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A FILIPPOV'S THEOREM AND TOPOLOGICAL STRUCTURE OF SOLUTION SETS FOR IMPLICIT FRACTIONAL q-DIFFERENCE INCLUSIONS

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ABSTRACT. In this paper we present some existence results and topological structure of the solution set for a class of Caputo implicit fractional q-difference inclusions in Banach spaces. Firstly, using the set-valued analysis, we study some global existence results and we present a new version of Filippov's Theorem. Further, we obtain results in the cases where the nonlinearity is upper as well as lower semi-continuous with respect to the second argument by using Mönch's and Schauder-Tikhonov fixed point theorems and the concept of measure of noncompactness. In the last section, we illustrate our results by an example.

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1. Introduction

Numerous mathematicians and physicists have shown a greater interest in fractional equations and inclusions, which give an efficient way to describe several practical dynamical developments in engineering and other applied sciences [1,3,4,6–8,14, 25,36,44,45,48,51]. Recently, many substantial and interesting results on initial and

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boundary value problems for fractional differential equations with Riemann-Liouville and Caputo fractional derivatives have been obtained [2, 3, 34, 42, 43, 50].

q-difference equations were established in the early nineteenth century [5, 19], and have gained a considerable interest recently. We suggest the papers [10, 11, 26, 52] and references therein, for some important results on q-difference and fractional q-difference equations and inclusions.

Filippov's solutions for various classes of integer or fractional order differential inclusions have been considered in the literature; see for instance [21–23, 31].

Implicit fractional differential equations have been studied by numerous researchers. For more information, we refer the readers to the papers [18, 46, 47, 49].

In this paper, we shall be concerned with a Filippov's theorem, existence of solutions and the topological structure of solution sets for the following fractional q-difference problem:

$$({}^cD_q^\zeta \mathfrak{w})(\vartheta) \in \Psi\left(\vartheta, \mathfrak{w}(\vartheta), ({}^cD_q^\zeta \mathfrak{w})(\vartheta)\right), \ \vartheta \in \Theta := [0, \kappa], \tag{1.1}$$

$$\mathfrak{w}(0) = \mathfrak{w}_0 \in \Xi, \tag{1.2}$$

where $(\Xi, \|\cdot\|)$ is a separable real or complex Banach space, $q \in (0,1)$, $\zeta \in (0,1]$, $\kappa > 0$, $\Psi : \Theta \times \Xi \times \Xi \to \mathcal{P}(\Xi)$ is a multivalued map, $\mathcal{P}(\Xi)$ is the family of all nonempty subsets of Ξ, ${}^cD_q^\zeta$ is the Caputo fractional q-difference derivative of order ζ.

2. **Preliminaries**

By $\mathfrak{F}(\Theta) := C(\Theta, \Xi)$, we denote the Banach space of continuous functions from Θ into Ξ with the norm

$$\|\mathfrak{w}\|_\infty := \sup_{\vartheta \in \Theta} \|\mathfrak{w}(\vartheta)\|.$$

Consider the space $L^1(\Theta)$ of measurable functions $\mathfrak{w} : \Theta \to \Xi$ which are Bochner integrable with the norm

$$\|\mathfrak{w}\|_1 = \int_\Theta \|\mathfrak{w}(\vartheta)\| d\vartheta.$$

Let us revisit some fractional q-calculus definitions and properties. For $\beta_1 \in \mathbb{R}$, we set

$$[\beta_1]_q = \frac{1 - q^{\beta_1}}{1 - q}.$$

The q-analogue of the power $(\beta_1 - \beta_2)^\alpha$ is

$$(\beta_1 - \beta_2)^{(0)} = 1, \ (\beta_1 - \beta_2)^{(\alpha)} = \Pi_{\xi=0}^{\alpha-1}(\beta_1 - \beta_2 q^\xi); \ \beta_1, \beta_2 \in \mathbb{R}, \ \alpha \in \mathbb{N}.$$

In general,

$$(\beta_1 - \beta_2)^{(\zeta)} = {\beta_1}^{\zeta}\Pi_{\xi=0}^{\infty}\left(\frac{\beta_1 - \beta_2 q^{\xi}}{\beta_1 - \beta_2 q^{\xi+\zeta}}\right);\ \beta_1, \beta_2, \zeta \in \mathbb{R}.$$

Definition 2.1. [33] The q-gamma function is given by

$$\Gamma_q(\varepsilon) = \frac{(1-q)^{(\varepsilon-1)}}{(1-q)^{\varepsilon-1}};\ \varepsilon \in \mathbb{R} - \{0, -1, -2, \dots\},$$

where $\Gamma_q(1+\varepsilon) = [\varepsilon]_q \Gamma_q(\varepsilon)$.

Definition 2.2. [33] The q-derivative of order $\alpha \in \mathbb{N}$ of a function $\mathfrak{w} : \Theta \to \Xi$ is given by $(D_q^0 \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta)$,

$$(D_q \mathfrak{w})(\vartheta) := (D_q^1 \mathfrak{w})(\vartheta) = \frac{\mathfrak{w}(\vartheta) - \mathfrak{w}(q\vartheta)}{(1-q)\vartheta};\ \vartheta \neq 0,\quad (D_q \mathfrak{w})(0) = \lim_{\vartheta \to 0} (D_q \mathfrak{w})(\vartheta),$$

and

$$(D_q^{\alpha} \mathfrak{w})(\vartheta) = (D_q D_q^{\alpha-1} \mathfrak{w})(\vartheta);\ \vartheta \in \Theta,\ \alpha \in \{1, 2, \dots\}.$$

Set $\Theta_{\vartheta} := \{\vartheta q^{\alpha} : \alpha \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3. [33] The q-integral of a function $\mathfrak{w} : \Theta_{\vartheta} \to \Xi$ is defined by

$$(I_q \mathfrak{w})(\vartheta) = \int_0^{\vartheta} \mathfrak{w}(\varrho) d_q \varrho = \sum_{\alpha=0}^{\infty} \vartheta(1-q) q^{\alpha} \psi(\vartheta q^{\alpha}).$$

It should be noted that $(D_q I_q \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta)$, while if $\mathfrak{w}$ is continuous at 0, then

$$(I_q D_q \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta) - \mathfrak{w}(0).$$

Definition 2.4. [9] The Riemann-Liouville fractional q-integral of order $\zeta \in \mathbb{R}_+ := [0, \infty)$ of a function $\mathfrak{w} : \Theta \to \Xi$ is given by $(I_q^0 \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta)$, and

$$(I_q^{\zeta} \mathfrak{w})(\vartheta) = \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{w}(\varrho) d_q \varrho;\ \vartheta \in \Theta.$$

Lemma 2.5. *[40] For $\zeta \in \mathbb{R}_+ := [0, \infty)$ and $\varpi \in (-1, \infty)$ we have*

$$(I_q^{\zeta} (\vartheta - a)^{(\varpi)})(\vartheta) = \frac{\Gamma_q(1+\varpi)}{\Gamma(1+\varpi+\zeta)} (\vartheta - a)^{(\varpi+\zeta)};\ 0 < a < \vartheta < \kappa.$$

In particular,

$$(I_q^{\zeta} 1)(\vartheta) = \frac{1}{\Gamma_q(1+\zeta)} \vartheta^{(\zeta)}.$$

Definition 2.6. [41] The Riemann-Liouville fractional q-derivative of order $\zeta \in \mathbb{R}_+$ of a function $\mathfrak{w} : \Theta \to \Xi$ is given by $(D_q^0 \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta)$, and

$$(D_q^{\zeta} \mathfrak{w})(\vartheta) = (D_q^{[\zeta]} I_q^{[\zeta]-\zeta} \mathfrak{w})(\vartheta);\ \vartheta \in \Theta,$$

where $[\zeta]$ is the integer part of ζ.

Definition 2.7. [41] The Caputo fractional q-derivative of order $\zeta \in \mathbb{R}_+$ of a function $\mathfrak{w} : \Theta \to \Xi$ is defined by $({}^C D_q^0 \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta)$, and

$$({}^C D_q^\zeta \mathfrak{w})(\vartheta) = (I_q^{[\zeta]-\zeta} D_q^{[\zeta]} \mathfrak{w})(\vartheta); \ \vartheta \in \Theta.$$

Lemma 2.8. *[41] Let $\zeta \in \mathbb{R}_+$. Then the following holds:*

$$(I_q^\zeta \, {}^C D_q^\zeta \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta) - \sum_{\xi=0}^{[\zeta]-1} \frac{\vartheta^\xi}{\Gamma_q(1+\xi)} (D_q^\xi \mathfrak{w})(0).$$

In particular, if $\zeta \in (0,1)$, then

$$(I_q^\zeta \, {}^C D_q^\zeta \mathfrak{w})(\vartheta) = \mathfrak{w}(\vartheta) - \mathfrak{w}(0).$$

We also use the subsets of $\mathcal{P}(\Xi)$ that follow (see [31] for more details):

$P_{cl}(\Xi) = \{\Phi \in \mathcal{P}(\Xi) : \Phi \text{ is closed}\}$,
$P_b(\Xi) = \{\Phi \in \mathcal{P}(\Xi) : \Phi \text{ is bounded}\}$,
$P_{cp}(\Xi) = \{\Phi \in \mathcal{P}(\Xi) : \Phi \text{ is compact}\}$
$P_{cv}(\Xi) = \{\Phi \in \mathcal{P}(\Xi) : \Phi \text{ is convex}\}$
$P_{cp,cv}(\Xi) = P_{cp}(\Xi) \cap P_{cv}(\Xi)$.

We denote by $Fix\mathfrak{S}$ the fixed point set of the multivalued operator $\mathfrak{S}$.

Definition 2.9. A multivalued map $\mathfrak{S} : \Theta \to P_{cl}(\Xi)$ is said to be measurable if for every $\mathfrak{z}_1 \in \Xi$, the function:

$$\vartheta \to d(\mathfrak{z}_1, \mathfrak{S}(\vartheta)) = \inf\{|\mathfrak{z}_1 - \mathfrak{z}_2| : \mathfrak{z}_2 \in \mathfrak{S}(\vartheta)\}$$

is measurable.

Lemma 2.10. *[31,32] Let $\mathfrak{S}$ be a completely continuous multivalued map with nonempty compact values, then $\mathfrak{S}$ is upper semi-continuous (u.s.c.) if and only if $\mathfrak{S}$ has a closed graph.*

Definition 2.11. A multi-valued map $\Psi : \Theta \times \Xi \times \Xi \to \mathcal{P}(\Xi)$ is Carathéodory if:

(1) $\vartheta \to \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})$ is measurable for each $\mathfrak{w}, \mathfrak{y} \in \Xi$;
(2) $\mathfrak{w} \to \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})$ is upper semicontinuous for almost all $\vartheta \in \Theta$.

Ψ is called L^1-Carathéodory if (1), (2) and the following requirements are met:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(\Theta, \mathbb{R}^+)$ where

$\|\Psi(\vartheta, \mathfrak{w}, \mathfrak{y})\|_{\mathcal{P}} = \sup\{|\mathfrak{z}_2| : \mathfrak{z}_2 \in \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})\} \leq \varphi_q$ for all $|\mathfrak{w}|, |\mathfrak{y}| \leq q$ and for a.e. $\vartheta \in \Theta$.

For each $\mathfrak{z}_1 \in \mathfrak{F}(\Theta)$, define the set of selections of Ψ by

$$S_{\Psi \circ \mathfrak{z}_1} = \{\mathfrak{z}_2 \in L^1(\Theta) : \mathfrak{z}_2(\vartheta) \in \Psi(\vartheta, \mathfrak{z}_1(\vartheta), {}^c D_q^\zeta \mathfrak{z}_1(\vartheta)) \text{ a.e. } \vartheta \in \Theta\}.$$

Let (Ξ, d) be a metric space induced from the normed space $(\Xi, |\cdot|)$. The function $\mathcal{H}_d : \mathcal{P}(\Xi) \times \mathcal{P}(\Xi) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$\mathcal{H}_d(\Phi_1, \Phi_2) = \max\{ \sup_{\beta_1 \in \Phi_1} d(\beta_1, \Phi_2), \sup_{\beta_2 \in \Phi_2} d(\Phi_1, \beta_2)\}$$

is referred to as the Hausdorff-Pompeiu metric. For further details on multivalued maps see works by Hu and Papageorgiou [32].

The symbol $\mathcal{M}_{\bar{\Xi}}$ stands for the class of all bounded subsets of a metric space $\bar{\Xi}$.

Definition 2.12. Let $\bar{\Xi}$ be a complete metric space. A function $\mu : \mathcal{M}_{\bar{\Xi}} \to [0, \infty)$ is said to be a measure of noncompactness on $\bar{\Xi}$ if the following conditions are verified for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{M}_{\bar{\Xi}}$.

(a) Regularity, i.e., $\mu(\Omega) = 0$ if and only if Ω is precompact,
(b) invariance under closure, i.e., $\mu(\Omega) = \mu(\overline{\Omega})$,
(c) semi-additivity, i.e., $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$.

Definition 2.13. [16] Let Ξ be a Banach space and denote by Ω_Ξ the family of bounded subsets of Ξ. the map $\mu : \Omega_\Xi \to [0, \infty)$ defined by

$$\mu(\tilde{\Phi}) = \inf\{\nu > 0 : \tilde{\Phi} \subset \cup_{j=1}^m \tilde{\Phi}_j, \mathrm{diam}(\tilde{\Phi}_j) \leq \nu\},\ \tilde{\Phi} \in \Omega_\Xi,$$

is called the Kuratowski measure of noncompactness.

Theorem 2.14. *[30] Let Ξ be a Banach space. Let $\tilde{\Omega} \subset L^1(\Theta)$ be a countable set with $|\mathfrak{w}(\vartheta)| \leq \delta(\vartheta)$ for a.e. $\vartheta \in \Theta$ and every $\mathfrak{w} \in \tilde{\Omega}$, where $\delta \in L^1(\Theta, \mathbb{R}_+)$. Then $\mu(\tilde{\Omega}(\vartheta)) \in L^1(\Theta, \mathbb{R}_+)$ and verifies*

$$\mu\left(\left\{\int_0^\kappa \mathfrak{w}(\varrho)\, d\varrho : \mathfrak{w} \in \tilde{\Omega}\right\}\right) \leq 2 \int_0^\kappa \mu(\tilde{\Omega}(\varrho))\, d\varrho,$$

where μ is the Kuratowski measure of noncompactness on the set Ξ.

Lemma 2.15. *[35] Let Θ be a compact real interval. Let Ψ be a Carathéodory multivalued map and let $\mathfrak{S}$ be a linear continuous map from $L^1(\Theta) \to \mathfrak{F}(\Theta)$. Then the operator*

$$\mathfrak{S} \circ S_{\Psi\circ\mathfrak{w}} : \mathfrak{F}(\Theta) \to \mathcal{P}_{cv,cp}(\mathfrak{F}(\Theta)), \quad \mathfrak{w} \mapsto (\mathfrak{S} \circ S_{\Psi\circ\mathfrak{w}})(\mathfrak{w}) = \mathfrak{S}(S_{\Psi\circ\mathfrak{w}})$$

is a closed graph operator in $\mathfrak{F}(\Theta) \times \mathfrak{F}(\Theta)$.

Definition 2.16. Let $\bar{\Xi}$ be Banach space. A multivalued mapping $\mathfrak{S} : \bar{\Xi} \to \mathcal{P}_{cl,b}(\bar{\Xi})$ is $\xi-$set- Lipschitz if there exists a constant $\xi > 0$, where $\mu(\mathfrak{S}(\Omega)) \leq \xi\mu(\Omega)$ for all $\Omega \in \mathcal{P}_{cl,b}(\Xi)$ with $\mathfrak{S}(\Omega) \in \mathcal{P}_{cl,b}(\Xi)$. If $\xi < 1$, then $\mathfrak{S}$ is said to be a $\xi-$set-contraction on $\bar{\Xi}$.

Theorem 2.17. *(Mönch fixed point theorem) [38] Let Ξ be Banach space and $\Omega_1 \subset \Xi$ be a closed and convex set. Also, let Ω_2 be a relatively open subset of Ω_1 and $\mathfrak{S} : \overline{\Omega_2} \to$*

$\mathcal{P}_c(\Omega_1)$. *Suppose that* $\mathfrak{S}$ *maps compact sets into relatively compact sets,* graph($\mathfrak{S}$) *is closed and for some* $x_0 \in \Omega_2$, *we have*

(2.1)

$$\operatorname{conv}(x_0 \cup \mathfrak{S}(\Phi)) \supset \Phi \subset \overline{\Omega_2} \text{ and } \overline{\Phi} = \overline{\Omega_2} \ (\tilde{\Omega} \subset \Phi \text{ countable}) \text{ imply } \overline{\Phi} \text{ is compact}$$

and

$$x \notin (1-\varpi)x_0 + \varpi\mathfrak{S}(x) \quad \forall x \in \overline{\Omega_2}\backslash\Omega_2, \ \varpi \in (0,1). \tag{2.2}$$

Then there exists $x \in \overline{\Omega_2}$ *with* $x \in \mathfrak{S}(x)$.

Also, we recall the Schauder-Tikhonov fixed point theorem:

Theorem 2.18. *(Schauder-Tikhonov fixed point theorem) [15] Let* $\bar{\Xi}$ *be a locally convex space,* $\tilde{\Omega}$ *a convex closed subset of* $\bar{\Xi}$ *and* $\mathfrak{S} : \tilde{\Omega} \to \tilde{\Omega}$ *is a continuous, compact map. Then* $\mathfrak{S}$ *has at least one fixed point in* $\tilde{\Omega}$.

3. Filippov's Theorem

Consider $\mathfrak{T} : \mathfrak{F}(\Theta) \to \mathcal{P}(\mathfrak{F}(\Theta))$, the operator defined by:

$$\mathfrak{T}(\mathfrak{w}) = \left\{ \delta \in \mathfrak{F}(\Theta) : \delta(\vartheta) = \mathfrak{w}_0 + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}(\varrho) d_q\varrho; \ \mathfrak{z} \in S_{\Psi\circ\mathfrak{w}} \right\}. \tag{3.1}$$

It is clear that the fixed points of $\mathfrak{T}$ are solutions of (1.1)-(1.2). First, we state the definition of a solution of the problem (1.1)-(1.2).

Definition 3.1. By a solution of the problem (1.1)-(1.2) we mean a function $\delta \in \mathfrak{F}(\Theta)$ that verifies

$$\delta(\vartheta) = \mathfrak{w}_0 + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}(\varrho) d_q\varrho,$$

where $\mathfrak{z} \in S_{\Psi\circ\mathfrak{w}}$.

Lemma 3.2. *[39] Let* $\mathfrak{S} : \Theta \to \mathcal{P}_{cl}(\Xi)$ *be a measurable multifunction and* $\mathfrak{w} : \Theta \to \Xi$ *be a measurable function. Assume that there exists* $p \in L^1(\Theta, \Xi)$ *such that* $\mathfrak{S}(\vartheta) \subset p(\vartheta)\Omega_0$, *where* $\Omega_0 := \Omega(0,1)$ *denotes the closed ball in* Ξ. *Then there exists a measurable selection* $\varkappa$ *of* $\mathfrak{S}$ *such that for a.e.* $\vartheta \in \Theta$,

$$\|\mathfrak{w}(\vartheta) - \varkappa(\vartheta)\| \leq d(\mathfrak{w}(\vartheta), \mathfrak{S}(\vartheta)).$$

Let $x_0 \in \Xi$, $\varkappa \in L^1(\Theta, \Xi)$, and let $x \in \mathfrak{F}(\Theta)$ be a solution of the fractional q-difference problem:

$$\begin{cases} ({}^cD_q^{\zeta}x)(\vartheta) = \varkappa(\vartheta), \ \vartheta \in \Theta, \\ x(0) = x_0. \end{cases} \tag{3.2}$$

The hypotheses:

($\mathcal{A}_1$) The multivalued map $\Psi : \Theta \times \Xi \times \Xi \to \mathcal{P}(\Xi)$ satisfies:
($\mathcal{A}_{1a}$) the map $\vartheta \mapsto \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})$ is measurable; for all $\mathfrak{w}, \mathfrak{y} \in \Xi$,
($\mathcal{A}_{1b}$) the map $\varpi : \vartheta \mapsto d\left(\psi(\vartheta), \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))\right)$ is integrable.
($\mathcal{A}_2$) There exists a function $\omega_1 \in L^\infty(\Theta, \mathbb{R}_+)$ such that

$$\mathcal{H}_d(\Psi(\vartheta, \mathfrak{w}, \mathfrak{y}), \Psi(\vartheta, \mathfrak{z}, \bar{\mathfrak{y}})) \le \omega_1(\vartheta)\|\mathfrak{w} - \mathfrak{z}\|;$$

for a.e. $\vartheta \in \Theta$, and each $\mathfrak{w}, \mathfrak{z}, \mathfrak{y}, \bar{\mathfrak{y}} \in \Xi$.

Remark 3.3. From Assumptions ($\mathcal{A}_{1a}$) and ($\mathcal{A}_{1b}$), the multi-function $\vartheta \mapsto \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})$ is measurable, and by Lemmas 1.4 and 1.5 from [27], $\mathfrak{z}(\vartheta) = d\left(\psi(\vartheta), \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))\right)$ is measurable.

Set

$$\omega_1{}^* = esssup_{\vartheta\in\Theta}\omega_1(\vartheta).$$

Theorem 3.4. *If ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are met, then the (1.1)-(1.2) has at least one solution $\mathfrak{w}$ defined on Θ. Moreover, for a.e. $\vartheta \in \Theta$, $\mathfrak{w}$ satisfies the estimates:*

$$\|\mathfrak{w}(\vartheta) - x(\vartheta)\| \le \|\mathfrak{w}_0 - x_0\| + \frac{\omega_1{}^*\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^\vartheta \sum_{i=2}^\infty \|x_i(\varrho) - x_{i-1}(\varrho)\| d_q\varrho,$$

and

$$\|({}^cD_q^\zeta \mathfrak{w})(\vartheta) - \varkappa(\vartheta)\| \le \omega_1{}^* \sum_{i=2}^\infty \|x_i(\vartheta) - x_{i-1}(\vartheta)\|,$$

where

$$\|x_\alpha(\vartheta) - x_{\alpha-1}(\vartheta)\| \le \left(\frac{\omega_1{}^*\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\right)^{\alpha-1} \int_0^\vartheta \int_0^{\varrho_1} \int_0^{\varrho_2} \cdots \int_0^{\varrho_{\alpha-2}} \Big(\|\mathfrak{w}_0 - x_0\| + \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\varrho_{\alpha-1}} \varpi(\tau) d_q\tau\Big) d_q\varrho_{\alpha-1} d_q\varrho_{\alpha-2} \cdots d_q\varrho_1.$$

Proof. First, we establish a sequence of functions $(\mathfrak{w}_\alpha)_{\alpha\in\mathbb{N}}$ which will be demonstrated to converges to a solution of (1.1)-(1.2) on Θ.
Let $\psi_0 = \varkappa$ on Θ. So, we have

$$x(\vartheta) = x_0 + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \psi_0(\varrho) d_q\varrho.$$

Define the multi-valued map $\Lambda_1 : \Theta \to \mathcal{P}(\Xi)$ by

$$\Lambda_1(\vartheta) + \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta)) \cap (\psi_0(\vartheta) + \varpi(\vartheta)\Omega_0).$$

Since ψ_0 and ϖ are measurable, the ball $(\psi_0(\vartheta) + \varpi(\vartheta)\Omega_0)$ is measurable from Theorem III.4.1 in [20]. Moreover $\Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))$ is measurable and Λ_1 is nonempty. It is clear that for a.e. $\vartheta \in \Theta$,

$$\begin{aligned} & d(0, \Psi(\vartheta, 0, 0)) \\ & \leq d(0, \psi_0(\vartheta)) + d(\psi_0(\vartheta), \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))) + \mathcal{H}_d(\Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta)), \Psi(\vartheta, 0, 0)) \\ & \leq \|\psi_0(\vartheta)\| + \varpi(\vartheta) + \omega_1(\vartheta)\|x(\vartheta)\|. \end{aligned}$$

Hence for all $\mathfrak{d} \in \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))$, we have

$$\begin{aligned} \|\mathfrak{d}\| &\leq d(0, \Psi(\vartheta, 0, 0)) + \mathcal{H}_d(\Psi(\vartheta, 0), \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))) \\ &\leq \|\psi_0(\vartheta)\| + \varpi(\vartheta) + 2p(\vartheta)\|x(\vartheta)\| := \gamma(\vartheta). \end{aligned}$$

This implies that

$$\Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta)) \subset \gamma(\vartheta)\Omega_0;\ \vartheta \in \Theta.$$

From Lemma 3.2, there exists $\mathfrak{w}$ which is a measurable selection of $\Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))$ such that

$$\|\mathfrak{w}(\vartheta) - \psi_0(\vartheta)\| \leq d(\psi_0(\vartheta), \Psi(\vartheta, x(\vartheta), {}^cD_q^\zeta x(\vartheta))) = \varpi(\vartheta).$$

Then $\mathfrak{w} \in \Lambda_1(\vartheta)$. We conclude that the intersection multivalued operator $\Lambda_1(\vartheta)$ is measurable (see [20, 39]). By Kuratowski-Ryll-Nardzewski selection theorem, there exists a function $\vartheta \to \psi_1(\vartheta)$ which is a measurable selection for Λ_1. Suppose

$$x_1(\vartheta) = \mathfrak{w}_0 + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \psi_1(\varrho) d_q\varrho.$$

For each $\vartheta \in \Theta$, we have

$$\|x_1(\vartheta) - x(\vartheta)\| \leq \|\mathfrak{w}_0 - x_0\| + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \|\psi_1(\varrho) - \psi_0(\varrho)\| d_q\varrho$$

$$\leq \|\mathfrak{w}_0 - x_0\| + \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^\vartheta \varpi(\varrho) d_q\varrho. \tag{3.3}$$

Next, from Lemma 1.4 in [27], $\Psi(\vartheta, x_1(\vartheta), {}^cD_q^\zeta x_1(\vartheta))$ is measurable. The ball $(\psi_1(\vartheta) + \omega_1(\vartheta)\|x_1(t) - x(\vartheta)\|\Omega_0)$ is also measurable. The set $\Lambda_2(\vartheta) = \Psi(\vartheta, x_1(\vartheta), {}^cD_q^\zeta x_1(\vartheta)) \cap (\psi_1(\vartheta) + \omega_1(\vartheta)|x_1(\vartheta) - x(\vartheta)\|\Omega_0)$ is nonempty. Since ψ_1 is a measurable function, Lemma 3.2 yields a measurable selection $\mathfrak{w}$ of $\Psi(\vartheta, x_1(\vartheta), {}^cD_q^\zeta x_1(\vartheta))$ such that

$$\|\mathfrak{w}(\vartheta) - \psi_1(\vartheta)\| \leq d(\psi_1(\vartheta), \Psi(\vartheta, x_1(\vartheta), {}^cD_q^\zeta x_1(\vartheta))).$$

Then using $(\mathcal{A}_2)$, we get

$$\begin{aligned} \|\mathfrak{w}(\vartheta) - \psi_1(\vartheta)\| &\leq d(\psi_1(\vartheta), \Psi(\vartheta, x_1(\vartheta), {}^cD_q^\zeta x_1(\vartheta))) \\ &\leq \mathcal{H}_d(\Psi(\vartheta, x(\vartheta)), \Psi(\vartheta, x_1(\vartheta), {}^cD_q^\zeta x_1(\vartheta))) \\ &\leq \omega_1(\vartheta)\|x(\vartheta) - x_1(\vartheta)\|. \end{aligned}$$

Thus, $\mathfrak{w} \in \Lambda_2(\vartheta)$. Further, as the intersection multi-valued operator Λ_2 given previously is measurable, there exists a measurable selection $\psi_2(\vartheta) \in \Lambda_2(\vartheta)$. Thus

$$\|\psi_2(\vartheta) - \psi_1(\vartheta)\| \leq \omega_1(\vartheta)\|x_1(\vartheta) - x(\vartheta)\|. \tag{3.4}$$

Consider

$$x_2(\vartheta) = x_0 + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \psi_2(\varrho) d_q\varrho.$$

Using (3.3) and (3.4), for every $\vartheta \in \Theta$,

$$\begin{aligned}
\|x_2(\vartheta) - x_1(\vartheta)\| &\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \|\psi_2(\varrho) - \psi_1(\varrho)\| d_q\varrho \\
&\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \omega_1(\varrho)\|x_1(\varrho) - x(\varrho)\| d_q\varrho \\
&\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \omega_1(\varrho) \left(\|\mathfrak{w}_0 - x_0\| + \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\varrho} \varpi(\tau) d_q\tau \right) d_q\varrho \\
&\leq \frac{{\omega_1}^* \kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \left(\|\mathfrak{w}_0 - x_0\| + \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\varrho} \varpi(\tau) d_q\tau \right) d_q\varrho.
\end{aligned}$$

Let $\Lambda_3(\vartheta) = \Psi(\vartheta, x_2(\vartheta), {}^cD_q^{\zeta} x_2(\vartheta)) \cap (\psi_2(\vartheta) + \omega_1(\vartheta)\|x_2(\vartheta) - x_1(\vartheta)\|\Omega_0)$. Similarly to Λ_2, we may demonstrate that Λ_3 is a measurable multi-valued map with nonempty values; so there exists a measurable selection $\psi_3(\vartheta) \in \Lambda_3(\vartheta)$. This gives us the ability to express the following:

$$x_3(\vartheta) = x_0 + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \psi_3(\varrho) d_q\varrho.$$

Then, for each $\vartheta \in \Theta$,

$$\begin{aligned}
\|x_3(\vartheta) - x_2(\vartheta)\| &\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \|\psi_3(\varrho) - \psi_2(\varrho)\| d_q\varrho \\
&\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \omega_1(\varrho)\|x_2(\varrho) - x_1(\varrho)\| d_q\varrho \\
&\leq \frac{{\omega_1}^* \kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \left(\frac{{\omega_1}^* \kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\varrho_1} (\|\mathfrak{w}_0 - x_0\| \right. \\
&\left. + \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\varrho_2} \varpi(\tau) d_q\tau \Big) d_q\varrho_2 \right) d_q\varrho_1 \\
&\leq \left(\frac{{\omega_1}^* \kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \right)^2 \int_0^{\vartheta} \int_0^{\varrho_1} \left(\|\mathfrak{w}_0 - x_0\| + \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\varrho_2} \varpi(\tau) d_q\tau \right) d_q\varrho_2 d_q\varrho_1.
\end{aligned}$$

Repeating the process for $\alpha = 1, 2, \cdots$, for each $\vartheta \in \Theta$,

$$\begin{aligned}
\|x_\alpha(\vartheta) - x_{\alpha-1}(\vartheta)\| &\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \|\psi_\alpha(\varrho) - \psi_{\alpha-1}(\varrho)\| d_q\varrho \\
&\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)} \int_0^{\vartheta} \omega_1(\varrho)\|x_\alpha(\varrho) - x_{\alpha-1}(\varrho)\| d_q\varrho.
\end{aligned}$$

Hence, we get

$$\|x_\alpha(\vartheta)-x_{\alpha-1}(\vartheta)\| \leq \left(\frac{\omega_1{}^*\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\right)^{\alpha-1}\int_0^\vartheta\int_0^{\varrho_1}\int_0^{\varrho_2}\cdots\int_0^{\varrho_{\alpha-2}}(\|\mathfrak{w}_0-x_0\|$$

$$+\frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\varrho_{\alpha-1}}\varpi(\tau)d_q\tau\right)d_q\varrho_{\alpha-1}d_q\varrho_{\alpha-2}\cdots d_q\varrho_1. \tag{3.5}$$

By induction, assume that (3.5) holds for some α and check (3.5) for $\alpha+1$. Let $\Lambda_{\alpha+1}(\vartheta)=\Psi(\vartheta,x_\alpha(\vartheta),{}^cD_q^\zeta x_\alpha(\vartheta))\cap(\psi_\alpha+\omega_1(\vartheta)\|x_\alpha(\vartheta)-x_{\alpha-1}(\vartheta)\|\Omega_0)$. Since $\Lambda_{\alpha+1}$ is a nonempty measurable set, there exists a measurable selection $\psi_{\alpha+1}(\vartheta)\in\Lambda_{\alpha+1}(\vartheta)$, it enables us to define $\alpha\in\mathbb{N}$,

$$x_{\alpha+1}(\vartheta)=x_0+\int_0^\vartheta\frac{(\vartheta-q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)}\psi_{\alpha+1}(\varrho)d_q\varrho. \tag{3.6}$$

Thus, for a.e. $\vartheta\in\Theta$, we obtain

$$\begin{aligned}\|x_{\alpha+1}(\vartheta)-x_\alpha(\vartheta)\| &\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^\vartheta\|\psi_{\alpha+1}(\varrho)-\psi_\alpha(\varrho)\|d_q\varrho\\ &\leq \frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^\vartheta\omega_1(\varrho)\|x_{\alpha+1}(\varrho)-x_\alpha(\varrho)\|d_q\varrho\\ &\leq \left(\frac{\omega_1{}^*\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\right)^{\alpha-1}\int_0^\vartheta\int_0^{\varrho_1}\int_0^{\varrho_2}\cdots\int_0^{\varrho_{\alpha-1}}(\|\mathfrak{w}_0-x_0\|\\ &+\frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\varrho_\alpha}\varpi(\tau)d_q\tau\right)d_q\varrho_\alpha d_q\varrho_{\alpha-1}\cdots d_q\varrho_1.\end{aligned}$$

Consequently, (3.5) is true for all $\alpha\in\mathbb{N}$. We deduce that $\{x_\alpha\}_\alpha$ is a Cauchy sequence in $\mathfrak{F}(\Theta)$, converging uniformly to a limit function $\mathfrak{w}\in\mathfrak{F}(\Theta)$.
Furthermore, from the definition of $\{\Lambda_\alpha\}_\alpha$, we get

$$\|\psi_{\alpha+1}-\psi_\alpha\|\leq\omega_1(\vartheta)\|x_\alpha-x_{\alpha-1}\|;\quad a.e.\ \vartheta\in\Theta,$$

Thus, for almost every $\vartheta\in\Theta$, $\{\psi_\alpha(\vartheta)\}_\alpha$ is also a Cauchy sequence in Ξ and then converges almost everywhere to some measurable function $\psi(\cdot)$ in Ξ. And, since $\psi_0=\varkappa$, we have for a.e. $\vartheta\in\Theta$,

$$\begin{aligned}\|\psi_\alpha(\vartheta)\| &\leq \sum_{i=1}^{\alpha}\|\psi_i(\vartheta)-\psi_{i-1}(\vartheta)\|+\|\psi_0(\vartheta)\|\\ &\leq \omega_1(\vartheta)\sum_{i=2}^{\infty}\|x_i(\vartheta)-x_{i-1}(\vartheta)\|+\|\mathfrak{w}_0-x_0\|+\|\psi_0(\vartheta)\|.\end{aligned}$$

We can now conclude that $\{\psi_\alpha\}_\alpha$ converges to $\psi\in L^1(\Theta,\Xi)$. Passing to the limit in (3.6), we obtain

$$\mathfrak{w}(\vartheta)=\mathfrak{w}_0+\int_0^\vartheta\frac{(\vartheta-q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)}\psi(\varrho)d_q\varrho,$$

is a solution of problem (1.1)-(1.2).

Further, for a.e. $\vartheta \in \Theta$, we get

$$\begin{aligned}\|\mathfrak{w}(\vartheta)-x(\vartheta)\| &\leq \|\mathfrak{w}_0-x_0\|+\frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\vartheta}\|\psi(\varrho)-\psi_0(\varrho)\|d_q\varrho\\ &\leq \|\mathfrak{w}_0-x_0\|+\frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\vartheta}\|\psi(\varrho)-\psi_\alpha(\varrho)\|d_q\varrho\\ &+\frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\vartheta}\|\psi_\alpha(\varrho)-\psi_0(\varrho)\|d_q\varrho\\ &\leq \|\mathfrak{w}_0-x_0\|+\frac{\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\vartheta}\|\psi(\varrho)-\psi_\alpha(\varrho)\|d_q\varrho\\ &+\frac{{\omega_1}^*\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\vartheta}\sum_{i=2}^{\infty}\|x_i(\varrho)-x_{i-1}(\varrho)\|d_q\varrho.\end{aligned}$$

As $\alpha\to\infty$, we get

$$\|\mathfrak{w}(\vartheta)-x(\vartheta)\| \leq \|\mathfrak{w}_0-x_0\|+\frac{{\omega_1}^*\kappa^{(\zeta-1)}}{\Gamma_q(\zeta)}\int_0^{\vartheta}\sum_{i=2}^{\infty}\|x_i(\varrho)-x_{i-1}(\varrho)\|d_q\varrho.$$

Next, we give an estimate for $\|({}^cD_q^{\zeta}\mathfrak{w})(\vartheta)-\varkappa(\vartheta)\|$ for $\vartheta\in\Theta$. We have

$$\begin{aligned}\|({}^cD_q^{\zeta}\mathfrak{w})(\vartheta)-\varkappa(\vartheta)\| &= \|\psi(\vartheta)-\psi_0(\vartheta)\|\\ &\leq \|\psi_\alpha(\vartheta)-\psi_0(\vartheta)\|+\|\psi_\alpha(\vartheta)-\psi(\vartheta)\|\\ &\leq \|\psi_\alpha(\vartheta)-\psi(\vartheta)\|+{\omega_1}^*\sum_{i=2}^{\infty}\|x_i(\vartheta)-x_{i-1}(\vartheta)\|.\end{aligned}$$

As $\alpha\to\infty$, we get

$$\|({}^cD_q^{\zeta}\mathfrak{w})(\vartheta)-\varkappa(\vartheta)\| \leq {\omega_1}^*\sum_{i=2}^{\infty}\|x_i(\vartheta)-x_{i-1}(\vartheta)\|.$$

4. Topological Structure of Solution Sets

4.1. **The upper semi-continuous case.** In this part, we provide a global existence result and demonstrate the compactness of our solution set by combining Mönch's fixed point theorem for multivalued maps with the measure of noncompactness.

The hypotheses:

($\mathcal{B}_1$) The multivalued map $\Psi:\Theta\times\Xi\times\Xi\to\mathcal{P}_{cp,c}(\Xi)$ is Carathéodory.

($\mathcal{B}_2$) There exists a function $\omega_1\in L^{\infty}(\Theta,\mathbb{R}_+)$ such that

$$\|\Psi(\vartheta,\mathfrak{w},\mathfrak{y})\|_{\mathcal{P}}=\sup\{\|\mathfrak{z}\|_C:\mathfrak{z}(\vartheta)\in\Psi(\vartheta,\mathfrak{w},\mathfrak{y})\}\leq\omega_1(\vartheta);$$

for a.e. $\vartheta\in\Theta$, and each $\mathfrak{w},\mathfrak{y}\in\Xi$.

($\mathcal{B}_3$) For each bounded sets $\Omega\subset\Xi$ and for each $\vartheta\in\Theta$, we have

$$\mu(\Psi(\vartheta,\Omega,({}^cD_q^{\zeta}\Omega)))\leq\omega_1(\vartheta)\mu(\Omega).$$

$(\mathcal{B}_4)$ The function $\tilde{\Psi} \equiv 0$ is the unique solution in $\mathfrak{F}(\Theta)$ of the inequality

$$\tilde{\Psi}(\vartheta) \leq 2\omega_1{}^*(I_q^\zeta \tilde{\Psi})(\vartheta).$$

Theorem 4.1. *If* $(\mathcal{B}_1) - (\mathcal{B}_4)$ *are met, then (1.1)-(1.2) has at least one solution defined on* Θ*. Furthermore, the solution set*

$$S_\Psi(\mathfrak{w}_0) = \{\mathfrak{w} \in \mathfrak{F}(\Theta) : \mathfrak{w} \textit{ is a solution of problem } (1.1) - (1.2)\},$$

is compact and the multivalued map $S_\Psi : \mathfrak{w}_0 \to (S_\Psi)(\mathfrak{w}_0)$ *is u.s.c.*

Proof. Consider the operator $\mathfrak{T} : \mathfrak{F}(\Theta) \to \mathcal{P}(\mathfrak{F}(\Theta))$ defined in (3.1).

Step 1. Existence of solutions.
From Theorem 5 in [13], the operator $\mathfrak{T}$ verifies all the requirements of Theorem 2.17, and we deduce that $\mathfrak{T}$ has at least one fixed point $\mathfrak{w} \in \mathfrak{F}(\Theta)$ which is a solution of (1.1)-(1.2).

Step 2. Compactness of the solution set.
For each a $\mathfrak{w}_0 \in \Xi$, we consider the set $S_\Psi(\mathfrak{w}_0)$. From Step 1, there exists $\gamma > 0$ such that for every $\mathfrak{w} \in S_\Psi(\mathfrak{w}_0) : \|\mathfrak{w}\|_\infty \leq \gamma$. Since $\mathfrak{T}$ is completely continuous, $\mathfrak{T}(S_\Psi(\mathfrak{w}_0))$ is relatively compact in $\mathfrak{F}(\Theta)$. Let $\mathfrak{w} \in S_\Psi(\mathfrak{w}_0)$; then $\mathfrak{w} \in \mathfrak{T}(\mathfrak{w})$. Hence $S_\Psi(\mathfrak{w}_0) \subset \mathfrak{T}(S_\Psi(\mathfrak{w}_0))$. Now, let us demonstrate that $S_\Psi(\mathfrak{w}_0)$ is a closed subset in $\mathfrak{F}(\Theta)$. Let $\{\mathfrak{w}_\alpha : \alpha \in \mathbb{N}\} \subset S_\Psi(\mathfrak{w}_0)$ be such that the sequence $(\mathfrak{w}_\alpha)_{\alpha\in\mathbb{N}}$ converges to $\mathfrak{w}$. For every $\alpha \in \mathbb{N}$, there exists $\mathfrak{z}_\alpha$ such that $\mathfrak{z}_\alpha(\vartheta) \in \Psi(\vartheta, \mathfrak{w}_\alpha(\vartheta), ({}^cD_q^\zeta \mathfrak{w}_\alpha)(\vartheta));$ *a.e.* $\vartheta \in \Theta$, and

$$\mathfrak{w}_\alpha(\vartheta) = \mathfrak{w}_0 + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}_\alpha(\varrho) d_q\varrho.$$

Since $\mathfrak{w}_\alpha \to \mathfrak{w}$, Lemma 2.15 implies that there exists $\mathfrak{z}$, where $\mathfrak{z}(\vartheta) \in \Psi(\vartheta, \mathfrak{w}(\vartheta));$ *a.e.* $\vartheta \in \Theta$, and

$$\mathfrak{w}(\vartheta) = \mathfrak{w}_0 + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}(\varrho) d_q\varrho. \tag{4.1}$$

Therefore $\mathfrak{w} \in S_\Psi(\mathfrak{w}_0)$ which yields that $S_\Psi(\mathfrak{w}_0)$ is closed, hence compact in $\mathfrak{F}(\Theta)$.

Step 3. $S_\Psi(\cdot)$ is u.s.c.
To do this, we prove that the graph Γ_{S_Ψ} of S_Ψ is closed. We have

$$\Gamma_{S_\Psi} = \{(\mathfrak{w}_0, \mathfrak{w}) : \mathfrak{w} \in S_\Psi(\mathfrak{w}_0)\},$$

Let $(\mathfrak{w}_{0n}, \mathfrak{w}_\alpha) \in \Gamma_{S_\Psi}$ be such that $(\mathfrak{w}_{0n}, \mathfrak{w}_\alpha) \to (\mathfrak{w}_0, \mathfrak{w});$ *as* $\alpha \to \infty$. Since $\mathfrak{w}_\alpha \in S_\Psi(\mathfrak{w}_{0n})$, there exists $\mathfrak{z}_\alpha \in L^1(\Theta)$ such that

$$\mathfrak{w}_\alpha(\vartheta) = \mathfrak{w}_{0n} + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}_\alpha(\varrho) d_q\varrho. \tag{4.2}$$

From Lemma 2.15, we can show that there exists $\mathfrak{z} \in S_{\Psi\circ\mathfrak{w}}$ where $\mathfrak{w}$ verifies (4.1). Thus, $\mathfrak{w} \in S_{\Psi}(\mathfrak{w}_0)$. Now, we demonstrate that S_{Ψ} maps bounded sets into relatively compact sets of $\mathfrak{F}(\Theta)$. Let Ω be a bounded set in $\Xi$and let $\{\mathfrak{w}_\alpha\} \subset S_{\Psi}(\Omega)$. Then there exists $\{\mathfrak{w}_{0n}\} \subset \Omega$ and $\mathfrak{z}_\alpha \in S_{\Psi\circ\mathfrak{w}_\alpha}$; $\alpha \in \mathbb{N}$ such that (4.2) is satisfied. Since $\{\mathfrak{w}_{0n}\}$ bounded sequence, there exists a subsequence of $\{\mathfrak{w}_{0n}\}$ converging to $\mathfrak{w}_0$. As in the proof of Theorem 5 in [13], we can show that $\{\mathfrak{w}_\alpha\}$ is compact on Θ. We deduce that there exists a subsequence of $\{\mathfrak{w}_\alpha\}$ converging to $\mathfrak{w}$ in $\mathfrak{F}(\Theta)$. Also; from Lemma 2.15, we can prove that $\mathfrak{w}$ satisfies (4.1) for some $\mathfrak{z} \in S_{\Psi\circ\mathfrak{w}}$. Hence, $S_{\Psi}(\mathfrak{w}_0)$ is u.s.c.

4.2. **The lower semi-continuous case.** The following existence result for problem (1.1)-(1.2) addresses the situation in which the nonlinearity is lower semi-continuous with concerning the second parameter which does not have convex values. We will apply Mönch's fixed point theorem for multivalued maps in conjunction with a selection theorem for lower semi-continuous (l.s.c.) multivalued maps with decomposable variables.

The preceding assumption is required for the sequel.

($\mathcal{B}_5$) The multivalued map Ψ is nonempty compact valued where
 (a) the mapping $(\vartheta, \mathfrak{w}) \to \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})$ is $\mathcal{L} \otimes \mathcal{B}$ measurable
 (b) The mapping $\mathfrak{w} \to \Psi(\vartheta, \mathfrak{w}, \mathfrak{y})$ is l.s.c. for each $\vartheta \in \Theta$.

Let us we state the celebrated selection theorem of Fryszkowski.

Lemma 4.2. *[29] Let $\bar{\Xi}$ be a separable metric space and let Ξ be a Banach space. Then every l.s.c. multivalued operator $\mathfrak{T} : \bar{\Xi} \to \mathcal{P}_{cl}(L^1(\Theta, \Xi))$ with nonempty closed decomposable values has a continuous selection, i.e. there exists a continuous single-valued function $\psi : \bar{\Xi} \to L^1(\Theta, \Xi)$ such that $\psi(\mathfrak{z}) \in \mathfrak{T}(\mathfrak{z})$ for every $\mathfrak{z} \in \bar{\Xi}$.*

Lemma 4.3. *[28] Let $\mathfrak{T} : \Theta \times \Xi \times \Xi \to \mathcal{P}_{cp}(L^1(\Theta, \Xi))$ be a locally integrably bounded multivalued map satisfying ($\mathcal{B}_5$). Then $\mathfrak{T}$ is of l.s.c. type.*

Theorem 4.4. *If ($\mathcal{B}_2$) and ($\mathcal{B}_5$) are met, then (1.1)-(1.2) has at least one solution defined on Θ.*

Proof. By Lemma 4.3, Ψ is of l.s.c. type. From Lemma 4.2, there exists a continuous selection $\psi : \mathfrak{F}(\Theta) \to L^1(\Theta)$ such that $\psi(\mathfrak{w}) \in S_{\Psi}(\mathfrak{w})$ for every $\mathfrak{w} \in \mathfrak{F}(\Theta)$. Consider the problem

$$
\begin{cases} ({}^cD_q^\zeta \mathfrak{w})(\vartheta) = (\psi\mathfrak{w})(\vartheta); \ \vartheta \in \Theta, \\ \mathfrak{w}(0) = \mathfrak{w}_0 \in \Xi, \end{cases} \tag{4.3}
$$

and the operator $\mathfrak{S} : \mathfrak{F}(\Theta) \to \mathfrak{F}(\Theta)$ defined by

$$
(\mathfrak{S}\mathfrak{w})(\vartheta) = \mathfrak{w}_0 + \int_0^\vartheta \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} (\psi\mathfrak{w})(\varrho) d_q\varrho.
$$

It is clear that the fixed points of $\mathfrak{S}$ are solutions of problem (1.1)-(1.2).

Let $\mathfrak{w} \in \mathfrak{F}(\Theta)$. Then for each $\vartheta \in \Theta$ we have

$$(\mathfrak{S}\mathfrak{w})(\vartheta) = \mathfrak{w}_0 + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}(\varrho) d_q\varrho,$$

for some $\mathfrak{z} \in S_{\Psi\circ\mathfrak{w}}$. On the other hand,

$$\begin{aligned}
\|\delta(\vartheta)\| &\leq \|\mathfrak{w}_0\| + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} (\psi\mathfrak{w})(\varrho) d_q\varrho \\
&\leq \|\mathfrak{w}_0\| + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \omega_1(\varrho) d_q\varrho \\
&\leq \|\mathfrak{w}_0\| + \frac{{\omega_1}^* \kappa^{(\zeta)}}{\Gamma_q(1+\zeta)} \\
&= R.
\end{aligned}$$

Hence $\|(\mathfrak{S}\mathfrak{w})(\mathfrak{w})\|_\infty \leq R$, and so $\mathfrak{S}(\Omega_R) \subset \Omega_R$, where $\Omega_R := \{\mathfrak{w} \in \mathfrak{F}(\Theta) : \|\mathfrak{w}\|_\infty \leq R\}$ be the bounded, closed and convex ball of $\mathfrak{F}(\Theta)$. We will demonstrate that $\mathfrak{S} : \Omega_R \to \Omega_R$ verifies all the requirements of Theorem 2.18. Now, proving that $\mathfrak{S}(\Omega_R)$ is relatively compact.

Let (δ_α) by any sequence in $\mathfrak{S}(\Omega_R)$. By, Arzéla-Ascoli compactness criterion in $\mathfrak{F}(\Theta)$, we demonstrate (δ_α) has a convergent subsequence. As $\delta_\alpha \in \mathfrak{S}(\Omega_R)$ there are $\mathfrak{w}_\alpha \in \Omega_R$ and $\mathfrak{z}_\alpha \in S_{\Psi\circ\mathfrak{w}_\alpha}$ where

$$\delta_\alpha(\vartheta) = \mathfrak{w}_0 + \int_0^{\vartheta} \frac{(\vartheta - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \mathfrak{z}_\alpha(\varrho) d_q\varrho.$$

We can show that $\{\delta_\alpha(\vartheta) : \alpha \geq 1\}$ is relatively compact for each $\vartheta \in \Theta$. And, for each ϑ_1 and ϑ_2 from Θ, with $\vartheta_1 < \vartheta_2$, we get

$$\begin{aligned}
&\|\delta_\alpha(\vartheta_2) - \delta_\alpha(\vartheta_1)\| \\
&\leq \left\| \int_0^{\vartheta_2} \frac{(\vartheta_2 - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \omega_1(\varrho) d_q\varrho - \int_0^{\vartheta_1} \frac{(\vartheta_1 - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \omega_1(\varrho) d_q\varrho \right\| \\
&\leq \int_{\vartheta_1}^{\vartheta_2} \frac{(\vartheta_2 - q\varrho)^{(\zeta-1)}}{\Gamma_q(\zeta)} \omega_1(\varrho) d_q\varrho \\
&\quad + \int_0^{\vartheta_1} \frac{|(\vartheta_2 - q\varrho)^{(\zeta-1)} - (\vartheta_1 - q\varrho)^{(\zeta-1)}|}{\Gamma_q(\zeta)} \omega_1(\varrho) d_q\varrho \qquad (4.4) \\
&\leq \frac{{\omega_1}^* \kappa^{\zeta}}{\Gamma_q(1+\zeta)} (\vartheta_2 - \vartheta_1)^{\zeta} \\
&\quad + {\omega_1}^* \int_0^{\vartheta_1} \frac{|(\vartheta_2 - q\varrho)^{(\zeta-1)} - (\vartheta_1 - q\varrho)^{(\zeta-1)}|}{\Gamma_q(\zeta)} d_q\varrho \\
&\quad \to 0 \ as \ \vartheta_1 \longrightarrow \vartheta_2.
\end{aligned}$$

This shows that $\{\delta_\alpha : \alpha \geq 1\}$ is equicontinuous. Consequently, by the Arzéla-Ascoli theorem, $\{\delta_\alpha : \alpha \geq 1\}$ is relatively compact in Ω_R. By Theorem 2.18, we deduce that $\mathfrak{G}$ has at least one fixed point, which is a solution of (1.1)-(1.2).

5. **An Example**

Let

$$\Xi = l^1 = \left\{ \mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\alpha, \ldots), \sum_{\alpha=1}^{\infty} |\mathfrak{w}_\alpha| < \infty \right\}$$

be the Banach space with the norm

$$\|\mathfrak{w}\|_E = \sum_{\alpha=1}^{\infty} |\mathfrak{w}_\alpha|.$$

Consider now the following problem of fractional $\frac{1}{4}-$difference inclusion

$$\begin{cases} ({}^cD_{\frac{1}{3}}^{\frac{1}{2}}\mathfrak{w}_\alpha)(\vartheta) \in \Psi_\alpha\left(\vartheta, \mathfrak{w}(\vartheta), ({}^cD_{\frac{1}{3}}^{\frac{1}{2}}\mathfrak{w}_\alpha)(\vartheta)\right);\ \vartheta \in [0, e], \\ \mathfrak{w}(0) = (1, 0, \ldots, 0, \ldots), \end{cases} \tag{5.1}$$

where

$$\Psi_\alpha(\vartheta, \mathfrak{w}(\vartheta)) = \frac{\vartheta^2 e^{-5-\vartheta}}{1 + \|\mathfrak{w}(\vartheta)\|_E + \|({}^cD_{\frac{1}{3}}^{\frac{1}{2}}\mathfrak{w}_\alpha)(\vartheta)\|_E}[\mathfrak{w}_\alpha(\vartheta) - 1, \mathfrak{w}_\alpha(\vartheta)];\ \vartheta \in \Theta,$$

with $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\alpha, \ldots)$. Set $\zeta = \frac{1}{2}$, and $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_\alpha, \ldots)$.
For each $\mathfrak{w} \in \Xi$ and $\vartheta \in \Theta$, we have

$$\|\Psi(\vartheta, \mathfrak{w})\|_{\mathcal{P}} \leq c\vartheta^2 e^{-\vartheta-5}.$$

Thus, the condition $(\mathcal{B}_2)$ is verified with ${\omega_1}^* = ce^{-3}$. We can easily show that all requirements of Theorem 4.1 are verified. Hence, (5.1) has at least one solution defined on Θ. Moreover, the solution set $S_\Psi(\mathfrak{w}_0)$ is compact and the multivalued map $S_\Psi : \mathfrak{w}_0 \to (S_\Psi)(\mathfrak{w}_0)$ is u.s.c.

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