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OSCILLATORY BEHAVIOR FOR NONLINEAR DELAY DIFFERENCE EQUATION WITH NON-MONOTONE ARGUMENTS

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ABSTRACT. The aim of this paper is to obtain the some new oscillatory conditions for all solutions of nonlinear difference equation

$$\Delta x(n)+\sum_{i=1}^{m} p_i(n) f_i\left(x\left(\tau_i(n)\right)\right)=0,\ n=0,1,\cdots, \tag{$\star$}$$

where, for $i=1,2,\ldots,m$, $(p_i(n))$ are sequences of nonnegative real numbers and $(\tau_i(n))$ are not necessarily monotone sequences, $f_i\in C(\mathbb{R},\mathbb{R})$ and $xf_i(x)>0$ $(i=1,2,\ldots,m)$ for $x\neq 0$.

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1. INTRODUCTION

Oscillation theory of difference equations has attracted many researchers. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of delay difference equations. For these oscillatory and nonoscillatory results, we refer, for instance, [1 − 15]. So, in the present paper, our aim is to obtain some new oscillatory conditions of all solutions for first order nonlinear delay difference equation. Consider the nonlinear delay difference equation with non-monotone arguments

$$\Delta x(n)+\sum_{i=1}^{m} p_i(n) f_i\left(x\left(\tau_i(n)\right)\right)=0,\ n=0,1,\cdots, \tag{1.1}$$

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where, for $i = 1, 2, \ldots, m$, $(p_i(n))$ are sequences of nonnegative real numbers and $(\tau_i(n))$ are sequences of integers such that

$$\tau_i(n) \leq n \text{ for all } n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \tag{1.2}$$

and

$$f_i \in C(\mathbb{R}, \mathbb{R}) \quad \text{and } x f_i(x) > 0 \text{ for } x \neq 0. \tag{1.3}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$.
Define, for $i = 1, 2, \ldots, m$

$$r = -\min_{n\geq 0} \tau_i(n). \quad \text{(Clearly, } r \text{ is a positive integer).}$$

By a solution of the difference equation (1.1), we mean a sequence of real numbers $(x(n))_{n\geq -r}$ which satisfies (1.1) for all $n \geq 0$.
A solution $(x(n))_{n\geq -r}$ of the difference equation (1.1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.
For $1 \leq i \leq m$, if $f_i(x) = x$, then equation (1.1) takes the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x\left(\tau_i(n)\right) = 0, \; n = 0, 1, \cdots . \tag{1.4}$$

In 2006, Berezansky and Braverman [1] established the following result for equation (1.4). If $(\tau_i(n))(i = 1, \ldots, m)$ are not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \;\text{ and }\; \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) > \frac{1}{e}, \tag{1.5}$$

where $\tau(n) = \max_{1\leq i\leq m} \tau_i(n)$, then all solutions of (1.4) oscillate.
In 2013, Chatzarakis et al. [3], studied the equation (1.4) and proved that, if $(\tau_i(n))$ are nondecreasing and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} \sum_{i=1}^{m} p_i(j) > 1, \tag{1.6}$$

where $\tau(n) = \max_{1\leq i\leq m} \tau_i(n)$, then all solutions of (1.4) oscillate.
Set

$$h_i(n) := \max_{s\leq n} \tau_i(s), \quad n \geq 0 \text{ and } h(n) = \max_{1\leq i\leq m} h_i(n). \tag{1.7}$$

Clearly, $(h_i(n))(i = 1, \ldots, m)$ are nondecreasing and $\tau_i(n) \leq h_i(n) \leq h(n)$ for all $n \geq 0$ and $1 \leq i \leq m$.
In 2015, Braverman et al. [2], analyzed the equation (1.4) and proved that, if $(\tau_i(n))(i = 1, \ldots, m)$ are not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) > 1, \tag{1.8}$$

where $h(n)$ is defined by (1.7), then all solutions of (1.4) oscillate.
In 2020, Kılıç and Öcalan [9], studied the equation (1.4) and proved that, if $(\tau_i(n))(i = 1, \ldots, m)$ are not necessarily monotone and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) > \frac{1}{e}, \tag{1.9}$$

where $h(n)$ is defined by (1.7), then all solutions of (1.4) oscillate.
For $m = 1$, equation (1.1) reduces to

$$\Delta x(n) + p(n) f\left(x\left(\tau(n)\right)\right) = 0, \;\; n = 0, 1, \cdots . \tag{1.10}$$

In 2018, Öcalan et al. [11], studied the equation (1.10) and obtained some new oscillatory conditions for all solutions of equation (1.10) to be oscillatory.

2. MAIN RESULTS

In this section, we investigated the oscillatory behavior of all solutions of equation (1.1). We obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1) under the assumption that the argument $(\tau_i(n))\,(i = 1, \ldots, m)$ are not necessarily monotone.
We assume that f_i $(i = 1, \ldots, m)$ hold the following condition;

$$\limsup_{x\to 0} \frac{x}{f_i(x)} = M_i, \quad 0 \le M_i < \infty. \tag{2.1}$$

The following result was given in [5].

Lemma 2.1. *Assume that* (1.1) *holds and* $\alpha > 0$. *Then we have*

$$\alpha = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \sum_{i=1}^{m} p_i(j) = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j),$$

where $h(n)$ *is defined by* (1.7) *and* $\tau(n) = \max_{1\le i\le m} \tau_i(n)$.

Lemma 2.2. *Assume that* $(x(n))$ *is an eventually positive solution of* (1.1). *If*

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) > 0, \tag{2.2}$$

where $h(n)$ *is defined by* (1.7), *then* $\lim_{n\to\infty} x(n) = 0$.
Also, assume that $(x(n))$ *is an eventually negative solution of* (1.1). *If* (2.2) *holds, then* $\lim_{n\to\infty} x(n) = 0$.

Proof. Let $(x(n))$ be an eventually positive solution of (1.1). Then, there exists $n_1 > n_0$ such that $x(n)$, $x\left(\tau(n)\right) > 0$ for all $n \ge n_1$. Thus, from (1.1), we get

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) f_i\left(x\left(\tau_i(n)\right)\right) \le 0, \text{ for all } n \ge n_1,$$

which means that $x(n)$ is nondecreasing and has a limit $l \geq 0$. Now, we claim that $l = 0$. Otherwise, $l > 0$. Summing up (1.1) from $h(n)$ to n, then we have

$$x(n+1) - x(h(n)) + \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(n) f_i\left(x\left(\tau_i(n)\right)\right) = 0, \ n \geq n_1. \tag{2.3}$$

Since $f_i (i = 1, \ldots, m)$ are continuous, then $\lim_{n\to\infty} f_i\left(x\left(\tau_i(n)\right)\right) = f_i(l) > 0$ for $1 \leq i \leq m$. So, there exists a n_2 such that $f_i\left(x\left(\tau_i(n)\right)\right) \geq d_i > 0$ for $n \geq n_2$ and $1 \leq i \leq m$. By using this fact and (2.3), we get the following inequality

$$x(n+1) - x(h(n)) + d \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) \leq 0, \ n \geq n_2, \tag{2.4}$$

where $d = \min_{1 \leq i \leq m}\{d_i\} > 0$. Then, (2.2) implies that there exists at least one sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} \sum_{j=h(n_k)}^{n_k} \sum_{i=1}^{m} p_i(j) > 0. \tag{2.5}$$

By writing $n \to n_k$ and taking limit as $k \to \infty$ in (2.4), we get

$$d \lim_{k\to\infty} \sum_{j=h(n_k)}^{n_k} \sum_{i=1}^{m} p_i(j) \leq 0,$$

but this contradicts to (2.5).
By using same process, it is easy to see that when $(x(n))$ is an eventually negative solution of (1.1) under assumption that (2.2), then $\lim_{n\to\infty} x(n) = 0$. □

Theorem 2.3. *Assume that* (1.2), (1.3) *and* (2.1) *hold. If* $(\tau_i(n))$ *are not necessarily monotone and*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) > \frac{M^*}{e}, \tag{2.6}$$

where $M^* = \max_{1 \leq i \leq m}\{M_i\}$, *then all solutions of* (1.1) *oscillate.*

Proof. Assume, for the sake of contradiction, that $(x(n))$ is an eventually positive solution of (1.1). If there exists an eventually negative solution $(x(n))$ of (1.1), then the proof can be done similarly as below. Then there exists $n_1 \geq n_0$ such that $x(n),\ x\left(\tau_i(n)\right),\ x\left(h(n)\right) > 0$ for all $n \geq n_1$ and $1 \leq i \leq m$. Thus, from Eq.(1.1) we have

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) f_i\left(x\left(\tau_i(n)\right)\right) \leq 0, \quad \text{for all } n \geq n_1,$$

which means that $(x(n))$ is eventually nondecreasing. Thus, condition (2.6) and Lemma 2.2 imply that $\lim_{n\to\infty} x(n) = 0$.

Now, suppose that $M_i > 0$ for $1 \le i \le m$. Then, in view of (2.1) we can choose $n_2 \ge n_1$ so large that

$$f_i(x(n)) \ge \frac{1}{2M_i}x(n) \ \ge \frac{1}{2M^*}x(n) \ \text{ for } n \ge n_2. \tag{2.7}$$

Since $h(n) \ge \tau_i(n)$ for $1 \le i \le m$ and $(x(n))$ is nonincreasing , by using (1.1) and (2.7) we have

$$\Delta x(n) + \frac{1}{2M^*}\sum_{i=1}^{m} p_i(n)x(h(n)) \le 0, \qquad n \ge n_2. \tag{2.8}$$

Also, from (2.6) and Lemma 2.1, it follows that there exists a constant $c > 0$ such that

$$\sum_{j=h(n)}^{n}\sum_{i=1}^{m} p_i(j) \ge \sum_{j=h(n)}^{n-1}\sum_{i=1}^{m} p_i(j) \ge c > \frac{M^*}{e}, \quad n \ge n_3 \ge n_2. \tag{2.9}$$

So, from (2.9), there exists an integer $n^* \in [h(n), n]$, for all $n \ge n_3$ such that

$$\sum_{j=h(n)}^{n^*}\sum_{i=1}^{m} p_i(j) > \frac{M^*}{2e} \quad \text{and} \quad \sum_{j=n^*}^{n}\sum_{i=1}^{m} p_i(j) > \frac{M^*}{2e}. \tag{2.10}$$

Summing up (2.8) from $h(n)$ to n^* and using $(x(n))$ is nonincreasing, then we have

$$x(n^*+1) - x(h(n)) + \frac{1}{2M^*}\sum_{j=h(n)}^{n^*}\sum_{i=1}^{m} p_i(j)x(h(j)) \le 0,$$

or

$$x(n^*+1) - x(h(n)) + \frac{1}{2M^*}x(h(n^*))\sum_{j=h(n)}^{n^*}\sum_{i=1}^{m} p_i(j) \le 0.$$

Thus, by (2.10), we have

$$-x(h(n)) + \frac{1}{2M^*}x(h(n^*))\frac{M^*}{2e} < 0. \tag{2.11}$$

Summing (2.8) from n^* to n and using the same facts , we get

$$x(n+1) - x(n^*) + \frac{1}{2M^*}\sum_{j=n^*}^{n}\sum_{i=1}^{m} p_i(j)x(h(j))\, ds \le 0.$$

Thus, by (2.10), we have

$$-x(n^*) + \frac{1}{2M^*}x(h(n))\frac{M^*}{2e} < 0. \tag{2.12}$$

Combining the inequalities (2.11) and (2.12), we obtain

$$x(n^*) > x(h(n))\frac{1}{4e} > x(h(n^*))\left(\frac{1}{4e}\right)^2,$$

and hence we have

$$\frac{x(h(n^*))}{x(n^*)} < (4e)^2 \quad \text{for } n \ge n_3.$$

Let

$$w = \liminf_{n\to\infty} \frac{x(h(n^*))}{x(n^*)} \geq 1, \tag{2.13}$$

and because of $1 \leq w \leq (4e)^2$, w is finite.

Now dividing (1.1) with $x(n)$ and then summing up from $h(n)$ to $n-1$ we obtain

$$\sum_{j=h(n)}^{n-1} \frac{\Delta x(j)}{x(j)} + \sum_{j=h(n)}^{n-1} \sum_{i=1}^{m} p_i(j) \frac{f_i(x(\tau_i(j)))}{x(j)} = 0. \tag{2.14}$$

It is well known that

$$\ln \frac{x(n)}{x(h(n))} \leq \sum_{j=h(n)}^{n-1} \frac{\Delta x(j)}{x(j)}. \tag{2.15}$$

So, by (2.14) and (2.15), we have

$$\ln \frac{x(n)}{x(h(n))} + \sum_{j=h(n)}^{n-1} \sum_{i=1}^{m} p_i(j) \frac{f_i(x(\tau_i(j)))}{x(\tau_i(j))} \frac{x(\tau_i(j))}{x(j)} \leq 0.$$

Since $h(n) \geq \tau_i(n)$ for $1 \leq i \leq m$ and $(x(n))$ is nonincreasing, we get

$$\ln \frac{x(h(n))}{x(n)} \geq \sum_{j=h(n)}^{n-1} \sum_{i=1}^{m} p_i(j) \frac{f(x(\tau_i(j)))}{x(\tau_i(j))} \frac{x(h(j))}{x(j)}. \tag{2.16}$$

Also, there exists an integer μ such that $h(n) \leq \mu \leq n$. Then, from (2.16), we have

$$\ln \frac{x(h(n))}{x(n)} \geq \sum_{i=1}^{m} \frac{f(x(\tau_i(\mu)))}{x(\tau_i(\mu))} \frac{x(h(\mu))}{x(\mu)} \sum_{j=h(n)}^{n-1} p_i(j). \tag{2.17}$$

Taking lower limits on both of (2.17) and using (2.1), (2.6) and (2.13), we obtain $\ln w > \frac{w}{e}$. But this is impossible since $\ln x \leq \frac{x}{e}$ for all $x > 0$.

Now, we consider the case where $M_i = 0$ for $1 \leq i \leq m$. In this case, it is clear that by (2.1), we have

$$\lim_{x\to 0} \frac{x}{f_i(x)} = 0 \quad \text{for } 1 \leq i \leq m. \tag{2.18}$$

Since $\frac{x}{f_i(x)} > 0$, by (2.17), for sufficiently large integers, we get

$$\frac{x}{f_i(x)} < \epsilon_i \leq \epsilon^* \quad \text{for } 1 \leq i \leq m,$$

or

$$\frac{f_i(x)}{x} > \frac{1}{\epsilon^*} \quad \text{for } 1 \leq i \leq m, \tag{2.19}$$

where $\epsilon^* = \max_{1\le i\le m}\{\epsilon_i\} > 0$ is an arbitrary real number. Thus, since $\tau_i(n) \le h(n)$ for $1 \le i \le m$ and $(x(n))$ is nonincreasing, by (1.1) and (2.19), we have

$$\Delta x(n) + \frac{1}{\epsilon^*}\sum_{i=1}^{m} p_i(n)x(h(n)) < 0, \quad n \ge n_1. \tag{2.20}$$

Summing up (2.19) from $h(n)$ to n, and using $(h(n))$ is nondecreasing we obtain

$$x(n+1) - x(h(n)) + \frac{1}{\epsilon^*}\sum_{j=h(n)}^{n}\sum_{i=1}^{m} p_i(j)x(h(j)) < 0,$$

and so, we get

$$-x(h(n)) + \frac{1}{\epsilon^*}x(h(n))\sum_{j=h(n)}^{n}\sum_{i=1}^{m} p_i(j) < 0. \tag{2.21}$$

Thus, by (2.9) and (2.21), we can write

$$\frac{c}{\epsilon^*} < 1$$

or

$$\epsilon^* > c.$$

Since ϵ^* is an arbitrary real number, this contradicts to $\lim_{x\to 0}\frac{x}{f(x)} = 0$. The proof of the theorem is completed. □

Theorem 2.4. *Assume that* (1.2), (1.3), (1.7) *and* (2.1) *hold with* $0 < M^* < \infty$. *If*

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n}\sum_{i=1}^{m} p_i(j) > M^*, \tag{2.22}$$

where $h(n)$ *is defined by* (1.7) *and* $M^* = \max_{1\le i\le m}\{M_i\}$, *then all solutions of* (1.1) *oscillate.*

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $(x(n))$ of (1.1). Then there exists $n_1 \ge n_0$ such that $x(n),\ x(\tau_i(n)),\ x(h(n)) > 0$ for all $n \ge n_1$ and $1 \le i \le m$. In view of Theorem 2.3, $(x(n))$ is eventually nondecreasing and also, from (2.22) and Lemma 2.2, we have $\lim_{n\to\infty} x(n) = 0$.

On the other hand, by (2.1) for $\theta > 1$, we get the following inequality

$$f_i(x(n)) \ge \frac{1}{\theta M_i}x(n) \ge \frac{1}{\theta M^*}x(n) \quad \text{for } 1 \le i \le m. \tag{2.23}$$

From (2.22), there exists a constant $K > 0$ such that

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n}\sum_{i=1}^{m} p_i(j) = K > M^*. \tag{2.24}$$

Since $K > M^*$, we have $M^* < \frac{K+M^*}{2} < K$. Also, with the help of (2.23) and (1.1), we get

$$\Delta x(n) + \frac{1}{\theta M^*} \sum_{i=1}^{m} p_i(j) x(\tau_i(n)) \leq 0.$$

As $h(n) \geq \tau_i(n)$ for $1 \leq i \leq m$ and $(x(n))$ is nonincreasing, we obtain

$$\Delta x(n) + \frac{1}{\theta M^*} \sum_{i=1}^{m} p_i(j) x(h(n)) \leq 0. \tag{2.25}$$

Summing up (2.25) from $h(n)$ to n, and using the fact that $(h(n))$ is nondecreasing

$$x(n+1) - x(h(n)) + \frac{1}{\theta M^*} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) x(h(j)) \leq 0$$

or

$$-x(h(n)) + \frac{1}{\theta M^*} x(h(n)) \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) < 0.$$

This implies that

$$-x(h(n)) \left[1 - \frac{1}{\theta M^*} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) \right] < 0 \quad \text{for } n \geq n_2,$$

and hence

$$\sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) < \theta M^*.$$

Therefore, we obtain

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) \leq \theta M^*.$$

Since $\theta > 1$ and $\frac{K+M^*}{2M^*} > 1$, we can choose this term instead of θ. If the term $\theta = \frac{K+M^*}{2M^*} > 1$ is replaced in the last inequality, we get

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) = K \leq \frac{K+M^*}{2}.$$

But, this contradicts to $K > \frac{K+M^*}{2}$, then the proof of the theorem is completed. □

Now, we present an example to show that the significance of our results.

Example 2.5. Consider the nonlinear delay difference equation

$$\Delta x(n) + \frac{0.2}{e} x(\tau_1(n)) \ln(10 + |x(\tau_1(n))|) + \frac{0.5}{e} x(\tau_2(n)) \ln(5 + |x(\tau_2(n))|) = 0, \quad n \geq 0, \tag{2.26}$$

where

$$\tau_1(n) = \begin{cases} n-1, & \text{if } n \in [3k, 3k+1] \\ -3n+12k+3, & \text{if } n \in [3k+1, 3k+2] \\ 5n-12k-13, & \text{if } n \in [3k+2, 3k+3] \end{cases}, \qquad k \in \mathbb{N}_0,$$

and

$$\tau_2(n) = \tau_1(n) - 1.$$

By (1.7), we see that

$$h_1(n) := \max_{s \le n} \tau_1(s) = \begin{cases} n-1, & \text{if } n \in [3k, 3k+1] \\ 3k, & \text{if } n \in [3k+1, 3k+2.6] \\ 5n-12k-13, & \text{if } n \in [3k+2.6, 3k+3] \end{cases}, \qquad k \in \mathbb{N}_0,$$

and

$$h_2(n) = h_1(n) - 1.$$

Therefore,

$$h(n) = \max_{1 \le i \le 2} \{h_i(n)\} = h_1(n).$$

On the other hand,

$$M_1 = \limsup_{x \to 0} \frac{x}{f_1(x)} = \limsup_{x \to 0} \frac{x}{x \ln(10 + |x|)} = \frac{1}{\ln 10},$$

and

$$M_2 = \limsup_{x \to 0} \frac{x}{f_2(x)} = \limsup_{x \to 0} \frac{x}{x \ln(5 + |x|)} = \frac{1}{\ln 5}.$$

So, we have

$$M^* = \max_{1 \le i \le 2} \{M_i\} = M_2$$

and

$$\liminf_{n \to \infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) = \frac{0.7}{e} > \frac{M^*}{e} = \frac{1}{e \ln 5},$$

that is, all conditions of Theorem 2.3 are satisfied and therefore all solutions of (2.26) oscillate.

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