A_4 = 1+\lambda,\ A_5 = \frac{\lambda T^\rho}{\Gamma(\rho+1)}, \tag{3.5}$$

and it is assumed that

$$\Delta = T\lambda A_2 - \eta A_2 A_4 + A_1 A_3 A_4 - A_1 A_5 \neq 0. \tag{3.6}$$

Proof. Applying I^α to both sides of (3.1) and using Proposition 2.4 (iv), we obtain

$$({}^cD^\rho + \mu)y(t) + c_0 = I^\alpha \chi(t), \tag{3.7}$$

where $c_0 \in \mathbb{R}$ is an unknown constant. Now operating I^ρ to both sides of (3.7) and using Proposition 2.4 (i) and (iv), we obtain

$$y(t) + c_1 + c_2 t + \mu I^\rho y(t) + \frac{c_0 t^\rho}{\Gamma(\rho+1)} = I^{\rho+\alpha}\chi(t), \tag{3.8}$$

where c_1 and $c_2 \in \mathbb{R}$ are unknown constants. From (3.8), we have

$$y'(t) + c_2 + \mu I^{\rho-1} y(t) + \frac{c_0 t^{\rho-1}}{\Gamma(\rho)} = I^{\rho+\alpha-1}\chi(t), \tag{3.9}$$

$${}^cD^\delta y(t) + c_2 \frac{t^{1-\delta}}{\Gamma(2-\delta)} + \mu I^{\rho-\delta} y(t) + \frac{c_0 t^{\rho-\delta}}{\Gamma(\rho-\delta+1)} = I^{\rho+\alpha-\delta}\chi(t). \tag{3.10}$$

Using (3.9) and (3.10) in the boundary condition $y'(0) = \sigma\, {}^cD^\delta y(T)$, we find that

$$\left(\frac{\sigma T^{1-\delta}}{\Gamma(2-\delta)} - 1\right) c_2 + \left(\frac{\sigma T^{\rho-\delta}}{\Gamma(\rho-\delta+1)}\right) c_0 = B_1, \tag{3.11}$$

where

$$B_1 = -\sigma\mu I^{\rho-\delta} y(T) + \sigma I^{\rho+\alpha-\delta}\chi(T).$$

Combining (3.8) with the conditions $y(\eta) = 0$ and $y(0) + \lambda y(T) = \omega g(y)$ yields

$$c_1 + \eta c_2 + \left(\frac{\eta^\rho}{\Gamma(\rho+1)}\right) c_0 = B_2, \tag{3.12}$$

$$(1+\lambda) c_1 + \lambda T c_2 + \left(\frac{\lambda T^\rho}{\Gamma(\rho+1)}\right) c_0 = B_3, \tag{3.13}$$

where

$$B_2 = -\mu I^\rho y(\eta) + I^{\rho+\alpha}\chi(\eta),\ B_3 = -\lambda\mu I^\rho y(T) + \lambda I^{\rho+\alpha}\chi(T) - \omega g(y).$$

Using the notation (3.5) in (3.11), (3.12) and (3.13), we get the following system:

$$\begin{cases} A_1 c_2 + A_2 c_0 &= B_1, \\ c_1 + \eta c_2 + A_3 c_0 &= B_2, \\ A_4 c_1 + \lambda T c_2 + A_5 c_0 &= B_3. \end{cases} \tag{3.14}$$

Solving the system (3.14) for c_0, c_1 and c_2, we obtain

$$c_0 = \frac{1}{\Delta}\Big\{(T\lambda - \eta A_4)B_1 + A_1 A_4 B_2 - A_1 B_3\Big\},$$

$$c_1 = \frac{1}{\Delta}\Big\{(\eta A_5 - T\lambda A_3)B_1 + (T\lambda A_2 - A_1A_5)B_2 + (A_1A_3 - \eta A_2)B_3\Big\},$$
$$c_2 = \frac{1}{\Delta}\Big\{(A_3A_4 - A_5)B_1 - A_2A_4B_2 + A_2B_3\Big\},$$

where Δ is given by (3.6). Substituting the values of c_0, c_1 and c_2 in (3.8) together with the notation (3.4) and (3.5), we get the solution (3.3). The converse follows by direct computation. This completes the proof. □

4. A UNIQUENESS RESULT

For computational convenience, we will use the following notations:

$$\Omega_1 = \frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta}\widehat{\varpi_1}\,|\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha}\widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha}\,|\lambda|\,\widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)}, \tag{4.1}$$

$$\Omega_2 = \frac{|\mu|\,T^{\rho}}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_1}\,|\sigma|\,|\mu|\,T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} + \frac{|\mu|\,\widehat{\varpi_2}\eta^{\rho}}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_3}|\lambda||\mu|T^{\rho}}{\Gamma(\rho+1)}, \tag{4.2}$$

where

$$\widehat{\varpi_i} = \sup\{|\varpi_i(t)| : t\in[0,T]\},\ i=1,2,3.$$

In the following theorem, we prove the existence and uniqueness of solutions for the problem (1.1) - (1.2) with the help of Banach fixed point theorem.

Theorem 4.1. *Assume that:*

- (A_1)**:** $f_i : [0,T]\times\mathbb{R}\to\mathbb{R},\ i=1,\dots,m$ *are continuous functions satisfying the Lipschitz condition:* $|f_i(t,x)-f_i(t,y)|\le L_i|x-y|, \forall t\in[0,T],\ x,y\in\mathbb{R},\ L_i>0;$
- (A_2)**:** *There exists a positive number* ε *such that* $|g(y_1)-g(y_2)|\le\varepsilon|y_1-y_2|$, $y_1,y_2\in\mathbb{R}$.

Then the boundary value problem (1.1)-(1.2) has a unique solution on $[0,T]$ *if*

$$\widehat{L}\Omega_1 + \Omega_2 + \widehat{\varpi_3}|\omega|\varepsilon < 1, \tag{4.3}$$

where $\Omega_i,\ i=1,2$ *are defined in (4.1), (4.2) respectively and*

$$\widehat{L} = \sum_{i=1}^{m}|a_i|\,L_i. \tag{4.4}$$

Proof. In relation to the problem (1.1)-(1.2), we introduce an operator $\mathcal{N}:\mathcal{C}\to\mathcal{C}$ by Lemma 3.1 as

$$\begin{aligned}
&(\mathcal{N}y)(t)\\
= &\int_0^t \frac{(t-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^{m}a_if_i(s,y(s))ds - \mu\int_0^t\frac{(t-s)^{\rho-1}}{\Gamma(\rho)}y(s)\,ds\\
&+\sigma\varpi_1(t)\left[-\mu\int_0^T\frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)}y(s)\,ds + \int_0^T\frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)}\sum_{i=1}^{m}a_if_i(s,y(s))ds\right]
\end{aligned}$$

$$(4.5)+\varpi_2(t)\left[-\mu\int_0^\eta\frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}y(s)\,ds+\int_0^\eta\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m a_if_i(s,y(s))ds\right]$$
$$+\lambda\varpi_3(t)\left[\int_0^T\frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m a_if_i(s,y(s))ds-\mu\int_0^T\frac{(T-s)^{\rho-1}}{\Gamma(\rho)}y(s)\,ds\right]$$
$$-\varpi_3(t)\,\omega g(y),$$

where $\mathcal{C}=C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ endowed with norm $\|y\|=\sup_{t\in[0,T]}|y(t)|$. Observe that the fixed points of the operator $\mathcal{N}$ are solutions of the problem (1.1)-(1.2). Now we show that the operator $\mathcal{N}$ has a unique fixed point by applying Banach fixed point theorem. We verify the hypotheses of Banach fixed point theorem in two steps.

Step 1. Setting $\sup_{t\in[0,T]}|f_i(t,0)|=M_i,\ i=1,2,\ldots,m$, we note that

$$\begin{aligned}|f_i(t,y)| &= |f_i(t,y)-f_i(t,0)+f_i(t,0)|\\ &\le L_i\|y\|+M_i,\qquad M_i>0\end{aligned}$$

and thus

$$|F(t,y(t))|\le\sum_{i=1}^m|a_i|\,(L_i\|y\|+M_i)\le\widehat{L}r+\widehat{M},\tag{4.6}$$

where $\widehat{L}$ is given by (4.4) and

$$\widehat{M}=\sum_{i=1}^m M_i.$$

Now we show that $\mathcal{N}B_r\subset B_r$, where $B_r=\{y\in\mathcal{C}:\|y\|\le r\}$ with

$$r>\frac{\widehat{M}\Omega_1+\widehat{\varpi_3}|\omega||g(0)|}{1-\widehat{L}\Omega_1-\Omega_2-\widehat{\varpi_3}|\omega|\varepsilon}.$$

Now, using (4.6) in (4.5), we get

$$\|\mathcal{N}y\|$$
$$\le\sup_{t\in[0,T]}\left\{\int_0^t\frac{(t-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\left(\sum_{i=1}^m|a_if_i(s,y(s))|\right)ds+|\mu|\int_0^t\frac{(t-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|\,ds\right.$$
$$+|\sigma|\,|\varpi_1(t)|\left[|\mu|\int_0^T\frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)}|y(s)|\,ds\right.$$
$$\left.+\int_0^T\frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)}\left(\sum_{i=1}^m|a_if_i(s,y(s))|\right)ds\right]$$
$$+|\varpi_2(t)|\left[|\mu|\int_0^\eta\frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds+\int_0^\eta\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\left(\sum_{i=1}^m|a_if_i(s,y(s))|\right)ds\right]$$
$$+|\lambda||\varpi_3(t)|\left[\int_0^T\frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\left(\sum_{i=1}^m|a_if_i(s,y(s))|\right)ds\right.$$

$$
\begin{aligned}
& +|\mu| \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} |y(s)| ds \Bigg] + |\varpi_3(t)| |\omega| (|g(y) - g(0)| + |g(0)|) \Bigg\} \\
\leq\; & (\widehat{L}r + \widehat{M}) \left\{ \frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta} \widehat{\varpi_1} |\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha} \widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha} |\lambda| \widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)} \right\} \\
& + r \left\{ \frac{|\mu| T^{\rho}}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_1} |\sigma| |\mu| T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} + \frac{|\mu| \widehat{\varpi_2} \eta^{\rho}}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_3} |\lambda| |\mu| T^{\rho}}{\Gamma(\rho+1)} \right\} + \widehat{\varpi_3} |\omega| (\varepsilon r + |g(0)|) \\
\leq\; & (\widehat{L}r + \widehat{M}) \Omega_1 + r\Omega_2 + \widehat{\varpi_3} |\omega| (\varepsilon r + |g(0)|) \leq r,
\end{aligned}
$$

which shows that $\mathcal{N} B_r \subset B_r$.

Step 2. $\mathcal{N}$ is a contraction. For $x, y \in \mathcal{C}$ and $\forall t \in [0, T]$, we have

$$
\begin{aligned}
& \|(\mathcal{N}x) - (\mathcal{N}y)\| \\
\leq\; & \sup_{t \in [0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \left(\sum_{i=1}^m |a_i [f_i(s, x(s)) - f_i(s, y(s))]| \right) ds \\
& + |\mu| \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} |x(s) - y(s)| ds \\
& + |\sigma| |\varpi_1(t)| \Bigg[|\mu| \int_0^T \frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)} |x(s) - y(s)| ds \\
& + \int_0^T \frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)} \left(\sum_{i=1}^m |a_i [f_i(s, x(s)) - f_i(s, y(s))]| \right) ds \Bigg] \\
& + |\varpi_2(t)| \Bigg[|\mu| \int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} |x(s) - y(s)| ds \\
& + \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \left(\sum_{i=1}^m |a_i [f_i(s, x(s)) - f_i(s, y(s))]| \right) ds \Bigg] \\
& + |\lambda| |\varpi_3(t)| \Bigg[\int_0^T \frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \left(\sum_{i=1}^m |a_i [f_i(s, x(s)) - f_i(s, y(s))]| \right) ds \\
& + |\mu| \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} |x(s) - y(s)| ds \Bigg] + |\varpi_3(t)| |\omega| |g(x) - g(y)| \Bigg\} \\
\leq\; & \widehat{L} \|x - y\| \left[\frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta} \widehat{\varpi_1} |\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha} \widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha} |\lambda| \widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)} \right] \\
& + \|x - y\| \left\{ \frac{|\mu| T^{\rho}}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_1} |\sigma| |\mu| T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} + \frac{|\mu| \widehat{\varpi_2} \eta^{\rho}}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_3} |\lambda| |\mu| T^{\rho}}{\Gamma(\rho+1)} \right\} \\
& + \widehat{\varpi_3} |\omega| \varepsilon \|x - y\| \\
& \leq \left(\widehat{L} \Omega_1 + \Omega_2 + \widehat{\varpi_3} |\omega| \varepsilon \right) \|x - y\|.
\end{aligned}
$$

Since $\widehat{L}\Omega_1 + \Omega_2 + \widehat{\varpi_3}|\omega|\varepsilon < 1$, therefore $\mathcal{N}$ is a contraction. Thus the operator $\mathcal{N}$ has a unique fixed point by Banach fixed point theorem, and hence there exists a unique solution for the boundary value problem (1.1)-(1.2) on $[0,T]$. This completes the proof. □

Example 4.2. Consider the following Langevin boundary value problem:

$$\begin{aligned}
&{}^cD^{\frac{1}{3}}\left({}^cD^2+\tfrac{1}{45}\right)y(t)=\textstyle\sum_{i=1}^3 a_if_i(t,y(t)),\ t\in[0,T],\\
&y'(0)=\tfrac{1}{8}{}^cD^{\frac{1}{2}}y(2),\ y\left(\tfrac{2}{3}\right)=0,\ y(0)+\tfrac{1}{5}y(2)=g(y)=\tfrac{|y|^3}{(t+4)(1+|y|^3)}+\tfrac{1}{8}.
\end{aligned}\tag{4.7}$$

Here $\alpha=1/3,\rho=2,\mu=1/45,\omega=1,m=3,T=2,\epsilon=1/4,\delta=\frac{1}{2},\lambda=1/5,\sigma=1/8,\eta=\frac{2}{3}$ and

$$\begin{aligned}
f_1(t,y)&=\frac{1}{\sqrt{t^2+900}}\frac{|y|}{|y|+1}+\frac{e^t}{9},\ f_2(t,y)=\frac{1}{t^2+25}\tan^{-1}y+\frac{t^2}{16},\\
f_3(t,y)&=\frac{1}{10}\left(\frac{1}{1+t^2}\right)\cos y+\frac{1}{1+2e^{t^2}}.
\end{aligned}$$

It is easy to find that $|f_i(t,x)-f_i(t,y)|\le L_i|x-y|, i=1,2,3$, with $L_1=1/30, L_2=1/25, L_3=1/10$ and $a_1=1/2, a_2=1/4, a_3=1/6$ using togather(4.4) with $m=3$ Furthermore, we find $\widehat{L}=0.043333$, $|\Delta|=T\lambda A_2-\eta A_2A_4+A_1A_3A_4-A_1A_5\approx 0.13298075$, $\widehat{\varpi_1}=\max_{t\in[0,T]}|\varpi_1(t)|\approx 6.6843801$, $\widehat{\varpi_2}=\max_{t\in[0,T]}|\varpi_2(t)|\approx 1.4000082$, $\widehat{\varpi_3}=\max_{t\in[0,T]}|\varpi_3(t)|\approx 1.0000057$ and $\Omega_1=4.099085, \Omega_2=0.0997534$. Clearly $\widehat{L}\Omega_1+\Omega_2+\widehat{\varpi_3}|\omega|\epsilon\approx 0.527380475305<1$. Thus all the assumptions of Theorem 4.1 are satisfied. Therefore the problem (4.7) has a unique solution on $t\in[0,T]$.

5. EXISTENCE RESULTS

In this section we establish existence results for the boundary value problem (1.1)-(1.2) by using Krasnoselskii fixed point theorem [20] and Leray-Schauder nonlinear alternative [21]. We start by proving an existence result via Krasnoselskii fixed point theorem.

Theorem 5.1. *Assume that $g: C([0,T],\mathbb{R})\to\mathbb{R}$ is a continuous function satisfying assumption (A_2). In addition we suppose that:*

(A_3) *The functions $f_i:[0,T]\times\mathbb{R}\to\mathbb{R},\ i=1,2,\dots,m$ are continuous and satisfy the conditions*

$$|f_i(t,y)|\le\phi_i(t),\ \textit{for all }(t,y)\in[0,T]\times\mathbb{R}.$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,T]$, provided that

$$\Omega_2+\widehat{\varpi_3}|\omega|\varepsilon<1.\tag{5.1}$$

Proof. We decompose the operator $\mathcal{N}:\mathcal{C}\to\mathcal{C}$ as

$$(\mathcal{N}y)(t)=(\mathcal{N}_1y)(t)+(\mathcal{N}_2y)(t),\ t\in[0,T],\tag{5.2}$$

where

$$\begin{aligned}
&\left(\mathcal{N}_1 y\right)(t) \\
= &\int_0^t \frac{(t-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \sum_{i=1}^m a_i f_i(s,y(s))ds + \sigma\varpi_1(t) \int_0^T \frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)} \sum_{i=1}^m a_i f_i(s,y(s))ds \\
&+\varpi_2(t) \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \sum_{i=1}^m a_i f_i(s,y(s))ds \\
&+\lambda\varpi_3(t) \int_0^T \frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \sum_{i=1}^m a_i f_i(s,y(s))ds,
\end{aligned}$$

and

$$\begin{aligned}
&\left(\mathcal{N}_2 y\right)(t) \\
= &-\mu \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} y(s)\,ds - \mu\sigma\varpi_1(t) \int_0^T \frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)} y(s)\,ds \\
&-\mu\varpi_2(t) \int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} y(s)\,ds - \mu\lambda\varpi_3(t) \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} y(s)\,ds \\
&-\varpi_3(t)\,\omega g(y).
\end{aligned}$$

Consider a closed ball $B_z = \{y \in \mathcal{C} : \|y\| \leq z\}$, with $z \geq \dfrac{\Omega_1 \sum_{i=1}^m \|\phi_i\| + \widehat{\varpi_3}|\omega||g(0)|}{1-\Omega_2-\widehat{\varpi_3}|\omega|\varepsilon}$.

(i) For $y_1, y_2 \in B_z$ we will prove that $\mathcal{N}_1 y_1 + \mathcal{N}_2 y_2 \in B_z$. Indeed, we have

$$\begin{aligned}
&\|\mathcal{N}_1 y_1 + \mathcal{N}_2 y_2\| \\
\leq &\sup_{t\in[0,T]} \left\{ \int_0^t \frac{(t-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \sum_{i=1}^m |a_i f_i(s,y_1(s))|ds + |\mu| \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} |y_2(s)|\,ds \right. \\
&+|\sigma|\,|\varpi_1(t)| \left[|\mu| \int_0^T \frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)} |y_2(s)|\,ds + \int_0^T \frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)} \sum_{i=1}^m |a_i f_i(s,y_1(s))|ds \right] \\
&+|\varpi_2(t)| \left[|\mu| \int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} |y_2(s)|ds + \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \sum_{i=1}^m |a_i f_i(s,y_1(s))|ds \right] \\
&+|\lambda||\varpi_3(t)| \left[\int_0^T \frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)} \sum_{i=1}^m |a_i f_i(s,y_1(s))|ds + |\mu| \int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)} |y_2(s)|ds \right] \\
&\left. +|\varpi_3(t)||\omega||g(y_2)| \right\} \\
\leq &\sum_{i=1}^m \|\phi_i\| \left\{ \frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta}\widehat{\varpi_1}|\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha}\widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha}|\lambda|\widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)} \right\} \\
&+\|y_2\| \left\{ \frac{|\mu|T^\rho}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_1}|\sigma||\mu|T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} + \frac{|\mu|\widehat{\varpi_2}\eta^\rho}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_3}|\lambda||\mu|T^\rho}{\Gamma(\rho+1)} \right\} + \widehat{\varpi_3}|\omega|(\varepsilon\|y_2\| + |g(0)|) \\
\leq &\sum_{i=1}^m \|\phi_i\| \left\{ \frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta}\widehat{\varpi_1}|\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha}\widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha}|\lambda|\widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)} \right\} \\
&+z \left\{ \frac{|\mu|T^\rho}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_1}|\sigma||\mu|T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} + \frac{|\mu|\widehat{\varpi_2}\eta^\rho}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_3}|\lambda||\mu|T^\rho}{\Gamma(\rho+1)} \right\} + \widehat{\varpi_3}|\omega|(\varepsilon z + |g(0)|)
\end{aligned}$$

$$= \sum_{i=1}^{m}\|\phi_i\|\Omega_1 + \widehat{\varpi_3}|\omega||g(0)| + z\Omega_2 + \widehat{\varpi_3}|\omega|\varepsilon z < z.$$

Hence $\mathcal{N}_1 y_1 + \mathcal{N}_2 y_2 \in B_z$.

(ii) $\mathcal{N}_1$ is compact and continuous. Continuity of $\mathcal{N}_1$ follows from that of F. Also $\mathcal{N}_1$ is uniformly bounded on B_z, as

$$\begin{aligned}
\|\mathcal{N}_1 y\| &\leq \sum_{i=1}^{m}\|\phi_i\|\left\{\frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta}\widehat{\varpi_1}\,|\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha}\widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha}\,|\lambda|\,\widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)}\right\}\\
&= \sum_{i=1}^{m}\|\phi_i\|\Omega_1.
\end{aligned}$$

We will show the compactness of the operator $\mathcal{N}_1$. Let $0 < t_1 < t_2 < T$. Then we have

$$\begin{aligned}
&|(\mathcal{N}_1 y)(t_2) - (\mathcal{N}_1 y)(t_1)|\\
\leq\ & \int_0^{t_1}\frac{(t_2-s)^{\rho+\alpha-1} - (t_1-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^{m}|a_i f_i(s,y(s))|ds\\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^{m}|a_i f_i(s,y(s))|ds\\
&+|\sigma||\varpi_1(t_2) - \varpi_1(t_1)|\int_0^T\frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)}\sum_{i=1}^{m}|a_i f_i(s,y(s))|ds\\
&+|\varpi_2(t_2) - \varpi_2(t_1)|\int_0^{\eta}\frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^{m}|a_i f_i(s,y(s))|ds\\
&+|\lambda|\varpi_3(t_2) - \varpi_3(t_1)|\int_0^T\frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^{m}|a_i f_i(s,y(s))|ds\\
\leq\ & \frac{\sum_{i=1}^{m}\|\phi_i\|}{\Gamma(\rho+\alpha+1)}\Big(|t_2^{\rho+\alpha} - t_1^{\rho+\alpha}| + 2(t_2-t_1)^{\rho+\alpha}\Big)\\
&+|\sigma||\varpi_1(t_2) - \varpi_1(t_1)|\frac{T^{\rho+\alpha-\delta}}{\Gamma(\rho+\alpha-\delta+1)}\\
&+|\varpi_2(t_2) - \varpi_2(t_1)|\frac{\eta^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + |\lambda|\varpi_3(t_2) - \varpi_3(t_1)|\frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)},
\end{aligned}$$

which tends to zero independently of y as $t_1 \to t_2$. Thus $\mathcal{N}_1$ is equicontinuous on B_z. By Arzelá-Ascoli theorem, the operator $\mathcal{N}_1$ is compact ob B_z.

(iii) $\mathcal{N}_2$ is a contraction. Let $y_1, y_2 \in B_z$ and $t \in [0,T]$. Then we have

$$\begin{aligned}
&\|(\mathcal{N}_2 y_2) - (\mathcal{N}_2 y_1)\|\\
\leq\ & \sup_{t\in[0,T}\Bigg\{|\mu|\int_0^t\frac{(t-s)^{\rho-1}}{\Gamma(\rho)}|y_2(s) - y_1(s)|ds\\
&+|\mu||\sigma||\varpi_1(t)|\int_0^T\frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)}|y_2(s) - y_1(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&+|\mu||\varpi_2(t)|\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|y_2(s)-y_1(s)|ds \\
&+|\mu||\lambda||\varpi_3(t)|\int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}|y_2(s)-y_1(s)|ds \\
&+|\varpi_3(t)||\omega|g(y_2)-g(y_1)|\Bigg\} \\
\leq\ & \|x-y\|\left\{\frac{|\mu|T^\rho}{\Gamma(\rho+1)}+\frac{\widehat{\varpi_1}|\sigma||\mu|T^{\rho-\delta}}{\Gamma(\rho-\delta+1)}+\frac{|\mu|\widehat{\varpi_2}\eta^\rho}{\Gamma(\rho+1)}+\frac{\widehat{\varpi_3}|\lambda||\mu|T^\rho}{\Gamma(\rho+1)}\right\} \\
&+\widehat{\varpi_3}|\omega|\varepsilon\|x-y\| \\
=\ & \Big(\Omega_2+\widehat{\varpi_3}|\omega|\varepsilon\Big)\|x-y\|,
\end{aligned}
$$

which in view of (5.1), implies that $\mathcal{N}_2$ is a contraction. Thus, all the conditions of Krasnoselskii's fixed point theorem are satisfied and consequently the boundary value problem (1.1)-(1.2) has at least one solution $[0,T]$. The proof is finished. □

Example 5.2. Using the data for the boundary value problem (4.7), we find that $\Omega_2+\widehat{\varpi_3}|\omega|\epsilon \approx 0.349754825 < 1$. The other conditions of Theorem 5.1 can easily be verified. Therefore there exists at least one solutions for the problem (4.7) as an application of Theorem 5.1.

Now, we show the existence result for the boundary value problem (1.1)-(1.2) by applying Leray-Schauder nonlinear alternative.

Theorem 5.3. *Suppose that* $f_i:[0,T]\times\mathbb{R}\to\mathbb{R}, i=1,2,\dots,m$ *and* $g:C([0,T],\mathbb{R})\to\mathbb{R}$ *are continuous functions. Assume that hypothesis* (A_2) *and the condition (5.1) hold. In addition we suppose that the following conditions are satisfied:*

(A_4) *there exist functions* $p_i\in C([0,T],\mathbb{R})$ *and nondecreasing functions* $\Phi_i:\mathbb{R}^+\to\mathbb{R}^+$, $i=1,2,\dots,m$ *such that*

$$|f_i(t,y)|\leq p_i(t)\Phi_i(|y|),\ \forall(t,y)\in[0,T]\times\mathbb{R};$$

(A_5) *There exists a constant* $M>0$ *such that*

$$\frac{\Big(1-\Omega_2-\widehat{\varpi_3}|\omega|\varepsilon\Big)M}{\sum_{i=1}^m|a_i|\|p_i\|\Phi(M)\Omega_1+\widehat{\varpi_3}|\omega||g(0)|}>1. \tag{5.3}$$

where $\Phi(M)=\max\{\Phi_1(M),\Phi_2(M),\dots,\Phi_m(M)\}$.

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,T]$.

Proof. We first show that the operator $\mathcal{N}:\mathcal{C}\to\mathcal{C}$ defined by (4.5) maps bounded sets into bounded sets in $\mathcal{C}$. For $\theta>0$ let $B_\theta=\{y\in\mathcal{C}:\|y\|\leq\theta\}$ be a bounded set $\mathcal{C}$. Then for $y\in B_\theta$ we have

$$\|\mathcal{N}y\|=\sup_{t\in[0,T]}|(\mathcal{N}y)(t)|$$

$$\leq \sup_{t\in[0,T]}\left\{\int_0^t \frac{(t-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds + |\mu|\int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|\,ds\right.$$
$$+|\sigma|\,|\varpi_1(t)|\left[|\mu|\int_0^T \frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)}|y(s)|\,ds + \int_0^T \frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds\right]$$
$$+|\varpi_2(t)|\left[|\mu|\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds + \int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds\right]$$
$$+|\lambda||\varpi_3(t)|\left[\int_0^T \frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds + |\mu|\int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds\right]$$
$$\left.+|\varpi_3(t)||\omega||g(y)-g(0)|+|g(0)|\right\}$$
$$\leq \sum_{i=1}^m |a_i|\|p_i\|\Phi(\|y\|)\left\{\frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha-\delta}\widehat{\varpi_1}\,|\sigma|}{\Gamma(\rho+\alpha-\delta+1)} + \frac{\eta^{\rho+\alpha}\widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)} + \frac{T^{\rho+\alpha}\,|\lambda|\,\widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)}\right\}$$
$$+\|y\|\left\{\frac{|\mu|\,T^\rho}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_1}\,|\sigma|\,|\mu|\,T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} + \frac{|\mu|\,\widehat{\varpi_2}\eta^\rho}{\Gamma(\rho+1)} + \frac{\widehat{\varpi_3}|\lambda||\mu|T^\rho}{\Gamma(\rho+1)}\right\} + \widehat{\varpi_3}|\omega|(\varepsilon\|y\|+|g(0)|)$$
$$\leq \sum_{i=1}^m |a_i|\|p_i\|\Phi(\|y\|)\Omega_1 + \|y\|\Omega_2 + \widehat{\varpi_3}|\omega|(\varepsilon\|y\|+|g(0)|).$$

Consequently

$$\|\mathcal{N}y\| \leq \sum_{i=1}^m |a_i|\|p_i\|\Phi(\theta)\Omega_1 + \theta\Omega_2 + \widehat{\varpi_3}|\omega|(\varepsilon\theta+|g(0)|),$$

which implies that the operator $\mathcal{N}$ is bounded in $\mathcal{C}$.

Now, we show that $\mathcal{N}$ maps bounded sets into equicontinuous sets. Let $t_1, t_2 \in [0, T$ with $0 < t_1 < t_2$ and $y \in B_\theta$. Then we have

$$|(\mathcal{N}y)(t_2) - (\mathcal{N}y)(t_1)|$$
$$\leq \int_0^{t_1} \frac{(t_2-s)^{\rho+\alpha-1}-(t_1-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds$$
$$+\int_{t_1}^{t_2} \frac{(t_2-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds$$
$$+|\sigma||\varpi_1(t_2)-\varpi_1(t_1)|\int_0^T \frac{(T-s)^{\rho+\alpha-\delta-1}}{\Gamma(\rho+\alpha-\delta)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds$$
$$+|\varpi_2(t_2)-\varpi_2(t_1)|\int_0^\eta \frac{(\eta-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds$$
$$+|\lambda|\varpi_3(t_2)-\varpi_3(t_1)|\int_0^T \frac{(T-s)^{\rho+\alpha-1}}{\Gamma(\rho+\alpha)}\sum_{i=1}^m |a_i f_i(s,y(s))|ds$$
$$+|\mu|\left[\int_0^{t_1} \frac{(t_2-s)^{\rho-1}-(t_1-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds\right]$$

$$
\begin{aligned}
&+|\mu||\sigma||\varpi_1(t_2)-\varpi_1(t_1)|\int_0^T \frac{(T-s)^{\rho-\delta-1}}{\Gamma(\rho-\delta)}|y(s)|ds \\
&+|\mu||\varpi_2(t_2)-\varpi_2(t_1)|\int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds \\
&+|\mu||\lambda||\varpi_3(t_2)-\varpi_3(t_1)|\int_0^T \frac{(T-s)^{\rho-1}}{\Gamma(\rho)}|y(s)|ds+|\varpi_3(t_2)-\varpi_3(t_1)||\omega||g(y)| \\
\leq\ & \frac{\sum_{i=1}^m |a_i|\|p_i\|\Phi(\theta)}{\Gamma(\rho+\alpha+1)}\Big(|t_2^{\rho+\alpha}-t_1^{\rho+\alpha}|+2(t_2-t_1)^{\rho+\alpha}\Big) \\
&+|\sigma||\varpi_1(t_2)-\varpi_1(t_1)|\frac{T^{\rho+\alpha-\delta}}{\Gamma(\rho+\alpha-\delta+1)} \\
&+|\varpi_2(t_2)-\varpi_2(t_1)|\frac{\eta^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)}+|\lambda|\varpi_3(t_2)-\varpi_3(t_1)|\frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)} \\
&+\frac{|\mu|\theta}{\Gamma(\rho+1)}\Bigg\{\Big(|t_2^\rho-t_1^\rho|+2(t_2-t_1)^\rho\Big)+|\sigma||\varpi_1(t_2)-\varpi_1(t_1)|\frac{T^{\rho-\delta}}{\Gamma(\rho-\delta+1)} \\
&+|\varpi_2(t_2)-\varpi_2(t_1)|\frac{\eta^\rho}{\Gamma(\rho+1)}+|\lambda||\varpi_3(t_2)-\varpi_3(t_1)|\frac{T^\rho}{\Gamma(\rho+1)}\Bigg\} \\
&+|\varpi_3(t_2)-\varpi_3(t_1)||\omega|(\varepsilon\theta+|g(0)|).
\end{aligned}
$$

Notice that the right hand side of the above inequality tends to zero as $t_1 \to t_2$ independent of $y \in B_\theta$. This shows that $\mathcal{N}$ is equicontinuous. By Arzelá-Ascoli theorem, the operator $\mathcal{N}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

Finally, we show that the set of all solutions to equations $y=\zeta\mathcal{N}y$ for $0<\zeta<1$ is bounded. For $t\in[0,T]$ we have, by using the computations in the first step above,

$$
\begin{aligned}
&|y(t)|=|\zeta(\mathcal{N}y)(t)| \\
\leq\ & \sum_{i=1}^m |a_i|\|p_i\|\Phi(\|y\|)\left\{\frac{T^{\rho+\alpha}}{\Gamma(\rho+\alpha+1)}+\frac{T^{\rho+\alpha-\delta}\widehat{\varpi_1}|\sigma|}{\Gamma(\rho+\alpha-\delta+1)}+\frac{\eta^{\rho+\alpha}\widehat{\varpi_2}}{\Gamma(\rho+\alpha+1)}+\frac{T^{\rho+\alpha}|\lambda|\widehat{\varpi_3}}{\Gamma(\rho+\alpha+1)}\right\} \\
&+\|y\|\left\{\frac{|\mu|T^\rho}{\Gamma(\rho+1)}+\frac{\widehat{\varpi_1}|\sigma||\mu|T^{\rho-\delta}}{\Gamma(\rho-\delta+1)}+\frac{|\mu|\widehat{\varpi_2}\eta^\rho}{\Gamma(\rho+1)}+\frac{\widehat{\varpi_3}|\lambda||\mu|T^\rho}{\Gamma(\rho+1)}\right\}+\widehat{\varpi_3}|\omega|(\varepsilon\|y\|+|g(0)|) \\
\leq\ & \sum_{i=1}^m |a_i|\|p_i\|\Phi(\|y\|)\Omega_1+\|y\|\Omega_2+\widehat{\varpi_3}|\omega|(\varepsilon\|y\|+|g(0)|),
\end{aligned}
$$

which yields

$$
\|y\|\leq\sum_{i=1}^m |a_i|\|p_i\|\Phi(\|y\|)\Omega_1+\|y\|\Omega_2+\widehat{\varpi_3}|\omega|(\varepsilon\|y\|+|g(0)|),
$$

or

$$
\frac{\Big(1-\Omega_2-\widehat{\varpi_3}|\omega|\varepsilon\Big)\|y\|}{\sum_{i=1}^m |a_i|\|p_i\|\Phi(\|y\|)\Omega_1+\widehat{\varpi_3}|\omega||g(0)|}\leq 1.
$$

By (5.3), we can find a positive real number M such that $\|y\|\neq M$. Consider the set $\mathcal{U}=\{y\in\mathcal{C}:\|y\|<M\}$. Observe that the operator $\mathcal{N}:\bar{\mathcal{U}}\to\mathcal{C}$ is continuous and completely continuous.Thus, from the choice of $\mathcal{U}$, there is no $y\in\partial\mathcal{U}$ such that

$y = \zeta \mathcal{N} y$ for some $0 < \zeta, 1$. Therefore, by Leray-Schauder nonlinear alternative, the operator $\mathcal{N}$ has a fixed point $y \in \bar{\mathcal{U}}$, which implies that the boundary value problem (1.1)-(1.2) has at least one solution $[0, T]$. The proof of the theorem is completed. □

Example 5.4. Using the data for the boundary value problem (4.7), we find that $\|p_1\| = (9 + 10e^2)/270, \|p_2\| = (2\pi + 25)/100, \|p_3\| = 13/180, \Phi(M) = 1$. Then, by the condition (5.3), we find that $M > 0.4743405$. Obviously all the assumptions of Theorem 5.3 are satisfied. Hence the conclusion of Theorem 5.3 applies to the problem (4.7).

6. CONCLUSIONS

In this paper, we have obtained the criteria ensuring the existence and uniqueness of solutions for a Langevin type fractional boundary value problem involving finitely many nonlinearities and nonlocal nonlinear fractional boundary conditions. The main results rely on the tools (Banach contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative) of the fixed point theory. As a special case, our results correspond to the problem with the boundary conditions of the form: $y'(0) = 0, y(\eta) = 0, y(0) + \lambda y(T) = \omega g(y)$ by taking σ to be zero in the results of this paper. Here it is worth-mentioning that the consideration of Langevin equation (1.1) in the present setting provides a leverage to take into account different types of of nonlinearities, such as $\sum_{i=1}^{m} a_i \int_0^t k_i(t,s)y(s)ds$ or $\sum_{i=1}^{m} a_i \int_0^t g_i(s, y(s))ds$ or $\sum_{i=1}^{m} a_i I^p g_i(t, y(t))$ or $\sum_{i=1}^{m} a_i f_i(t, y(t)) + \sum_{j=1}^{\kappa} a_j I^{q_j} g_j(t, y(t))$, $q_j > 0$, or some of the functions may be non-Lipschitz type. In summary, our results are new in the given setting and contribute significantly to the present literature on the Langevin equation.

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