Dynamic Systems and Applications 31 (2022) No.2, 63-88

EXPLORING THE INFLUENCE OF TIME-DELAY ON THE DYNAMICS OF A TWO PREY-ONE PREDATOR SYSTEM WITH QUADRATIC SELF-INTERACTION

ABDUL HUSSAIN SUROSH[a,b], REZA KHOSHSIAR GHAZIANI[a,*], AND JAVAD ALIDOUSTI[a]

[a]Department of Mathematical Sciences, Shahrekord University, Shahrekord, P.O.Box 115, Iran
[b]Department of Mathematics, Baghlan University, Pol-e-Khomri, Baghlan–Afghanistan

ABSTRACT. This work is devoted to studying the effects of time-delay on the dynamics of a two prey-one predator system with quadratic self-interaction. The essential dynamical structures of the delayed system are analyzed by means of local stability analysis and bifurcation theory. Taking the time lag τ as a free parameter, the necessary conditions for the existence of the Hopf bifurcation around the interior equilibrium of the system has been derived both analytically and numerically. It is observed that a Hopf bifurcation occurs when the bifurcation parameter crosses a certain threshold value and it is found that the dynamics of the system can be effected significantly by the time delay which has both stabilizing and destabilizing impacts depending on the magnitude of the delay. Moreover, we derived the explicit formulas in order to determine the direction of the Hopf bifurcation and examine the nature of the bifurcating periodic solution by using the normal form method and the center manifold theorem. Eventually, some numerical simulations are given to verify the derived theoretical analysis.

AMS (MOS) Subject Classification. 34L10, 34B24, 47E05.

Key Words and Phrases. Time delay system, Predator-prey, Center manifold theorem, Stability analysis, Hopf bifurcation.

1. INTRODUCTION

In ecological systems, the interaction between prey and predator is one of the basic interspecies relations, which shapes the community structure and ecosystem stability. Usually, the species regarded as food is called the prey and the consuming

* Corresponding author
Email addresses: khoshsiar@sku.ac.ir (Reza Khoshsiar Ghaziani), abdulhussain-surosh@stu.sku.ac.ir (Abdul Hussain Surosh), alidousti@sku.ac.ir (Javad Alidousti)
Received December 4, 2021 ISSN 1056-2176(Print); ISSN 2693-5295 (online)

https://doi.org/10.46719/dsa202231.02.01
www.dynamicpublishers.org.

species the predator [7]. Also, a predation is known as a biological interaction in ecology where a predator feeds on a prey while the accessibility of prey for predation is recognized as a functional response. Among this interaction, the rate at which prey are consumed by the predator can be determined by the a particular functional response [1, 5, 6].

Thus, the predator–prey models are playing a fundamental role in mathematical ecology due to their importance and universal existence. The mathematical model of predator–prey which is originally introduced by Lotka and Volterra, describe the growth rate of the population involved in predation process and it is known as Lotka–Volterra model [1, 3]. There has been great attention with a huge number of studies in population models during the last few decades which amongst of them, the predator-prey systems play an important role in population dynamics. Many applied mathematicians and ecologists have focused on the stability of the predator–prey models and in particular, they have analyzed the stability and dynamical behavior of such systems by incorporating the time delays into the models. Hence, a system of predator–prey becomes more realistic by introducing time-delays into the system [8, 15].

Introducing of delay in a system makes the system infinite-dimensional and generally, delay differential equations (DDEs) exhibit very complicated dynamics than ordinary differential equations (ODEs) since a time delay destabilizes the system's equilibria and could cause fluctuation in the population. Therefore, one of the essential topics concerning predator-prey systems is to investigate the influence of time delays on the dynamical behaviors of the systems such as periodic structure, bifurcation and so on [4, 8, 9, 25]. Time-delay dynamical systems are generally described by DDEs which arise frequently in various models and have wide range of applications in science and engineering [10, 11]. For the earlier years, DDEs has attracted many researchers' attention in diverse subjects, including mathematics, physics, biology, economics, engineering, etc [19, 22, 23]. Also, many natural systems are mathematically modeled by nonlinear delay differential equations which contain one or more time delays. For example, several practical systems such as controlling systems, network communication systems, manufacturing processes, population dynamics, rocket motors, nuclear reactors, load balancing instability in parallel calculation, and other various physical phenomena can be described by delayed models [19, 22, 25]. Recently, the influence of time delay in the predator-prey system with functional response functions are studied by many scholars [16–18]. The stability analysis of time delay system has been an active field in the control community since time delay can considerably change the performance and stability of a control system [13]. The stability changes can take place when the time delay crosses a threshold, a stable limit cycle may emerge through a local Hopf bifurcation [5, 20]. Hence, Hopf bifurcation scheme has

been extensively used to acquire more information about periodic solution properties near an equilibrium point of a nonlinear system [21].
In [2], I.K. Aybar et al. discussed the two prey-one predator system consisting of quadratic self interaction in the prey equations. In fact, the predator species of the considered system interacts with two prey species described by:

$$\begin{aligned} \frac{dx}{dt} &= x(a_1 - b_1 x - c_1 y), \\ \frac{dy}{dt} &= y(-a_2 + b_2 x + c_2 z), \\ \frac{dz}{dt} &= z(a_3 - b_3 y - c_3 z), \end{aligned} \tag{1.1}$$

where x and z represent the population densities of prey-A and prey-B, respectively, and y denotes the predator's population density. In order to have physical and biological descriptions, all parameters and variables of system (1.1) are assumed to be nonnegative. The detailed biological meanings of parameters are shown in Table 1. The authors of [2] investigated the the singular points' stability of system (1.1), and

Table 1. Biological meaning of parameters.

Parameter	**Biological meaning**
a_1	Growth rate of prey-A in the absence of predator
a_2	The death rate of the predator in the absence of the preys
a_3	Growth rate of prey-B in the absence of predator
b_1	Quadratic self interaction rate of prey-A
b_2	Consumption rate of the predator over prey-A
b_3	The death rate of prey-B due to predation
c_1	The death rate of prey-A due to predation
c_2	Consumption rate of the predator over prey-B
c_3	Quadratic self interaction rate of prey-B

by means of numerical simulation, they showed that solutions trajectories of the system can be approached to the stable singular points under given conditions. They also introduced an approach for examining the existence of Hopf bifurcation.

Motivated by the recent work of I.K. Aybar et al. [2] with considering the fact that the more realistic models should consist of delay differential equations without instantaneous feedbacks, we focus on the following time delayed predator-prey system with a single delay consisting of three species which represent the population densities of two prey and one predator species:

$$
\begin{cases}
\dfrac{dx(t)}{dt} = x(t)\big(a_1 - b_1 x(t-\tau) - c_1 y(t)\big),\\
\dfrac{dy(t)}{dt} = y(t)\big(-a_2 + b_2 x(t-\tau) + c_2 z(t-\tau)\big),\\
\dfrac{dz(t)}{dt} = z(t)\big(a_3 - b_3 y(t) - c_3 z(t-\tau)\big).
\end{cases} \tag{1.2}
$$

The initial conditions of the delayed system (1.2) can be chosen as:

$$
\begin{aligned}
x(\theta) &= \phi_1(\theta), \quad y(\theta) = \phi_2(\theta),\\
z(\theta) &= \phi_3(\theta), \qquad \theta \in [-\tau, 0],
\end{aligned} \tag{1.3}
$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$. Here C represents the Banach space $C([-\tau, 0], \mathbb{R}^3)$ of continuous functions mapping from the interval $[-\tau, 0]$ into $\mathbb{R}^3 = \{(x, y, z) \,|x \geq 0, y \geq 0, z \geq 0\}$. For biological feasibility, the initial functions are considered as

$$\phi_i(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad i = 1, 2, 3.$$

The paper main concern is to study the possible stability switches of the predator–prey system (1.2) with time delay and investigate how the time lag effects on the dynamical behavior of this system. As the time delay τ is treated as a bifurcation parameter and when it is passed though its critical value, we observe that the positive equilibrium losses its stability and the system exhibits limit cycle oscillations, i.e. a Hopf bifurcation occurs.

The remainder parts of the paper are arranged as follows. In Section 2, by linearizing the system (1.2) around the positive equilibrium point, the associated characteristic equation is discussed and stability analysis is performed. Furthermore, by taking τ as a bifurcation parameter, the conditions for the existence of Hopf bifurcations at the positive equilibrium are obtained. In Section 3, by the help of normal form method and theory of center manifold discussed in [27], we derive an explicit statement in order to determine the direction of the Hopf bifurcations and stability of the bifurcation periodic solutions. In Section 4, some numerical simulations are performed based on the suitable parameters' values and time delay to examine the derived analytical results. A brief conclusion is included in Section 5.

2. STABILITY OF POSITIVE EQUILIBRIUM AND HOPF BIFURCATION ANALYSIS

In this section, we investigate the local stability and effect of delay on the dynamic behavior of system (1.2) around the positive equilibrium (i.e., coexistence equilibrium) based on its linearized form. So if the conditions

$$(\mathrm{H}_1) \qquad a_2 c_3 > a_3 c_2, \qquad a_1 b_2 > a_2 b_1, \qquad a_3 c_1 > a_1 b_3.$$

are fulfilled, then system (1.2) has a unique positive equilibrium $E^*(x^*, y^*, z^*)$, where

$$x^* = \frac{a_1 b_3 c_2 + c_1(a_2 c_3 - a_3 c_2)}{b_1 b_3 c_2 + b_2 c_1 c_3},$$
$$y^* = \frac{a_3 b_1 c_2 + c_3(a_1 b_2 - a_2 b_1)}{b_1 b_3 c_2 + b_2 c_1 c_3},$$
$$z^* = \frac{a_2 b_1 b_3 + b_2(a_3 c_1 - a_1 b_3)}{b_1 b_3 c_2 + b_2 c_1 c_3}.$$

For local stability analysis, we use the transformation $X(t) = x(t) - x^*$, $Y(t) = y(t) - y^*$ and $Z(t) = z(t) - z^*$ to obtain the linearized system. Therefore, the linearization of (1.2) around E^* takes the form:

$$\begin{cases} \dfrac{dX(t)}{dt} = a_{11}X(t) + a_{12}Y(t) + b_{11}X(t-\tau), \\ \dfrac{dY(t)}{dt} = a_{22}Y(t) + b_{21}X(t-\tau) + b_{23}Z(t-\tau), \\ \dfrac{dZ(t)}{dt} = a_{32}Y(t) + a_{33}Z(t) + b_{33}Z(t-\tau), \end{cases} \tag{2.1}$$

which can be written as

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \\ \dot{Z}(t) \end{bmatrix} = A_1 \begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} + A_2 \begin{bmatrix} X(t-\tau) \\ Y(t-\tau) \\ Z(t-\tau) \end{bmatrix}, \tag{2.2}$$

where

$$A_1 = \begin{bmatrix} a_1 - b_1 x^* - c_1 y^* & -c_1 x^* & 0 \\ 0 & -a_2 + b_2 x^* + c_2 z^* & 0 \\ 0 & -b_3 z^* & a_3 - b_3 y^* - c_3 z^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} -b_1 x^* & 0 & 0 \\ b_2 y^* & 0 & c_2 y^* \\ 0 & 0 & -c_3 z^* \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}.$$

The characteristic polynomial of the delayed system (1.2), depending on τ, can be described by

$$\det \begin{pmatrix} \lambda - (a_{11} + b_{11}e^{-\lambda\tau}) & -a_{12} & 0 \\ -b_{21}e^{-\lambda\tau} & \lambda - a_{22} & -b_{23}e^{-\lambda\tau} \\ 0 & -a_{32} & \lambda - (a_{33} + b_{33}e^{-\lambda\tau}) \end{pmatrix} = 0.$$

Then, a straightforward calculation leads to

(2.3)
$$\triangle(\lambda,\tau) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau} + (h_1\lambda + h_0)e^{-2\lambda\tau} = 0,$$

where

$$\begin{aligned}
&p_2 = -a_{33} - a_{22} - a_{11}, \quad p_1 = a_{22}a_{33} - (-a_{33} - a_{22})a_{11}, \quad p_0 = -a_{11}a_{22}a_{33},\\
&q_2 = -b_{33} - b_{11}, \quad q_1 = (b_{11} + b_{33})a_{22} - b_{23}a_{32} + b_{33}a_{11} - a_{12}b_{21} + b_{11}a_{33},\\
&q_0 = (-a_{11}b_{33} - b_{11}a_{33})a_{22} + a_{11}a_{32}b_{23} + a_{12}a_{33}b_{21}, \quad h_1 = b_{11}b_{33},\\
&h_0 = (a_{12}b_{21} - a_{22}b_{11})b_{33} + a_{32}b_{11}b_{23}.
\end{aligned}$$

Multiplying both sides of (2.3) by $e^{\lambda\tau}$, gives

$$(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)e^{\lambda\tau} + q_2\lambda^2 + q_1\lambda + q_0 + (h_1\lambda + h_0)e^{-\lambda\tau} = 0. \tag{2.4}$$

In order to analyze the distribution of roots of Eq. (2.4), we use the following result given in [26].

Lemma 2.1. *Consider the transcendental equation*

$$\begin{aligned}
P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = {} & \lambda^n + p_1^{(0)}\lambda^{(n-1)} + \cdots + p_{(n-1)}^{(0)}\lambda + p_n^{(0)}\\
& + \left[p_1^{(1)}\lambda^{(n-1)} + \cdots + p_{(n-1)}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1}\\
& + \cdots + \left[p_1^{(m)}\lambda^{(n-1)} + \cdots + p_{(n-1)}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0,
\end{aligned}$$

where $\tau_i \geq 0$ $(i = 1, 2, \cdots, m)$ *and* $p_j^{(i)}(i = 0, 1, \cdots, m; j = 1, 2, \cdots, n)$ *are constants. As* $(\tau_1, \tau_2, \cdots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

For analyzing the system (1.2) dynamically, we consider the following two cases.

Case I: $\tau = 0$**.** To examine the stability of E^* for $\tau = 0$, we substitue $\tau = 0$ into Eq. (2.4) which reduces to

$$\triangle(\lambda) = \lambda^3 + (p_2 + q_2)\lambda^2 + (p_1 + q_1 + h_1)\lambda + p_0 + q_0 + h_0 = 0. \tag{2.5}$$

So based on Routh–Hurwitz criterion [12], we know that all roots of Eq. (2.5) have negative real parts if the following conditions hold

$$(\mathrm{H}_2) \qquad p_2 + q_2 > 0, \qquad p_0 + q_0 + h_0 > 0, \qquad (p_2 + q_2)(p_1 + q_1 + h_1) > p_0 + q_0 + h_0.$$

Hence, we have the following lemma.

Lemma 2.2. *If the conditions* (H_1) *and* (H_2) *hold, then the equilibrium point* E^* *of system* (1.2) *becomes locally asymptotically stable for* $\tau = 0$*.*

Case II: $\tau \neq 0$**.** In this case, we apply the Hopf bifurcation theory i.e. a dynamical system undergos a Hopf bifurcation if its corresponding characteristic equation has a pair of complex conjugate pure imaginary roots. Obviously, $\pm \mathrm{i}\omega$ $(\omega > 0)$ are a pair of purely imaginary roots of Eq. (2.4) if and only if ω satisfies

$$(-\mathrm{i}\omega^3 - p_2\omega^2 + p_1\omega \mathrm{i} + p_0)e^{\mathrm{i}\omega\tau} - q_2\omega^2 + q_1\omega \mathrm{i} + q_0 + (h_1\omega \mathrm{i} + h_0)e^{-\mathrm{i}\omega\tau} = 0. \tag{2.6}$$

Then we get

$$\begin{aligned} &(p_0 + h_0 - p_2\omega^2)\cos(\omega\tau) + \omega(h_1 - p_1 + \omega^2)\sin(\omega\tau) = q_2\omega^2 - q_0, \\ &\omega(h_1 + p_1 - \omega^2)\cos(\omega\tau) + (p_0 - h_0 - p_2\omega^2)\sin(\omega\tau) = -q_1\omega. \end{aligned} \tag{2.7}$$

which leads to

$$Q_2\cos^2(\omega_0\tau) + Q_1\sin(\omega_0\tau)\cos(\omega_0\tau) + Q_0 = 0, \tag{2.8}$$

where

$$\begin{aligned} Q_2 &= 4h_1\omega_0^2 - 4(h_1p_1 - h_0p_2)\omega_0^2 + 4h_0p_0, \\ Q_1 &= 4\big((h_1p_2 - h_0)\omega_0^2 + h_0p_1 - h_1p_0\big)\omega_0, \\ Q_0 &= -\omega_0^6 + q_2^2\omega_0^4 + \big(q_1^2 - 2q_0q_2\big)\omega_0^2 + q_0^2 - (p_2^2 + 2h_1 - 2p_1)\omega_0^4 \\ &\quad - (h_1^2 + p_1^2 + 2h_0p_2 - 2h_1p_1 - 2p_0p_2)\omega_0^2 - (h_0 - p_0)^2. \end{aligned}$$

According to $\sin(\omega_0\tau) = \pm\sqrt{1 - \cos^2(\omega_0\tau)}$ and (2.8), we get

$$l_2\cos^4(\omega_0\tau) + l_1\cos^2(\omega_0\tau) + l_0 = 0, \tag{2.9}$$

where

$$l_2 = Q_1^2 + Q_2^2, \quad l_1 = 2Q_1Q_2 - Q_1^2, \quad l_0 = Q_0^2.$$

Let $\cos(\omega_0\tau) = r$, then we have

$$l_2r^4 + l_1r^2 + l_0 = 0. \tag{2.10}$$

Let $\Delta = l_1^2 - 4l_0l_2$, then the roots of (2.10) are as follows:

$$\begin{aligned} r_1 &= \frac{\sqrt{2}}{2}\frac{\sqrt{l_2(-l_1 + \sqrt{\Delta})}}{l_2}, \qquad r_2 = -\frac{\sqrt{2}}{2}\frac{\sqrt{l_2(-l_1 + \sqrt{\Delta})}}{l_2}, \\ r_3 &= \frac{\sqrt{2}\mathrm{i}}{2}\frac{\sqrt{l_2(l_1 + \sqrt{\Delta})}}{l_2}, \qquad r_4 = -\frac{\sqrt{2}\mathrm{i}}{2}\frac{\sqrt{l_2(l_1 + \sqrt{\Delta})}}{l_2}. \end{aligned}$$

According to the analysis above, the expression of $\cos(\omega_0\tau)$ can be derived as

$$\cos(\omega_0\tau) = f_1(\omega_0), \tag{2.11}$$

where $f_1(\omega_0)$ is a function w.r.t. ω_0. By substituting (2.11) into (2.8), the expression of $\sin(\omega_0\tau)$ can be obtained as

$$\sin(\omega_0\tau) = f_2(\omega_0), \tag{2.12}$$

where $f_2(\omega_0)$ is a function w.r.t. ω_0. Thus we can get

$$f_1^2(\omega_0) + f_2^2(\omega_0) = 1. \tag{2.13}$$

We can obtain ω_0 from (2.13) by using mathematical software. Then, based on (2.11), we have

$$\tau_k^{(j)} = \frac{1}{\omega_k}[\arccos(f_1(\omega_0)) + 2j\pi], \ (j = 0, 1, 2, \cdots). \tag{2.14}$$

Hence, we can derive a more clear explicit formula for determining the values of $\tau_k^{(j)}$ by direct computation of (2.7), i.e.

$$\tau_k^{(j)} = \begin{cases} \frac{1}{\omega_k}[\arccos(Q) + 2j\pi], & L \geq 0 \\ \frac{1}{\omega_k}[2\pi - \arccos(Q) + 2j\pi], & L < 0, \end{cases} \tag{2.15}$$

where

$$\begin{aligned} L &= \sin(\omega_k\tau_k) \\ &= \frac{\big(-q_2\omega_k^4 + \big((h_1+p_1)q_2 - p_2q_1 + q_0\big)\omega_k^2 - (p_1+h_1)q_0 + (p_0+h_0)q_1\big)\omega_k}{-\omega_k^6 + (2p_1 - p_2^2)\omega_k^4 + (h_1^2 + 2p_0p_2 - p_1^2)\omega_k^2 + h_0^2 - p_0^2}, \end{aligned}$$

$$Q = \cos(\omega_k\tau_k) = \frac{(p_2q_2 - q_1)\omega_k^4 + \big((p_1-h_1)q_1 + (h_0-p_0)q_2 - p_2q_0\big)\omega_k^2 - (h_0-p_0)q_0}{-\omega_k^6 + (2p_1 - p_2^2)\omega_k^4 + (h_1^2 + 2p_0p_2 - p_1^2)\omega_k^2 + h_0^2 - p_0^2}.$$

Lemma 2.3. *Let $\tau = \tau_k^{(j)}$ be defined by* (2.14) *or* (2.15)*. Assume the conditions* (H_1) *and* (H_2) *hold, then we deduce the following statements:*

- **(i):** *If $\tau \in [0, \tau_0)$, then Eq. (2.4) have the roots with strictly negative real parts.*
- **(ii):** *At $\tau = \tau_k^{(j)}$ $(k = 1, 2, 3; j = 0, 1, 2, \cdots)$, Eq. (2.4) has a pair of complex conjugate roots $\pm \mathrm{i}\omega_k$ and all other roots have negative real parts.*

Moreover, in order to discuss the existence of Hopf bifurcation with respect to the bifurcation parameter τ, we need further analysis, i.e, to verify the transversality condition.

Lemma 2.4. *Let $\lambda(\tau) = \eta(\tau) + \mathrm{i}\omega(\tau)$ be a root of (2.4) near $\tau = \tau_k^{(j)}$ satisfying $\eta(\tau_k^{(j)}) = 0$ and $\omega(\tau_k^{(j)}) = \omega_k$, then, the following cross-sectional condition holds*

$$\Big[\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Re}(\lambda(\tau))\Big]_{\lambda = i\omega_0, \tau = \tau_0} \neq 0,$$

where ω_0, τ_0 are the critical frequency and bifurcation point of system (1.2)*, respectively.*

Proof. Differentiating the characteristic Eq. (2.4) w.r.t. τ, we get

$$\begin{aligned} &\Bigg[(3\lambda^2 + 2p_2\lambda + p_1)e^{\lambda\tau} + (\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)\tau e^{\lambda\tau} + 2q_2\lambda + q_1 + h_1\mathrm{e}^{-\lambda\tau} \\ &\quad - (h_1\lambda + h_0)\tau e^{-\lambda\tau}\Bigg]\frac{d\lambda}{d\tau} = -\Big(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0\Big)\lambda e^{\lambda\tau} + (h_1\lambda + h_0)\lambda\mathrm{e}^{-\lambda\tau}, \end{aligned}$$

which can be written as

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2(p_2+q_2)\lambda + p_1 + q_1 + h_1 \mathrm{e}^{-\lambda\tau}}{-\lambda\big(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 - (h_1\lambda + h_0)\mathrm{e}^{-\lambda\tau}\big)} - \frac{\tau}{\lambda}. \tag{2.16}$$

By substituting $\lambda = \mathrm{i}\omega_0$ into (2.16) and straightforward computation, we obtain

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda=i\omega_0,\tau=\tau_0} = Re\Big(\frac{\chi_1 + i\chi_2}{\chi_3 + i\chi_4}\Big) = \frac{\chi_1\chi_3 + \chi_2\chi_4}{\chi_3^2 + \chi_4^2},$$

where

$$\begin{aligned}
\chi_1 =& \Big[\big(2\tau_0 p_2 + 6\big)\omega_0^2 - 2\tau_0 p_0 - 2p_1\Big]\cos^2(\omega_0\tau_0) + \Big[2\omega_0\big(-\omega_0^2\tau_0 + p_1\tau_0 + 2p_2\big)\cos(\omega_0\tau_0)\\
&+ 2q_2\omega_0\Big]\sin(\omega_0\tau_0) - q_1\cos(\omega_0\tau_0) + \big(-p_2\omega_0 - 3\big)\omega_0^2 + (h_0 + p_0)\tau_0 - h_1 + p_1,\\
\chi_2 =& -2\big(-\omega_0^2\tau_0 + p_1\tau_0 + 2p_2\big)\omega_0\cos^2(\omega\tau_0) + \Big[\big(\omega_0^2(6 + 2p_2\tau_0) - 2p_0\tau_0 - 2p_1\big)\\
&- q_1\Big]\sin(\omega_0\tau_0) - 2q_2\omega_0\cos(\omega_0\tau_0) + \big(-\omega_0^2\tau_0 + h_1\tau_0 + p_1\tau_0 + 2p_2\big)\omega_0\\
\chi_3 =& -2\big(-\omega_0^2 p_2 + p_0\big)\omega_0^2\cos^2(\omega_0\tau_0) - 2\omega_0\sin(\omega_0\tau_0)\big(-p_2\omega_0^2 + p_0\big)\cos(\omega_0\tau_0)\\
&+ \big(-\omega_0^2 + h_1 + p_1\big)\omega_0^2,\\
\chi_4 =& -2\big(p_2\omega_0^2 - p_0\big)\omega_0\cos^2(\omega_0\tau_0) - 2\omega_0^2\sin(\omega_0\tau_0)\big(-\omega_0^2 + p_1\big)\cos(\omega_0\tau_0)\\
&- \big(-p_2\omega_0^2 + h_0 + p_0\big)\omega_0.
\end{aligned}$$

Since $Re\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\Big)^{-1}\Big|_{\lambda=i\omega_0}$ and $\Big[\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\Big]_{\lambda=i\omega_0}$ have the same sign, therefore, if the condition $(H_{41}):\ \chi_1\chi_3 + \chi_2\chi_4 \neq 0$ holds, then we conclude that $\Big[\frac{\mathrm{d}(Re\lambda)}{\mathrm{d}\tau}\Big]_{\lambda=i\omega_0} \neq 0$. This implies that by increasing of the delay τ, all the roots pass through the imaginary axis from left to right at $\mathrm{i}\omega$. Hence, the transversality condition holds and accordingly a Hopf bifurcation occurs. This completes the proof. □

Define $\tau_0 = \tau_{k_0} = \min_{1\le k\le 3}\{\tau_k\}, \omega_0 = \omega_{k_0}$, and applying the theory of Hopf bifurcation for functional differential equations [27], we can conclude the following theorem for the existence of Hopf bifurcation.

Theorem 2.5. *For system (1.2) with $\tau \neq 0$, if the assumptions* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold, then we have the following results.*

- **(i):** *The positive equilibrium E^* is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.*
- **(ii):** *When $\tau = \tau_0$, a Hopf bifurcation occurs at E^* as τ passes through the critical value τ_0.*

3. PROPERTIES OF THE HOPF BIFURCATION

In this part, we investigate the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions arising through Hopf bifurcation. We employed the helpful analytical methods, i.e, the normal form and the center manifold theory proposed by Hassard et al. [27]. Throughout this section, we suppose that system (1.2) undergoes a Hopf bifurcation at the positive equilibrium E^* for $\tau = \tau_0$, and $\mathrm{i}\omega_0$ is a pure imaginary root of the corresponding characteristic equation at this equilibrium point.
By changing of variables $u_1(t) = x(t) - x^*, u_2(t) = y(t) - y^*, u_3(t) = z(t) - z^*$, we can rewrite the system (2.1) by adding the nonlinear parts as the following form:

$$\begin{cases} \dot{u}_1(t) = a_{11}u_1(t) + a_{12}u_2(t) + b_{11}u_1(t-\tau) + a_{13}u_1^2(t) \\ \qquad\quad +b_{12}u_1(t)u_1(t-\tau) + a_{14}u_1(t)u_2(t) + h.o.t., \\ \dot{u}_2(t) = a_{22}u_2(t) + b_{21}u_1(t-\tau) + b_{23}u_3(t-\tau) + a_{23}u_1(t)u_2(t) \\ \qquad\quad +b_{24}u_1(t-\tau)u_2(t) + a_{24}u_2(t)u_3(t) + b_{25}u_2(t)u_3(t-\tau) + h.o.t., \\ \dot{u}_3(t) = a_{32}u_2(t) + a_{33}u_3(t) + b_{33}u_3(t-\tau) \\ \qquad\quad +a_{34}u_2(t)u_3(t) + a_{35}u_3^2(t) + b_{34}u_3(t)u_3(t-\tau) + h.o.t., \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &a_{13} = b_{12} = -b_1, \qquad && a_{14} = -c_1, \qquad && a_{23} = b_{24} = b_2, \\ &a_{24} = b_{25} = c_2, && a_{34} = -b_3, && a_{35} = b_{34} = -c_3. \end{aligned}$$

Notice that the coefficients of linear part of (3.1) have been determined in Section 2. Without loss of generality, denote the critical value τ_0 by $\bar{\tau}$, and let $\tau = \bar{\tau} + \mu$, $(\mu \in \mathbb{R})$ in which the μ is a new bifurcation parameter. Then $\mu = 0$ is the bifurcating point of the system (1.2). For sake of simplicity, we use τ instead of $\bar{\tau}$. To reduce the system to its central manifold, it is necessary to convert the system into a functional differential equation. Therefore, by normalizing the delay by the scaling $t \longrightarrow t/\tau$, system (3.1) can be transformed into functional differential equation (FDE) in $\mathcal{C} = \mathcal{C}\big([-1,0], \mathbb{R}^3\big)$:

$$\dot{u}(t) = \mathcal{L}_\mu(u_t) + \mathcal{F}(\mu, u_t), \tag{3.2}$$

where

$$\begin{aligned} u_t(\theta) &= u(t+\theta) = \big(x(t+\theta), y(t+\theta), z(t+\theta)\big)^T \in \mathbb{R}^3, \\ \phi(t) &= \big(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)\big)^T \in \mathcal{C}, \end{aligned}$$

and $\mathscr{L}_\mu : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^3, \mathscr{F} : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^3$ are defined, respectively, by

$$\begin{aligned} \mathscr{L}_\mu(\phi) &= (\tau_0+\mu)\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ &+ (\tau_0+\mu)\begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}, \end{aligned} \tag{3.3}$$

and

$$\mathscr{F}(\mu,\phi) = (\tau_0+\mu)\begin{pmatrix} a_{13}\phi_1^2(0) + b_{12}\phi_1(0)\phi_1(-1) + a_{14}\phi_1(0)\phi_2(0) + h.o.t. \\ a_{23}\phi_1(0)\phi_2(0) + b_{24}\phi_1(-1)\phi_2(0) + a_{24}\phi_2(0)\phi_3(0) \\ +b_{25}\phi_2(0)\phi_3(-1) + h.o.t. \\ a_{34}\phi_2(0)\phi_3(0) + a_{35}\phi_3^2(0) + b_{34}\phi_3(0)\phi_3(-1) + h.o.t. \end{pmatrix}. \tag{3.4}$$

Based on Riesz representation theorem [27], there exists a function $\eta(\theta,\mu)$ whose components are bounded variation for $\theta \in [-1,0]$ such that the operator $\mathscr{L}_\mu$ can be defined in an integral form:

$$\mathscr{L}_\mu(\phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta), \quad \theta \in \mathcal{C}.$$

In fact, $\eta(\theta,\mu)$ can be written as

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}\delta(\theta) - (\tau_0+\mu)\begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & 0 & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}\delta(\theta+1),$$

where $\delta(\theta) = \begin{cases} 0, & \theta \neq 0 \\ 1, & \theta = 0. \end{cases}$

For $\phi \in C^1([-1,0],\mathbb{R}^3)$, we define the operator $\mathscr{A}(\mu)$ as

$$\mathscr{A}(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0) \\ \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta), & \theta = 0 \end{cases} \tag{3.5}$$

and

$$\mathscr{R}(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0) \\ \mathscr{F}(\mu,\phi), & \theta = 0. \end{cases} \tag{3.6}$$

Then system (3.2) is equivalent to the following operator equation

$$\dot{u}_t = \mathscr{A}(\mu)u_t + \mathscr{R}(\mu)u_t, \tag{3.7}$$

where $u_t(\theta) = u(t+\theta), \ \theta \in [-1,0]$. For $\psi \in C^1([0,1],(\mathbb{R}^3)^*)$, where $(\mathbb{R}^3)^*$ is the three-dimensional space of row vectors, we define the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}(0)$ as

$$\mathcal{A}^*(\mu)\psi(s) = \begin{cases} \frac{-d\psi(s)}{ds}, & s \in (0,1] \\ \int_{-1}^{0} \mathrm{d}\eta^T(s,\mu)\psi(-s), & s = 0 \end{cases} \tag{3.8}$$

where η^T is the transpose of the matrix η.
To normalize the eigenvector of $\mathcal{A}$ and its adjoint $\mathcal{A}^*$, we define a bilinear inner product for $\phi \in C^1([-1,0],\mathbb{R}^3)$ and $\psi \in C^1([0,1],(\mathbb{R}^3)^*)$ as

$$\langle \psi(s), \phi(\theta)\rangle = \overline{\psi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\, d\xi, \tag{3.9}$$

where $\eta(\theta) = \eta(\theta,0)$. Then $\mathcal{A}(0)$ and $\mathcal{A}^*(0)$ are a pair of adjoint operators.

From Section 2, we see that $\pm \mathrm{i}\omega_0\tau_0$ are eigenvalues of $\mathcal{A}(0)$ and therefore they are also eigenvalues of $\mathcal{A}^*(0)$. We first need to compute the eigenvector $q(\theta)$ of $\mathcal{A}(0)$ corresponding to $\mathrm{i}\omega_0\tau_0$ and eigenvector $q^*(s)$ of $\mathcal{A}^*$ corresponding to the eigenvalue $-\mathrm{i}\omega_0\tau_0$.

Let $q(\theta) = (1,\alpha_1,\alpha_2)^T e^{\mathrm{i}\omega_0\tau_0\theta}$ be the eigenvector of $\mathcal{A}(0)$ corresponding to $\mathrm{i}\omega_0\tau_0$, i.e. $\mathcal{A}(0)q(\theta) = \mathrm{i}\omega_0\tau_0 q(\theta)$. Then, it follows from the definition of $\mathcal{A}(0)$ and (3.3) that

(3.10)
$$\tau_0 \begin{pmatrix} \mathrm{i}\omega_0 - (a_{11} + b_{11}e^{-\mathrm{i}\omega_0\tau}) & -a_{12} & 0 \\ -b_{21}e^{-\mathrm{i}\omega_0\tau} & \mathrm{i}\omega_0 - a_{22} & -b_{23}e^{-\mathrm{i}\omega_0\tau} \\ 0 & -a_{32} & \mathrm{i}\omega_0 - (a_{33} + b_{33}e^{-\mathrm{i}\omega_0\tau}) \end{pmatrix} (q(0)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By direct calculation, we get

$$q(0) = (1,\alpha_1,\alpha_2)^T = \left(1, \frac{\mathrm{i}\omega_0 - a_{11} - b_{11}e^{-\mathrm{i}\omega_0\tau_0}}{a_{12}}, \frac{a_{32}\big(b_{11}e^{-\mathrm{i}\omega_0\tau_0} - \mathrm{i}\omega_0 + a_{11}\big)}{a_{12}\big(b_{33}e^{-\mathrm{i}\omega_0\tau_0} - \mathrm{i}\omega_0 + a_{33}\big)}\right)^T.$$

If $q^*(s) = B(1,\alpha_1^*,\alpha_2^*)^T e^{i\omega_0^*\tau_0 s}$ be the eigenvector of $\mathcal{A}^*$ related to $-i\omega_0^*\tau_0$, i.e. $\mathcal{A}^*(0)q^{*T}(\theta) = -i\omega_0\tau_0 q^{*T}(\theta)$, then

$$\alpha_1^* = \frac{(-\mathrm{i}\omega_0 - a_{11})e^{-\mathrm{i}\omega_0\tau_0} - b_{11}}{b_{21}}, \qquad \alpha_2^* = \frac{b_{23}(\mathrm{i}\omega_0 + a_{11} + b_{11}e^{\mathrm{i}\omega_0\tau_0})}{b_{21}(\mathrm{i}\omega_0 + a_{33} + b_{33}e^{\mathrm{i}\omega_0\tau_0})}.$$

To verify the conditions

$$\langle q^*(s), q(\theta)\rangle = 1 \quad \textit{and} \quad \langle q^*(s), \overline{q}(\theta)\rangle = 0,$$

we need to calculate the value of B. By Eq. (3.9), we have

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \overline{q}^*(0)q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{q^*}^T(\xi-\theta)\mathrm{d}\eta(\theta)q(\xi)\,d\xi \\ &= \bar{B}(1,\overline{\alpha_1^*},\overline{\alpha_2^*})(1,\alpha_1,\alpha_2)^T \\ &\quad - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{B}(1,\overline{\alpha_1^*},\overline{\alpha_2^*})e^{-\mathrm{i}\omega_0\tau_0(\xi-\theta)}\mathrm{d}\eta(\theta)\times(1,\alpha_1,\alpha_2)^T e^{\mathrm{i}\omega_0\tau_0\xi}\,d\xi \\ &= \bar{B}\Big\{1+\alpha_1\overline{\alpha_1^*}+\alpha_2\overline{\alpha_2^*} - \int_{-1}^{0}(1,\overline{\alpha_1^*},\overline{\alpha_2^*})\theta e^{i\omega_0\tau_0\theta}\mathrm{d}\eta(\theta)(1,\alpha_1,\alpha_2)^T\Big\} \\ &= \bar{B}\Big\{1+\alpha_1\overline{\alpha_1^*}+\alpha_2\overline{\alpha_2^*}+\tau_0\big(b_{11}+b_{21}\overline{\alpha_1^*}+b_{23}\overline{\alpha_1^*}\alpha_2+b_{33}\alpha_2\overline{\alpha_2^*}\big)e^{-\mathrm{i}\omega_0\tau_0}\Big\}.\end{aligned}$$

Hence, according to $\langle q^*(s), q(\theta)\rangle = 1$, we get

$$\bar{B} = \frac{1}{1+\alpha_1\overline{\alpha_1^*}+\alpha_2\overline{\alpha_2^*}+\tau_0\big(b_{11}+b_{21}\overline{\alpha_1^*}+b_{23}\overline{\alpha_1^*}\alpha_2+b_{33}\alpha_2\overline{\alpha_2^*}\big)e^{-\mathrm{i}\omega_0\tau_0}},$$

i.e.

$$B = \frac{1}{1+\overline{\alpha_1}\alpha_1^*+\overline{\alpha_2}\alpha_2^*+\tau_0\big(b_{11}+b_{21}\alpha_1^*+b_{23}\alpha_1^*\overline{\alpha_2}+b_{33}\overline{\alpha_2}\alpha_2^*\big)e^{-\mathrm{i}\omega_0\tau_0}}.$$

Next, we determine the stability of bifurcating periodic solution. So, we will follow the same notations as given in Hassard et al.[27] as well as same algorithms and computation given in [14, 15]. The bifurcating periodic solutions $z(t,\mu(\xi))$ have the amplitude $O(\xi)$ and nonzero Floquet exponent $\beta(\xi)$ with $\beta(0)=0$. Then, μ and β can be defined by

$$\begin{aligned}\mu &= \mu_2\xi_2+\mu_4\xi_4+\cdots, \\ \beta &= \beta_2\xi_2+\beta_4\xi_4+\cdots.\end{aligned}$$

Here, the sign of μ_2 denotes the direction of bifurcating periodic solution while β_2 specifies the stability of $z(t,\mu(\xi))$, which is stable if $\beta_2<0$ and unstable if $\beta_2>0$. The next step is to compute the coordinates of a local invariant manifold C_0 at $\mu=0$, which is attracting a two-dimensional manifold and known as a center manifold [15, 27]. Let u_t be a solution of Eq.(3.2) when $\mu=0$. Define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t(\theta) - 2Re\{z(t), q(t)\}. \tag{3.11}$$

On the center manifold C_0, we obtain $W(t,\theta)=W(z(t),\overline{z}(t),\theta)$, where

$$W(z,\overline{z},\theta) = W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+W_{30}(\theta)\frac{z^3}{6}+\cdots. \tag{3.12}$$

In Eq. (3.12), z and $\overline{z}$ represent the coordinates of center manifold C_0 in the direction of q^* and $\overline{q^*}$, respectively. In fact, it is the help of center manifold which we are enable to reduce (3.2) to the form of an ordinary differential equation with a single complex variable on $\mathcal{C}_0$. In (3.11) and (3.12), W is real if u_t is real. Therefore, we consider only the real solutions. For the solution $u_t \in \mathcal{C}_0$ of (3.2), we have

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}_t\rangle = \langle q^*, \mathcal{A}(u_t) + \mathcal{R}(u_t)\rangle \\ &= \langle q^*, \mathcal{A}(u_t)\rangle + \langle q^*, \mathcal{R}(u_t)\rangle \\ &= \langle \mathcal{A}(q^*), u_t\rangle + \langle q^*, \mathcal{R}(u_t)\rangle \\ &= \mathrm{i}\omega_0\tau_0 z + \overline{q^*}(\theta)\cdot \mathcal{F}\big(0, W(z,\overline{z},\theta) + 2Re[z(t)q(\theta)]\big) \\ &= \mathrm{i}\omega_0\tau_0 z + \overline{q^*}(0)\cdot \mathcal{F}\big(0, W(z,\overline{z},0) + 2Re[z(t)q(0)]\big) \\ &= \mathrm{i}\omega_0\tau_0 z + \overline{q^*}(0)\cdot \mathcal{F}_0\big(z,\overline{z}\big), \end{aligned} \tag{3.13}$$

which can be rewriten as

$$\dot{z}(t) = \mathrm{i}\omega_0\tau_0 z + \mathrm{g}(z,\overline{z}), \tag{3.14}$$

where

$$\mathrm{g}(z,\overline{z}) = \mathrm{g}_{20}(\theta)\frac{z^2}{2} + \mathrm{g}_{11}(\theta)z\overline{z} + \mathrm{g}_{02}(\theta)\frac{\overline{z}^2}{2} + \mathrm{g}_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots. \tag{3.15}$$

According to (3.13) and (3.14), we can obtain the coefficients of the expansion (3.15) as follows

$$\begin{aligned} \mathrm{g}(z,\overline{z}) &= \overline{q^*}^T(0)\mathcal{F}(z,\overline{z}) = \tau_0\overline{B}(1,\overline{\alpha_1^*},\overline{\alpha_2^*}) \\ &\times \begin{pmatrix} a_{13}u_{1t}^2(0) + b_{12}u_{1t}(0)u_{1t}(-1) + a_{14}u_{1t}(0)u_{2t}(0) \\ a_{23}u_{1t}(0)u_{2t}(0) + b_{24}u_{1t}(-1)u_{2t}(0) + a_{24}u_{2t}(0)u_{3t}(0) \\ +b_{25}u_{2t}(0)u_{3t}(-1) \\ a_{34}u_{2t}(0)u_{3t}(0) + a_{35}u_{3t}^2(0) + b_{34}u_{3t}(0)u_{3t}(-1) \end{pmatrix}. \end{aligned} \tag{3.16}$$

Notice that

$$\begin{aligned} u(t) &= (u_{1t}(\theta), u_{2t}(\theta), u_{3t}(\theta)) = W(t,\theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta), \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + (1,\alpha_1,\alpha_2)^T e^{\theta i\omega_0\tau_0} z \\ &+ (1,\overline{\alpha}_1,\overline{\alpha}_2)^T e^{\theta i\omega_0\tau_0}\overline{z} + \cdots, \end{aligned}$$

and $q(\theta) = (1, \alpha_1, \alpha_2)^T e^{i\omega_0\tau_0\theta}$, then we obtain

$$
\begin{aligned}
u_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{2t}(0) &= \alpha_1 z + \overline{\alpha_1 z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{3t}(0) &= \alpha_2 z + \overline{\alpha_2 z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{1t}(-1) &= ze^{-i\omega_0} + \overline{z}e^{i\omega_0} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z}\\
&\quad + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{2t}(-1) &= \alpha_1 ze^{-i\omega_0} + \overline{\alpha}_1\overline{z}e^{i\omega_0} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z}\\
&\quad + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
u_{3t}(-1) &= \alpha_2 ze^{-i\omega_0} + \overline{\alpha}_2\overline{z}e^{i\omega_0} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\overline{z}\\
&\quad + W_{02}^{(3)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3).
\end{aligned}
\tag{3.17}
$$

From (3.16) and (3.17), we get

$$
\begin{aligned}
\mathrm{g}(z,\overline{z}) &= \tau_0\bar{B}\Big[a_{13}u_{1t}^2(0) + b_{12}u_{1t}(0)u_{1t}(-1) + a_{14}u_{1t}(0)u_{2t}(0)\Big]\\
&\quad + \tau_0\bar{B}\overline{\alpha_1^*}\Big[a_{23}u_{1t}(0)u_{2t}(0) + b_{24}u_{1t}(-1)u_{2t}(0) + a_{24}u_{2t}(0)u_{3t}(0)\\
&\quad + b_{25}u_{2t}(0)u_{3t}(-1)\Big]\\
&\quad + \tau_0\bar{B}\overline{\alpha_2^*}\Big[a_{34}u_{2t}(0)u_{3t}(0) + a_{35}u_{3t}^2(0) + b_{34}u_{3t}(0)u_{3t}(-1)\Big].
\end{aligned}
\tag{3.18}
$$

It follows that

$$
\begin{aligned}
\mathrm{g}(z,\overline{z}) = \tau_0\bar{B}\Bigg\{& a_{13}\big(z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3)\big)^2\\
&+ b_{12}\big(z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3)\big)\\
&\times \big(ze^{-i\omega_0} + \overline{z}e^{i\omega_0} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3)\big)\\
&+ a_{14}\big(z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3)\big)\\
&\times \big[\alpha_1 z + \overline{\alpha_1 z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3)\big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\tau_0\bar{B}\Bigg\{\overline{\alpha_1^*}\Big[a_{23}\big(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)^2\\
&\times\big[\alpha_1 z+\overline{\alpha_1 z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big]\\
&+b_{24}\big(ze^{-i\omega_0}+\overline{z}e^{i\omega_0}+W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\overline{z}+W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)\\
&\times\big[\alpha_1 z+\overline{\alpha_1 z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big]\\
&+a_{24}\big[\alpha_1 z+\overline{\alpha_1 z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big]\\
&\times\big(\alpha_2 z+\overline{\alpha_2 z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)\\
&+b_{25}\big(\alpha_2 z+\overline{\alpha_2 z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)\\
&\times\big(\alpha_2 ze^{-i\omega_0}+\overline{\alpha}_2\overline{z}e^{i\omega_0}+W_{20}^{(3)}(-1)\frac{z^2}{2}+W_{11}^{(3)}(-1)z\overline{z}\\
&+W_{02}^{(3)}(-1)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)\Big]\Bigg\}\\
&+\tau_0\bar{B}\Bigg\{\overline{\alpha}_2^*\Big[a_{34}\Big(\alpha_1 z+\overline{\alpha_1 z}+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\overline{z}+W_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\Big)\\
&\times\big(\alpha_2 z+\overline{\alpha_2 z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)\\
&+a_{35}\big(\alpha_2 ze^{-i\omega_0}+\overline{\alpha}_2\overline{z}e^{i\omega_0}+W_{20}^{(3)}(-1)\frac{z^2}{2}+W_{11}^{(3)}(-1)z\overline{z}\\
&+W_{02}^{(3)}(-1)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)^2\\
&+b_{34}\big(\alpha_2 z+\overline{\alpha_2 z}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+O(|z,\overline{z}|^3)\big)\\
&\times\Big(\alpha_2 ze^{-i\omega_0}+\overline{\alpha}_2\overline{z}e^{i\omega_0}+W_{20}^{(3)}(-1)\frac{z^2}{2}+W_{11}^{(3)}(-1)z\overline{z}+W_{02}^{(3)}(-1)\frac{\overline{z}^2}{2}+\cdots\Big)\Big]\Bigg\}.
\end{aligned}
$$

Comparing the coefficients of above equation with (3.15), we obtain the following relevant parameters.

$$
\begin{aligned}
g_{20} &= 2\tau_0 \bar{B}\Big[a_{13} + b_{12}e^{-i\omega_0\tau_0} + a_{14}\alpha_1 + \overline{\alpha_1^*}\big(b_{25}\alpha_1\alpha_2 e^{-i\omega_0\tau_0} \\
&\quad + b_{24}\alpha_1 e^{-i\omega_0\tau_0} + a_{24}\alpha_1\alpha_2 + a_{23}\alpha_1\big) + \overline{\alpha_2^*}\big(b_{34}\alpha_2^2 e^{-i\omega_0\tau_0} \\
&\quad + a_{34}\alpha_1\alpha_2 + a_{35}\alpha_2^2\big)\Big], \\
g_{11} &= 2\tau_0 \bar{B}\Bigg[a_{13} + b_{12}\mathrm{Re}\{e^{-i\omega_0\tau_0}\} + a_{14}\mathrm{Re}\{\alpha_1\} \\
&\quad + \overline{\alpha_1^*}\Big(b_{25}\mathrm{Re}\{\alpha_1\alpha_2 e^{-i\omega_0\tau_0}\} + b_{24}\mathrm{Re}\{\alpha_1 e^{-i\omega_0\tau_0}\} \\
&\quad + a_{24}\mathrm{Re}\{\alpha_1\alpha_2\} + a_{23}\mathrm{Re}\{\alpha_1\}\Big) + \overline{\alpha_2^*}\Big(b_{34}\mathrm{Re}\{\alpha_1\alpha_2 e^{-i\omega_0\tau_0}\} \\
&\quad + a_{34}\mathrm{Re}\{\alpha_1\alpha_2\} + a_{35}|\alpha_2|^2\Big)\Bigg], \\
g_{02} &= 2\tau_0 \bar{B}\Big[a_{13} + b_{12}e^{i\omega_0\tau_0} + a_{14}\overline{\alpha_1} + \overline{\alpha_1^*}\big(b_{25}\overline{\alpha_1\alpha_2}e^{i\omega_0\tau_0} \\
&\quad + b_{24}\overline{\alpha_1}e^{i\omega_0\tau_0} + a_{23}\overline{\alpha_1} + a_{24}\overline{\alpha_1\alpha_2}\big) + \overline{\alpha_2^*}\big(b_{34}\overline{\alpha_2}^2 e^{i\omega_0\tau_0} \\
&\quad + a_{34}\overline{\alpha_1\alpha_2} + a_{35}\overline{\alpha_2}^2\big)\Big], \\
g_{21} &= 2\tau_0 \bar{B}\Bigg\{a_{13}\Big(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\Big) + b_{12}\Big(\frac{1}{2}W_{20}^{(1)}(-1) \\
&\quad + W_{11}^{(1)}(-1)e^{-i\omega_0\tau_0} + W_{11}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(-1)e^{i\omega_0\tau_0}\Big) \\
&\quad + a_{14}\Big(\frac{1}{2}W_{20}^{(2)}(0) + \alpha_1 W_{11}^{(1)}(0) + W_{11}^{(2)}(0) + \frac{1}{2}\overline{\alpha_1}W_{20}^{(1)}(0)\Big) \\
&\quad + \overline{\alpha_1^*}\Big[a_{23}\Big(\frac{1}{2}W_{20}^{(2)}(0) + \alpha_1 W_{11}^{(1)}(0) + W_{11}^{(2)}(0) + \frac{1}{2}\overline{\alpha_1}W_{20}^{(1)}(0)\Big) \\
&\quad + b_{24}\Big(\frac{1}{2}W_{20}^{(2)}(-1)e^{i\omega_0\tau_0} + W_{11}^{(2)}(-1)e^{-i\omega_0\tau_0} + \alpha_1 W_{11}^{(1)}(-1) \\
&\quad + \frac{1}{2}\overline{\alpha_1}W_{20}^{(1)}(-1)\Big) + a_{24}\Big(\frac{1}{2}\overline{\alpha_1}W_{20}^{(3)}(0) + \alpha_2 W_{11}^{(2)}(0) + \alpha_1 W_{11}^{(3)}(0) \\
&\quad + \frac{1}{2}\overline{\alpha_2}W_{20}^{(2)}(0)\Big) + b_{25}\Big(\frac{1}{2}\overline{\alpha_1}W_{20}^{(3)}(-1) + \alpha_2 W_{11}^{(2)}(-1)e^{-i\omega_0\tau_0} \\
&\quad + \alpha_1 W_{11}^{(3)}(-1) + \overline{\alpha_2}W_{20}^{(2)}(-1)e^{i\omega_0\tau_0}\Big)\Big], + \overline{\alpha_2^*}\Big[a_{34}\Big(\frac{1}{2}\overline{\alpha_1}W_{20}^{(3)}(0) \\
&\quad + \alpha_2 W_{11}^{(2)}(0) + \alpha_1 W_{11}^{(3)}(0) + \frac{1}{2}\overline{\alpha_2}W_{20}^{(2)}(0)\Big) + a_{35}\Big(\overline{\alpha_2}W_{20}^{(3)}(0) \\
&\quad + 2\alpha_2 W_{11}^{(3)}(0)\Big) + b_{34}\Big(\frac{1}{2}\overline{\alpha_2}W_{20}^{(3)}(-1) + \alpha_2 W_{11}^{(3)}(-1)e^{-i\omega_0\tau_0} \\
&\quad + \alpha_2 W_{11}^{(3)}(-1) + \frac{1}{2}\overline{\alpha_2}W_{11}^{(3)}(-1)e^{i\omega_0\tau_0}\Big)\Big]\Bigg\}.
\end{aligned} \tag{3.19}
$$

In order to obtain the expression of g_{21} we need to calculate $W_{20}(\theta) = \big(W_{20}^{(1)}, W_{20}^{(2)}, W_{20}^{(3)}\big)$ and $W_{11}(\theta) = \big(W_{11}^{(1)}, W_{11}^{(2)}, W_{11}^{(3)}\big)$. From Eqs.(3.7) and (3.11), we have

$$\begin{aligned}
\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q},\\
&= \mathcal{A}(\mu)u_t + \mathcal{R}u_t - \big[\mathrm{i}\omega_0\tau_0 z + \overline{q}^*(0)\mathcal{F}_0(z,\overline{z})\big]q - \big[-i\omega_0\tau_0\overline{z} + q^*(0)\overline{\mathcal{F}_0}(z,\overline{z})\big]\overline{q},\\
&= \mathcal{A}W + 2\mathcal{A}\,Re[zq] + \mathcal{R}u_t - 2Re[\overline{q}^*(0)\mathcal{F}_0(z,\overline{z})q(\theta)] - 2Re[\mathrm{i}\omega_0\tau_0 z q(\theta)],\\
&= \mathcal{A}W - 2Re[\overline{q}^*(0)\mathcal{F}_0(z,\overline{z})q(\theta)] + \mathcal{R}u_t,\\
&= \begin{cases} \mathcal{A}W - 2Re[\overline{q}^*(0)\mathcal{F}_0(z,\overline{z})q(\theta)], & -1 \le \theta < 0\\ \mathcal{A}W - 2Re[\overline{q}^*(0)\mathcal{F}_0(z,\overline{z})q(\theta)] + \mathcal{F}_t, & \theta = 0. \end{cases}
\end{aligned} \tag{3.20}$$

Let rewrite Eq. (3.20) as

$$\dot{W} = \mathcal{A}W + H(z,\bar{z},\theta), \tag{3.21}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + H_{21}(\theta)\frac{z^2\overline{z}}{2} + \cdots. \tag{3.22}$$

Differentiating of Eq. (3.12) w.r.t. t, we get the following expression on the center manifold C_0 near to the origin

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{3.23}$$

Using Eqs. (3.23), (3.21) and (3.22), we obtain

$$(\mathcal{A} - 2\mathrm{i}\omega_0\tau_0)W_{20}(\theta) = -H_{20}(\theta), \tag{3.24}$$

$$\mathcal{A}W_{11}(\theta) = -H_{11}(\theta). \tag{3.25}$$

By Eqs. (3.20) and (3.21), we see that for $\theta \in [-1,0)$,

$$\begin{aligned}
H(z,\bar{z},\theta) &= -2Re(\overline{q}^*(0)\mathcal{F}_0 q(\theta))\\
&= -\mathrm{g}(z,\bar{z})q(\theta) - \bar{\mathrm{g}}(z,\bar{z})\bar{q}(\theta).
\end{aligned} \tag{3.26}$$

Comparing the coefficients of Eq.(3.26) with Eq.(3.22) gives that

$$H_{20}(\theta) = -\mathrm{g}(20)q(\theta) - \bar{\mathrm{g}}_{02}\bar{q}(\theta), \tag{3.27}$$

and

$$H_{11}(\theta) = -\mathrm{g}(11)q(\theta) - \bar{\mathrm{g}}_{11}\bar{q}(\theta). \tag{3.28}$$

From Eqs.(3.24) - (3.28) and the definition of $\mathcal{A}$, we obtain

$$\begin{cases} \dot{W}_{20}(\theta) = 2\mathrm{i}\omega_0\tau_0 W_{20}(\theta) + \mathrm{g}_{20}q(\theta) + \bar{\mathrm{g}}_{02}\bar{q}(\theta),\\ \dot{W}_{11}(\theta) = \mathrm{g}_{11}q(\theta) + \bar{\mathrm{g}}_{11}\bar{q}(\theta). \end{cases} \tag{3.29}$$

where

$$
(3.30)\qquad \begin{cases} W_{20}(\theta) = \frac{\mathrm{i}g_{20}}{\omega_0\tau_0} q(0) e^{\mathrm{i}\theta\omega_0\tau_0} + \frac{\mathrm{i}\bar{g}_{20}}{3\omega_0\tau_0} \bar{q}(0) e^{-\mathrm{i}\theta\omega_0\tau_0} + E_1 e^{2\mathrm{i}\theta\omega_0\tau_0}, \\ W_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\omega_0\tau_0} q(0) e^{\mathrm{i}\theta\omega_0\tau_0} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0\tau_0} \bar{q}(0) e^{-\mathrm{i}\theta\omega_0\tau_0} + E_2. \end{cases}
$$

Here, $E_1 = \left(E_1^{(1)}, E_1^{(2)}, E_1^{(3)}\right)$ and $E_2 = \left(E_2^{(1)}, E_2^{(2)}, E_2^{(3)}\right)$ are constant vectors in $\mathbb{R}^3$. These vectors satisfy the following equations:

$$
(3.31)\qquad \begin{pmatrix} 2\mathrm{i}\omega_0 - (a_{11} + b_{11} e^{-2\mathrm{i}\omega_0\tau}) & -a_{12} & 0 \\ -b_{21} e^{-2\mathrm{i}\omega_0\tau} & 2\mathrm{i}\omega_0 - a_{22} & -b_{23} e^{-2\mathrm{i}\omega_0\tau} \\ 0 & -a_{32} & 2\mathrm{i}\omega_0 - (a_{33} + b_{33} e^{-2\mathrm{i}\omega_0\tau}) \end{pmatrix} \times \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} M_{11} \\ M_{21} \\ M_{31} \end{pmatrix},
$$

where

$$
\begin{aligned}
M_{11} &= a_{13} + b_{12} e^{-\mathrm{i}\omega_0\tau_0} + a_{14}\alpha_1, \\
M_{21} &= b_{25}\alpha_1\alpha_2 e^{-\mathrm{i}\omega_0\tau_0} + b_{24}\alpha_1 e^{-\mathrm{i}\omega_0\tau_0} + a_{24}\alpha_1\alpha_2 + a_{23}\alpha_1, \\
M_{31} &= b_{34}\alpha_2^2 e^{-\mathrm{i}\omega_0\tau_0} + a_{34}\alpha_1\alpha_2 + a_{35}\alpha_2^2,
\end{aligned}
$$

and

$$
(3.32)\qquad \begin{pmatrix} -a_{11} - b_{11} & -a_{12} & 0 \\ -b_{21} & -a_{22} & -b_{23} \\ 0 & -a_{32} & -a_{33} - b_{33} \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = 2 \begin{pmatrix} N_{11} \\ N_{21} \\ N_{31} \end{pmatrix},
$$

where

$$
\begin{aligned}
N_{11} &= a_{13} + b_{12}\mathrm{Re}\{e^{-\mathrm{i}\omega_0\tau_0}\} + a_{14}\mathrm{Re}\{\alpha_1\}, \\
N_{21} &= b_{25}\mathrm{Re}\{\alpha_1\alpha_2 e^{-\mathrm{i}\omega_0\tau_0}\} + b_{24}\mathrm{Re}\{\alpha_1 e^{-\mathrm{i}\omega_0\tau_0}\} + a_{24}\mathrm{Re}\{\alpha_1\alpha_2\} + a_{23}\mathrm{Re}\{\alpha_1\}, \\
N_{31} &= b_{34}\mathrm{Re}\{\alpha_1\alpha_2 e^{-\mathrm{i}\omega_0\tau_0}\} + a_{34}\mathrm{Re}\{\alpha_1\alpha_2\} + a_{35}|\alpha_2|^2.
\end{aligned}
$$

By the Cramer's rule, we can get the solution of (3.31) as

$$
E_1^{(1)} = \frac{2}{\bar{\Delta}} \begin{vmatrix} M_{11} & -a_{12} & 0 \\ M_{21} & 2\mathrm{i}\omega_0 - a_{22} & -b_{23} e^{-2\mathrm{i}\omega_0\tau} \\ M_{31} & -a_{32} & 2\mathrm{i}\omega_0 - (a_{33} + b_{33} e^{-2\mathrm{i}\omega_0\tau}) \end{vmatrix},
$$

$$
E_1^{(2)} = \frac{2}{\bar{\Delta}} \begin{vmatrix} 2\mathrm{i}\omega_0 - (a_{11} + b_{11} e^{-2\mathrm{i}\omega_0\tau}) & M_{11} & 0 \\ -b_{21} e^{-2\mathrm{i}\omega_0\tau} & M_{21} & -b_{23} e^{-2\mathrm{i}\omega_0\tau} \\ 0 & M_{31} & 2\mathrm{i}\omega_0 - (a_{33} + b_{33} e^{-2\mathrm{i}\omega_0\tau}) \end{vmatrix},
$$

$$
E_1^{(3)} = \frac{2}{\bar{\Delta}} \begin{vmatrix} 2\mathrm{i}\omega_0 - (a_{11} + b_{11} e^{-2\mathrm{i}\omega_0\tau}) & -a_{12} & M_{11} \\ -b_{21} e^{-2\mathrm{i}\omega_0\tau} & 2\mathrm{i}\omega_0 - a_{22} & M_{21} \\ 0 & -a_{32} & M_{31} \end{vmatrix},
$$

where

$$\bar{\Delta} = \begin{vmatrix} 2\mathrm{i}\omega_0 - (a_{11} + b_{11}e^{-2\mathrm{i}\omega_0\tau}) & -a_{12} & 0 \\ -b_{21}e^{-2\mathrm{i}\omega_0\tau} & 2\mathrm{i}\omega_0 - a_{22} & -b_{23}e^{-2\mathrm{i}\omega_0\tau} \\ 0 & -a_{32} & 2\mathrm{i}\omega_0 - (a_{33} + b_{33}e^{-2\mathrm{i}\omega_0\tau}) \end{vmatrix}.$$

Similarly, solution of (3.32) is given by

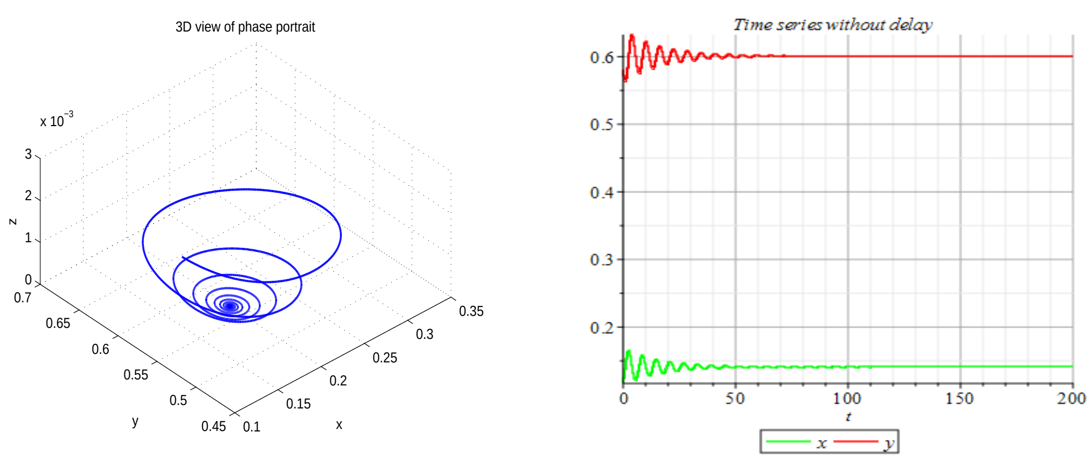
$$E_2^{(1)} = \frac{2}{\tilde{\Delta}} \begin{vmatrix} N_{11} & -a_{12} & 0 \\ N_{21} & -a_{22} & -b_{23} \\ N_{31} & -a_{32} & -a_{33} - b_{33} \end{vmatrix}, \quad E_2^{(2)} = \frac{2}{\tilde{\Delta}} \begin{vmatrix} -a_{11} - b_{11} & N_{11} & 0 \\ -b_{21} & N_{21} & -b_{23} \\ 0 & N_{31} & -a_{33} - b_{33} \end{vmatrix},$$

$$E_2^{(3)} = \frac{2}{\tilde{\Delta}} \begin{vmatrix} -a_{11} - b_{11} & -a_{12} & N_{11} \\ -b_{21} & -a_{22} & N_{21} \\ 0 & -a_{32} & N_{31} \end{vmatrix},$$

where

$$\tilde{\Delta} = \begin{vmatrix} -a_{11} - b_{11} & -a_{12} & 0 \\ -b_{21} & -a_{22} & -b_{23} \\ 0 & -a_{32} & -a_{33} - b_{33} \end{vmatrix}.$$

Therefore, all g_{ij} can be expressed in terms of parameters. By using Eq. (3.19), we



(A) (B)

FIGURE 1. System (1.2) is locally asymptotically stable in the absence of delay. (A) Three-dimensional view of phase portrait. (B) Time series solution. The initial condition is $(0.12, 0.54, 0.0024)$.

can compute the following values:

$$
\begin{aligned}
C_1(0) &= \frac{\mathrm{i}}{2\omega_0\tau_0}\Big(g_{11}g_{20} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\Big) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{Re(C_1(0))}{Re(\lambda'(\tau_0))},\\
\beta_2 &= 2Re(C_1(0)),\\
T_2 &= -\frac{1}{\omega_0\tau_0}\Big[Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}\Big].
\end{aligned} \tag{3.33}
$$

These expressions determine the quantities and qualitative behavior of bifurcating periodic solutions on the center manifold of system (1.2) for the critical value $\tau = \tau_0$. Therefore, we can conclude the following main result.

Theorem 3.1. *For system (1.2), the following conclusions hold:*

- **(i):** *If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical), i.e., the bifurcating periodic solutions exist for $\tau > \tau_0$ ($\tau < \tau_0$).*
- **(ii):** *The bifurcated periodic solutions are orbitally stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$.*
- **(iii):** *The period is increasing if $T_2 > 0$ and decreasing if $T_2 < 0$.*

4. NUMERICAL SIMULATIONS

In this section, some numerical simulation results are presented to verify our analytical analysis obtained in the previous section by using the calculation tools MATLAB 2013a and Maple 2017. For the system (1.2), we choose the parameter values as provided in Table 2, i.e. the system (1.2) has only one positive equilibrium $E^* = (0.14105, 0.60087, 0.00385)$, which is feasible for these set of parameters. By

Table 2. The set of parameter values for simulating the dynamics of the system (1.2).

Parameter	a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3
Value	3.12	0.82	5	0.35	2.48	0.05	0.25	0.32	15

Eq. (2.14) or direct computation of Eq. (2.15), we have obtained the critical value of τ i.e. $\tau_0 = 0.10544$. Thus based on Lemma 2.2 and Theorem 2.5, the interior equilibrium point E^* becomes asymptotically stable for $\tau = 0$ and $\tau = 0.065 < \tau_0$, that is, the population of prey-A, prey-B and the predator population have stable dynamical behavior (see Figs.1 and 2). Moreover, when the time delay τ crosses the critical value $\tau_0 = 0.10544$, E^* losses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from E^*. Therefore, E^* is destabilized when

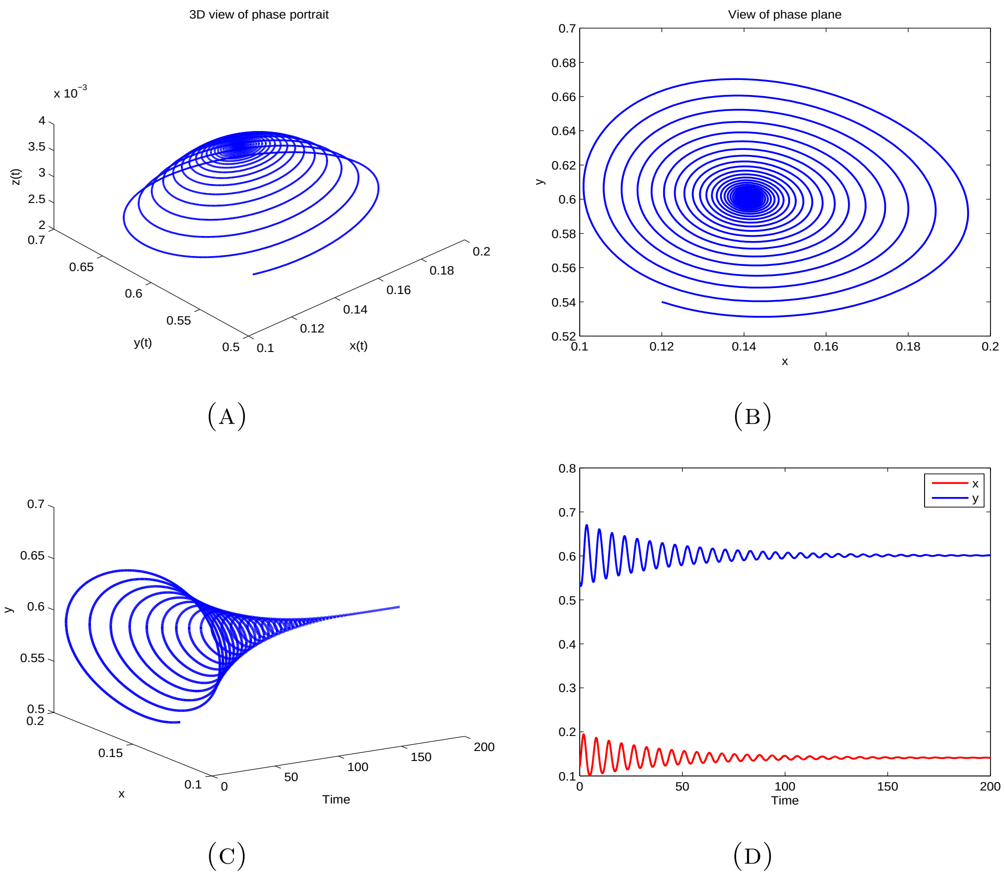


FIGURE 2. E^* is locally asymptotically stable when $\tau = 0.065 < \tau_0 = 0.10544$. (A, B) phase-portraits in xyz-space and in xy plane, (C) phase-portrait in txy-space, (D) time series evolution of species. The initial value is $(0.12, 0.54, 0.0024)$.

$\tau = 0.11 > \tau_0$ and a stable limit cycle appeared beyond of τ_0. Simulation results are depicted in Figure 3, indicate a stable periodic solution which emerges around the coexisting equilibrium point. According to Lemma 2.4, the transversality condition is verified, i.e.

$$\left[\frac{\mathrm{d}\big(Re\lambda(\tau)\big)}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_0} \approx 0.0134942 > 0, \qquad \left[\frac{\mathrm{d}\big(Re\lambda(\tau)\big)}{\mathrm{d}\tau}\right]_{\tau=\tau_0} \approx 74.1055468 > 0,$$

and by the algorithm derived in previous section, we get

$$\begin{aligned} \mathrm{g}_{20} &\approx -0.63006 + 0.5348\mathrm{i}, & \mathrm{g}_{11} &\approx 0.0076 + 0.5261\mathrm{i}, \\ \mathrm{g}_{02} &\approx 0.6453 + 0.5174\mathrm{i}, & \mathrm{g}_{21} &\approx -87437.69876 + 1492.66643\mathrm{i}, \\ \lambda'(\tau_0) &\approx 57.96749 - 30.58566\mathrm{i}, & \mathcal{C}_1(0) &\approx -43718.74485 + 745.99226\mathrm{i}, \\ \mu_2 &\approx 754.19420 > 0, & \beta_2 &\approx -87437.48970 < 0, \\ T_2 &\approx -15206.23002 < 0. \end{aligned}$$

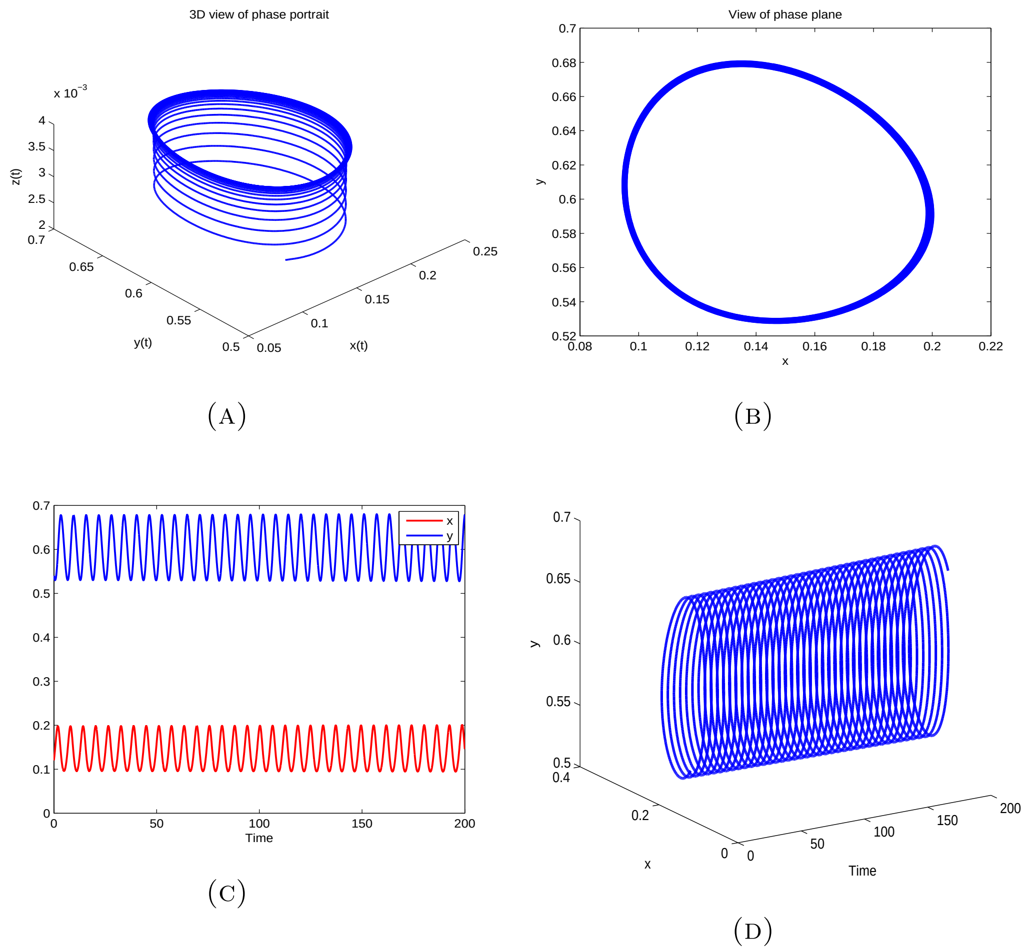


FIGURE 3. Simulating solutions of system (1.2): E^* is unstable when $\tau = 0.11$ which indicates that an stable periodic solution bifurcates from E^*. (A, B) existence of periodic solutions, (C) time series evolution of species, (D) phase-portrait in txy-space. The initial value is $(0.12, 0.54, 0.0024)$.

According to Theorem 3.1, we see that the Hopf bifurcation is supercritical. In addition, the numerical results imply that the bifurcated periodic solutions are orbitally asymptotically stable and its period is decreasing, as $\beta_2 < 0$ and $T_2 < 0$.

5. CONCLUSION

In this article we have analyzed the dynamical behaviors of a three species predator–prey model with time delay. It is shown that the time delay has great impact on the dynamics of the system and makes the system more realistic. Usually the dynamical system involving time delay can exhibit rich biological and dynamical properties. It is observed that when the magnitude of delay increases, the system dynamically becomes richer and leads to produce periodic solutions. In fact, this work extensively study the effect of time delay around the interior equilibrium point of a predator-prey system consisting of three species which represent the population densities of two prey and one predator species.

By analyzing the linearized system and related characteristic equation, some necessary and sufficient conditions for asymptotic stability of the interior equilibrium point and occurrence of Hopf bifurcation are proved. Based on the values of the selected fixed parameters, we theoretically investigate the value of the time delay and it has been shown that the existence of a time delay in a certain interval causes to change the system stability and exhibits limit cycle oscillations. This implies that the considered delayed system converges to a stable state when the delay parameter is less than its critical value and a Hopf bifurcation occurs when the parameter τ passes the critical value τ_0. Furthermore, an explicit algorithm with sufficient criteria for the stability, direction and other characterization of a Hopf bifurcation are derived by means of normal form method and center manifold argument. Our theoretical work as well as numerical simulations indicate that time delay plays a major role in destabilization of stable equilibrium point and has a great impact on the nature of the system dynamics.

REFERENCES

[1] Q. Din, Stability, bifurcation analysis and chaos control for a predator-prey system, *Journal of Vibration and Control*, 1–15, 2018.

[2] I. K. Aybar, O. O. Aybar, M. Dukaric and B. Fercec, Dynamical analysis of a two prey-one predator system with quadratic self interaction, *Applied Mathematics and Computation*, 33: 118–132, 2018.

[3] P. Andayani and W. M. Kusumawinahyu, Global stability analysis on a predator-prey model with omnivores, *Applied Mathematical Sciences*, 9 (36):1771 - 1782, 2015.

[4] X-Y. Meng, N-N. Qin and H-F. Huo, *Dynamics analysis of a predator–prey system with harvesting prey and disease in prey species*, JOURNAL OF BIOLOGICAL DYNAMICS, 12 (1): 342–374, 2018.

[5] P. Panday, S. Samanta, N. Pal and J. Chattopadhyay, *Delay induced multiple stability switch and chaos in a predator–prey model with fear effect*, Mathematics and Computers in Simulation, 172 (6): 134–158, 2020.

[6] C.-S. Chou and A. Friedman, *Introduction to Mathematical Biology, Modeling*, Analysis, and Simulations, Springer International Publishing Switzerland, 2016.

[7] D. S. Jones, M. J. Plank and B.D. Sleeman, *Differential Equations and Mathematical Biology*, Taylor & Francis Group, LLC, 2009.

[8] H. Hu and L. Huang, *Stability and Hopf bifurcation in a delayed predator-prey system with stage structure for prey*, Nonlinear Analysis: Real World Applications, 11: 2757-2769, 2010.

[9] C. Celik, *The stability and Hopf bifurcation for a predator–prey system with time delay*, Chaos, Solitons and Fractals, 37: 87–99, 2008.

[10] V. Yaman and B. Yilmaz, *Solutions of nonlinear delay differential equations by Daftardar-Jafari Method*, Turk. J. Math. Comput. Sci., 10: 95–106, 2018.

[11] J. Sieber, *Local bifurcations in differential equations with state-dependent delay*, Chaos 27: 1-27, 2017.

[12] P. Lancaster and M. Tismenetsky, *The Theory of Matrices with Applications*, Academic Press, London, 1985.

[13] C. Jin, K. Gu, S-I. Niculescu and I. Boussaada, *Stability analysis of systems with delay-dependent coefficients: An overview*, IEEE Access: Special Section on Analysis and Synthesis of Time-Delay Systems , 6: 27392–27407, 2018.

[14] X. Chen and X. Wang, *Qualitative analysis and control for predator-prey delays system*, Chaos, Solitons and Fractals, 123: 361-372, 2019.

[15] M. A. Abd-Rabo, M. Zakarya, C. Cesarano and S. Aly, *Bifurcation analysis of time-delay model of consumer with the advertising effect*, Symmetry, 13 (417): 1-21, 2021.

[16] D. P. Hu, Y. Y. Li, M. Liu and Y. Z. Bai, *Stability and Hopf bifurcation for a delayed predator-prey model with stage structure for prey and Ivlev type functional response*, Nonlinear Dynam., 99: 3323–3350, 2020.

[17] F. A. Rihan and C. Rajivganthi, *Dynamics of fractional-order delay differential model of prey-predator system with Holling-type III and infection among predators*, Chaos Soliton. Fract., 141: 110365, 2020.

[18] D. Mukherjee and C. Maji, *Bifurcation analysis of a Holling type II predator-prey model with refuge*, Chinese J. Phys., 65: 153–162, 2020.

[19] D. Olvera, A. Elías-Zúñiga, L. N. López de Lacalle and C.A. Rodríguez, *Approximate Solutions of Delay Differential Equations with Constant and Variable Coefficients by the Enhanced Multistage Homotopy Perturbation Method*. Hindawi Publishing Corporation: Abstract and Applied Analysis, 1–12, 2015.

[20] C. Sun, M. Han, Y. Lin and Y. Chen, *Global qualitative analysis for a predator–prey system with delay*, Chaos, Solitons and Fractals, 32: 1582–1596, 2007.

[21] C. Huang, J. Cao, M. Xiao, A. Alsaedi and F. E. Alsaadi, *Controlling bifurcation in a delayed fractional predator–prey system with incommensurate orders*, Applied Mathematics and Computation, 293: 293-310, 2017.

[22] P. Wahi and A. Chatterjee, *Asymptotics for the characteristic roots of delayed dynamic systems*, Journal of Applied Mechanics, 72: 475–483, 2005.

[23] T. Li, Y. Wang and X. Zhou, *Bifurcation analysis of a first time-delay chaotic system*, Advances in Difference Equations, 78: 1-18, 2019

[24] R. Rusinek, A. Weremczuk, K. Kecik and J. Warminski, *Dynamics of a time delayed Duffing oscillator, International Journal of Non-Linear Mechanics*, 65:

98–106, 2014.

[25] X. Li, J.C. Ji and C. H. Hansen, *Dynamics of two delay coupled van der Pol oscillators*, Mechanics Research Communications, 33: 614–627, 2006.

[26] S.G. Ruan and J.J. Wei, *On the zero of some transcendential functions with applications to stability of delay differential equations with two delays*, Dyn. Cont. Discrete Impulsive Syst. Ser. A, 10 (6): 863–874, 2003.

[27] B. Hassard, D. Kazarino and Y. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.

[28] J.K. Hale, *Theory of functional differential equationsTheory of functional differential equations*, Springer, New York, 1977.