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PROPORTIONAL CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Practical stability properties of generalized proportional Caputo fractional delay differential equations are presented and exponential practical stability is introduced and studied. This type of stability on one hand is deeply connected with the used generalized proportional derivative and on the other hand it is a generalization of classical practical stability. Three types of sufficient conditions are obtained. Since the considered fractional derivative is a generalization of the Caputo fractional derivative, the connection with sufficient conditions in the literature concerning Caputo fractional differential equations is discussed. Some examples are given illustrate our theory.

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1. INTRODUCTION

Recently fractional differential equations were studied extensively due to their applications in modeling in various fields of engineering and science (see, for example, the monographs [1, 2, 3], and the cited therein references). In the literature there are various types of fractional derivatives with different properties. The main common property of fractional derivatives is connected with the memory which differs from integer order derivatives. Recently ([4, 5]) generalized proportional integrals and derivatives were introduced and applied to differential equations (see, for examples, [6, 7, 8]). These integrals and derivatives generalize the classical Riemann-Liouville and Caputo fractional integrals and derivatives.

In the qualitative study for nonlinear systems stability properties are important. One of the approaches in studying stability is the application of Lyapunov functions.

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The presence of the fractional derivative in differential equations leads one to consider an appropriate application of the derivative of Lyapunov functions (see, for example, [9, 10, 11, 12]). LaSalle and Lefschetz [13] introduced the so called practical stability which do not provide stability of the equilibrium point but it is connected with its boundedness. This type of stability is studied for various types of differential equations in the literature (see, for example, [14, 15]).

The main goal of this paper is the study some stability properties of generalized proportional Caputo fractional differential equations. We define the exponential practical stability which has a deep connection with the considered fractional derivative. Also, this type of stability is a generalization of the known and studied practical stability in the literature. To the best of our knowledge, this is the first paper presenting practical stability properties of generalized proportional Caputo fractional differential equations. Three types of sufficient conditions are obtained based on the application of Lyapunov functions, their generalized proportional fractional derivatives, and comparison results. The connection of these results with the results in the literature for Caputo fractional differential equations is discussed. Some examples are provided to illustrate the application of the obtained sufficient conditions.

2. NOTES ON FRACTIONAL CALCULUS

In many applications in science and engineering, the fractional order q is often less than 1, so we restrict $q \in (0,1)$ everywhere in the paper.

Let the function $u : [a,b] \to \mathbb{R}$ with $a, b \in \mathbb{R},\ b \leq \infty$ (if $b = \infty$ then the interval is half open). Then the generalized proportional fractional integral is defined by (as long as all intebgrals are well defined, see [4])

$$({}_a\mathcal{I}^{q,\rho}u)(t) = \frac{1}{\rho^q\Gamma(q)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}\,(t-s)^{q-1}\,u(s)\;ds,\ \text{for } t \in (a,b],\quad q \geq 0,\ \rho \in (0,1],$$

and the generalized proportional Caputo fractional derivative is defined by (as long as all integrals are well defined, see [4])

$$({}_a^C\mathcal{D}^{q,\rho}u)(t) = \frac{1}{\rho^{1-q}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}\,(t-s)^{-q}\,(\mathcal{D}^{1,\rho}u)(s)\;ds, \tag{2.1}$$
$$\text{for}\ \ t \in (a,b],\quad q \in (0,1),\ \rho \in (0,1],$$

where $(\mathcal{D}^{1,\rho}u)(t) = (1-\rho)u(t) + \rho u'(t)$.

Remark 2.1. If $\rho = 1$, then the generalized Caputo proportional fractional derivative (2.1) is reduced to the classical Caputo fractional derivative of order $q \in (0,1)$: ${}_a^C\mathcal{D}^q u(t)$.

Remark 2.2. The generalized proportional Caputo fractional derivative of a constant is not zero for $\rho \in (0,1)$ and $q \in (0,1)$ (compare with the Caputo fractional derivative).

Remark 2.3. The relation

$$({}_{a}^{C}\mathcal{D}^{q,\rho}e^{\frac{\rho-1}{\rho}(.)})(t)=0 \quad \text{for} \quad t>a$$

is known from [4, Remark 3.2].

Lemma 2.4. *(Proposition 5.2[4]) For $\rho\in(0,1]$ and $q\in(0,1)$ we have*

$$({}_{a}^{C}\mathcal{D}^{q,\rho}(e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1})(t)=\frac{\rho^{q}\Gamma(\beta)}{\Gamma(\beta-q)}e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1-q}, \quad \beta>0.$$

We will use the explicit form of the solution of the initial value problem for the scalar linear generalized proportional Caputo fractional differential equation which is given in Example 5.7 [4] and which is (with necessary slight corrections) given in the following result.

Lemma 2.5. *The solution of the scalar linear generalized proportional Caputo fractional initial value problem*

$$({}_{a}^{C}\mathcal{D}^{q,\rho}u)(t)=\lambda u(t), \quad u(a)=u_0, \quad q\in(0,1), \ \rho\in(0,1]$$

has a solution

$$u(t)=u_0e^{\frac{\rho-1}{\rho}(t-a)}E_q(\lambda(\frac{t-a}{\rho})^{q}),$$

where $E_q(t)$ is the Mittag-Leffler function of one parameter.

Lemma 2.6. *([6]) Let the function $u\in C^1([a,b],\mathbb{R})$ with $a,b\in\mathbb{R}$, $b\leq\infty$ (if $b=\infty$ then the interval is half open), and $q\in(0,1)$, $\rho\in(0,1]$ be two reals. Then,*

$$({}_{a}^{C}\mathcal{D}^{q,\rho}u^2)(t)\leq 2u(t)({}_{a}^{C}\mathcal{D}^{q,\rho}u)(t), \quad t\in(a,b].$$

We recall the following result for generalized proportional Caputo fractional derivatives of continuous functions.

Lemma 2.7. *(Lemma 5 [7]) Let $u\in C([t_0,T],\mathbb{R})$, $T>t_0$, and suppose that there exist $t^*\in(t_0,T]$ such that $u(t^*)=0$, and $u(t)<0$, for $t_0\leq t<t^*$. Then, if the generalized proportional Caputo fractional derivative of u exists at t^*, then the inequality $(\ {}_{t_0}^{c}\mathcal{D}^{q,\rho}u)t|_{t=t^*}>0$ holds.*

The generalized proportional Caputo fractional derivatives for scalar functions could be easily generalized to the vector case by taking generalized proportional Caputo fractional derivatives with the same fractional order for all components.

3. STATEMENT OF THE PROBLEM

Consider the initial value problem (IVP) for a nonlinear system of generalized proportional Caputo fractional differential equations (FrDE)

$$\begin{aligned} &({}^{C}_{t_0}\mathcal{D}^{q,\rho}_{t} y)(t) = f(t, y(t)), \quad \text{for } t > t_0, \\ &y(t_0) = y_0, \end{aligned} \tag{3.1}$$

where $y_0 \in \mathbb{R}^n$, $t_0 \geq 0$ is the initial time, $({}^{C}_{t_0}D^{q,\rho}_{t} y)(t)$ denotes the generalized proportional Caputo fractional derivative for the state y, $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\rho \in (0, 1]$ and $q \in (0, 1)$.

We introduce the following assumption:

A1. The function f belongs to $C([t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t, 0) = 0$ for $t \geq t_0$ and for any $y_0 \in \mathbb{R}^n$, the IVP for the FrDE (3.1) has a solution $y(t; t_0, y_0)$.

Remark 3.1. We note that the change of the initial time t_0 leads to a change of FrDE(3.1) and not only on the initial condition (which is different to the case of ordinary differential equations).

We will illustrate Remark 3.1 with an Example.

Example 3.2. Consider the scalar linear generalized proportional Caputo fractional differential equation

$$({}^{C}_{a}\mathcal{D}^{q,\rho} u)(t) = \lambda u(t), \quad q \in (0, 1), \ \rho \in (0, 1],$$

where $\lambda \in \mathbb{R}$ is a given constant.

According to Lemma 2.5 it has a solution

$$u(t) = u(a) e^{\frac{\rho-1}{\rho}(t-a)} E_q(\lambda(\frac{t-a}{\rho})^q).$$

It is obvious that the change of the lower limit of the proportional fractional derivative leads to a change of the solution.

Remark 3.3. According to Remark 3.1 and Example 3.2 the meaning of uniform stability properties in fractional differential equations is changing and we will not consider and study these types of stability properties (which is different in the case of integer order differential equations).

Now we will define practical stability for the nonlinear Caputo FrDDE following the ideas for practical stability for ordinary differential equations ([13]).

Definition 3.4. The zero solution of FrDE (3.1) with zero initial function is called

- *practically stable* w.r.t. (λ, A), if for any initial value $y_0 \in \mathbb{R}^n : \|y_0\| < \lambda$, the inequality $\|y(t; t_0, y_0)\| < A$ for $t \geq t_0$, holds, where the real numbers (λ, A) with $0 < \lambda < A$ are given;

- *exponentially practically stable* w.r.t. (λ, A), if for any initial value $y_0 \in \mathbb{R}^n$: $\|y_0\| < \lambda$, the inequality $\|y(t; t_0, y_0)\| < Ae^{\frac{\rho-1}{\rho}(t-t_0)}$ for $t \geq t_0$, holds, where the real numbers (λ, A) with $0 < \lambda < A$ are given.

Here, $y(t; t_0, y_0)$ is a solution of (3.1).

Remark 3.5. Note that, from exponential practical stability of the zero solution of (3.1), we have the practical stability but the opposite is not true.

Remark 3.6. Note that, from stability properties of the zero solution of (3.1), we have practical stability but the opposite is not true.

Example 3.7. According to Remark 2.3 the function $u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-t_0)},\ u_0 \in \mathbb{R}$ is the unique solution of the IVP for scalar FrDE (3.1) with $n = 1,\ f(t, x) \equiv 0, y_0 = u_0$. Obviously, it is stable, it is exponentially practical stable for any couple $(\lambda, \lambda),\ \lambda > 0$ and it is practically stable.

Example 3.8. From Eq. (2.1) we get

$$\begin{aligned}
({}_0^C\mathcal{D}^{q,\rho}1)(t) &= \frac{(1-\rho)}{\rho^{1-q}\Gamma(1-q)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-q}\ ds \\
&= \frac{(1-\rho)}{\rho^{1-q}\Gamma(1-q)} (\frac{\rho}{1-\rho})^{1-q} \Big(\Gamma(1-q) - \Gamma(1-q, \frac{1-\rho}{\rho} t)\Big) \\
&= \frac{(1-\rho)^q}{\Gamma(1-q)} \Big(\Gamma(1-q) - \Gamma(1-q, \frac{1-\rho}{\rho} t)\Big) \\
&= \frac{(1-\rho)^q}{\Gamma(1-q)} \gamma(1-q, \frac{1-\rho}{\rho} t)
\end{aligned}$$

where $\gamma(.,.)$ is the lower incomplete gamma function.

Therefore, the constant function $u(t) = K \in \mathbb{R}$ is a solution of the FrDE (3.1) with $n = 1$, $t_0 = 0$, $f(t, x) = \frac{(1-\rho)^q}{\Gamma(1-q)} \gamma(1-q, \frac{1-\rho}{\rho} t) x,\ x \in \mathbb{R}$, $y_0 = u(0) = K$. Obviously, it is stable, it is not exponentially practically stable, and it is practically stable.

Examples 3.7 and 3.8 prove the necessity of studying exponential practical stability independent of stability of generalized proportional Caputo fractional differential equations.

We will use comparison results for the IVP for the scalar generalized proportional Caputo fractional differential equation (SFrDE)

$$ {}_{t_0}^{c}\mathcal{D}^{q,\rho} u(t) = g\,(t, u)\,, \qquad \text{for } t > t_0, \qquad u(t_0) = u_0, \tag{3.2}$$

where $u_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$.

We denote the solution of the IVP for the scalar SFrDE (3.2) by $u(t; t_0, u_0)$. In the case of non-uniqueness of the solution we will assume the existence of a maximal one.

We introduce the assumption:

A2. The function $g \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $g(t, 0) \equiv 0$, and for any $u_0 \in \mathbb{R}$, the IVP for the SFrDE (3.2) has a solution $u(t; t_0, u_0)$.

Remark 3.9. Practical stability properties of the SIVP (3.2) is defined similar to Definition 3.4.

Remark 3.10. We will study the practical stability of (3.1) or (3.2) in the case when the right side part depends on the unknown function. In the case $f(t, y) \equiv F(t)$ or $g(t, u) \equiv G(t)$, then the equation could not have a zero solution.

In this paper we study the connection between the practical stability properties of the zero solution of the system of FrDE (3.1) and the practical stability of the zero solution of the SFrDE (3.2) by applying an appropriate Lyapunov functions and their generalized proportional Caputo fractional derivatives.

We introduce the class Λ of Lyapunov-like functions which will be used to investigate the practical stability of the system FrDE (3.1).

Definition 3.11. Let $I \subset \mathbb{R}_+$ and $\mathcal{D} \subset \mathbb{R}^n$. We say that the function $V : I \times \mathcal{D} \to \mathbb{R}_+$ belongs to the class $\Lambda(I, \mathcal{D})$ if V is continuous and locally Lipschitzian with respect to its second argument in $I \times \mathcal{D}$.

We will apply the following comparison results:

Lemma 3.12. *(Lemma 6*[7]*) Assume the following conditions are satisfied:*

1. *The function* $\tilde{x}(.) = y(.; t_0, y_0) \in \Delta,\ \Delta \subset \mathbb{R}^n,\ 0 \in \Delta,$ *is a solution of IVP for FrDE* (3.1) *defined for* $t \in [t_0, T]$, $T > t_0$, $y_0 \in \Delta$.
2. *The function* $g \in C([t_0, T] \times \mathbb{R}_+, \mathbb{R})$.
3. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and, for any point* $t \in [t_0, T]$ *the inequality*

$$({}_{t_0}^{c}\mathcal{D}^{q,\rho} V(., \tilde{x}(.))(t) \leq g(t, V(t, \tilde{x}(t))) \tag{3.3}$$

holds.

4. *The function* $u^*(t) = u(t; t_0, u_0)$ *is the maximal solution of IVP for SFrDE* (3.2) *on* $[t_0, T]$.

Then, the inequality

$$V(t_0, y_0) \leq u_0$$

implies

$$V(t, \tilde{x}(t)) \leq u^*(t), \qquad \textit{for } t \in [t_0, T].$$

Corollary 3.13. *Let Condition 1 of Lemma 3.12 be satisfied and the function* $V \in \Lambda([t_0, T], \Delta)$ *be such that, for any point* $t \in [t_0, T]$ *the inequality*

$$({}_{t_0}^{c}\mathcal{D}^{q,\rho} V(., \tilde{x}(.)))(t) \leq 0 \tag{3.4}$$

holds.

Then, for $t \in [t_0, T]$, the inequality

$$V(t, \tilde{x}(t)) \leq V(t_0, y_0) e^{\frac{\rho-1}{\rho}(t-t_0)}$$

holds.

Proof. The proof of Corollary 3.13 follows from Lemma 3.12 and the fact that the corresponding IVP for the SFrDE (3.2) with $g(t,u) \equiv 0,\ u_0 = V(t_0, y_0)$ according to Remark 2.3 has a unique solution $u(t) = V(t_0, y_0) e^{\frac{\rho-1}{\rho}(t-t_0)}$ for $t \in [t_0, T]$. □

Remark 3.14. The result of Lemma 3.12 and Corollary 3.13 are also true on the half line.

4. Practical stability results.

We will give sufficient conditions for exponential practical stability and practical stability of the zero solution of FrDE (3.1) applying the generalized proportional Caputo fractional derivative of the Lyapunov function.

Define the following sets:

$$\begin{aligned}
\mathcal{K} &= \{a \in C(\mathbb{R}_+, \mathbb{R}_+) : a \text{ is strictly increasing and } a(0) = 0\};\\
\mathcal{M} &= \{a \in \mathcal{K} : \text{ there exists a function } b \in \mathcal{K} : \ a(Ce^{\frac{\rho-1}{\rho}s}) \leq b(C) e^{\frac{\rho-1}{\rho}s}\\
&\qquad \text{for any constant } C > 0 \text{ and } s \geq 0\};\\
S_A &= \{x \in \mathbb{R}^n : \ \|x\| \leq A\}, \quad A > 0.
\end{aligned}$$

We will prove sufficient conditions for the exponential practical stability of FrDE (3.1) depending on the exponential practical stability of the SFrDE (3.2) .

Theorem 4.1. *Let the following conditions be satisfied:*

1. *The conditions (A1) and (A2) are satisfied for the given t_0.*
2. *There exists a function $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ with $V(t, 0) = 0$ such that*
   (i) *the inequalities*

$$\alpha_1(\|x\|) \leq V(t,x) \text{ for } t \geq t_0,\ x \in \mathbb{R}^n,$$
$$V(t,x) \leq \alpha_2(\|x\|) \text{ for } t \geq t_0,\ x \in S_\lambda,$$

   hold, where $a_1 \in \mathcal{M}$ (with a function $b \in \mathcal{K}$), $\alpha_2 \in \mathcal{K},\ i = 1, 2,$ and $\lambda > 0$ is a given number;

   (ii) *there exists a constant $\alpha > b(a_2(\lambda)) > 0$ such that if the solution $y(t)$ of (3.1) satisfies $y(t) \in S_\alpha$ on an interval $[t_0, \tau]$ then the inequality*

$$({}_{t_0}^{C}\mathcal{D}^{q,\rho} V(., y(.)))(t) \leq g(t, V(t, y(t))), \quad t \in (t_0, \tau] \tag{4.1}$$

   holds.

3. *The zero solution of the SFrDE* (3.2) *is exponentially practically stable w.r.t.* $(\alpha_2(\lambda), A)$, *where* $A \in [\alpha_2(\lambda), b^{-1}(\alpha)]$ *is a given number.*

Then, the zero solution of FrDE (3.1) *is exponentially practically stable w.r.t.* $(\lambda, b(A))$.

Proof. From condition 3 of Theorem 4.1, for $u_0 \in \mathbb{R}$ with $|u_0| < \alpha_2(\lambda)$, we have

$$|u(t; t_0, u_0)| < Ae^{\frac{\rho-1}{\rho}(t-t_0)}, \text{ for } t \geq t_0, \tag{4.2}$$

where $u(t; t_0, u_0)$ is a solution of FrDE (3.2).

Let $y(t) = y(t; t_0, y_0)$ be any solution of FrDE (3.1) with $||y_0|| < \lambda$. From condition 2(i) it follows that $\alpha_2(\lambda) \geq \alpha_1(\lambda)$. Thus, $A \geq \alpha_1(\lambda)$, or $a_1^{-1}(A) \geq \lambda$. Therefore, $||y_0|| < a_1^{-1}(A) \leq b(A)$.

We will prove that

$$\|y\,(t)\| < b(A)e^{\frac{\rho-1}{\rho}(t-t_0)} \quad \text{for } t \geq t_0. \tag{4.3}$$

Assume the opposite, i.e., there exists a point $T > t_0$ such that

$$||y(s)|| < b(A)e^{\frac{\rho-1}{\rho}(s-t_0)}, \quad s \in [t_0, T), \qquad ||y(T)|| = b(A)e^{\frac{\rho-1}{\rho}(T-t_0)}.$$

From the choice of the point T we have that $||y(t)|| \leq b(A)$ for $t \in [t_0, T]$, i.e., $y(t) \in S_{b(A)} \subset S_\alpha$ on $[t_0, T]$. From condition 2(ii) with $\tau = T$ the inequality

$$(\,{}_{t_0}^{C}\mathcal{D}^{q,\rho}V(., y(.)))(t) \leq g(t, V(t, y(t))) \quad t \in (t_0, T] \tag{4.4}$$

holds.

Let $\tilde{u}_0 = V(t_0, y_0)$. Consider the maximal solution $u^*(t) = u(t; t_0, \tilde{u}_0)$ of SFrDE (3.2). Thus, the conditions of Lemma 3.12 are satisfied with $\Delta = S_{b(A)}$. According to Lemma 3.12, the inequality

$$V(t, y\,(t)) \leq u^*(t) \qquad \text{for } t \in [t_0, T] \tag{4.5}$$

holds.

From the choice of $\tilde{u}_0$ and condition 2(i) it follows that $|\tilde{u}_0| \leq a_2(||y_0||) < \alpha_2(\lambda)$, and therefore the inequality (4.2) holds for $u^*(t)$.

Then, from condition 2 (i) and inequalities (4.2), (4.5) we obtain

$$\alpha_1(\|y(T)\|) \leq V(T, y\,(T)) \leq u^*(T) < Ae^{\frac{\rho-1}{\rho}(t-t_0)},$$

which implies that

$$b(A)e^{\frac{\rho-1}{\rho}(T-t_0)} = \|y\,(T)\| < a_1^{-1}(Ae^{\frac{\rho-1}{\rho}(t-t_0)}) \leq b(A)e^{\frac{\rho-1}{\rho}(t-t_0)}.$$

The obtained contradiction proves the inequality (4.3). Therefore, according to Definition 3.4, the zero solution is exponentially practically stable w.r.t. $(\lambda, b(A))$. □

Corollary 4.2. *Let the conditions 1,2 of Theorem 4.1 be satisfied with* $g(t, u) \equiv 0$. *Then the zero solution of FrDE (3.1) is exponentially practically stable.*

The proof of Corollary 4.2 follows from Example 3.7 and the exponential practical stability of the zero solution of SFrDE (3.2) with $g(t,u)\equiv 0$.

We will prove sufficient conditions for the practical stability of FrDE (3.1) depending on the practical stability of the SFrDE (3.2) .

Theorem 4.3. *Let the following conditions be satisfied:*

1. *The conditions (A1) and (A2) are satisfied for the given* t_0.
2. *There exists a function* $V\in\Lambda([t_0,\infty),\mathbb{R}^n)$ *with* $V(t,0)=0$ *such that*
   (i) *the inequalities*
   $$\alpha_1(\|x\|)\le V(t,x) \text{ for } t\ge t_0,\ x\in\mathbb{R}^n,$$
   $$V(t,x)\le\alpha_2(\|x\|) \text{ for } t\ge t_0,\ x\in S_\lambda,$$
   hold, where $a_1,\alpha_2\in\mathcal{K},\ i=1,2,$ *and* $\lambda>0$ *is a given number;*
   (ii) *there exists a constant* $A\ge\alpha_1^{-1}(\alpha_2(\lambda))>0$ *such that if the solution* $y(t)$ *of (3.1) satisfies* $y(t)\in S_A$ *on an interval* $[t_0,\tau]$, $\tau>t_0$, *then the inequality*
   $$({}_{t_0}^{C}\mathcal{D}^{q,\rho}V(.,y(.)))(t)\le g(t,V(t,y(t))), \quad \text{for } t\in(t_0,\tau] \tag{4.6}$$
   holds.
3. *The zero solution of the SFrDE* (3.2) *is practically stable w.r.t.* $(\alpha_2(\lambda),\alpha_1(A))$.

Then, the zero solution of FrDE (3.1) *is practically stable w.r.t.* (λ,A).

Proof. From condition 3 of Theorem 4.3, for $u_0\in\mathbb{R}$ with $|u_0|<\alpha_2(\lambda)$, we have
$$|u(t;t_0,u_0)|<\alpha_1(A),\ \text{for } t\ge t_0, \tag{4.7}$$
where $u(t;t_0,u_0)$ is a solution of FrDE (3.2).

Note from condition 2(i) we have that $\alpha_1(\lambda)<\alpha_2(\lambda)$ and $\lambda<\alpha_1^{-1}(\lambda_2(\lambda))\le A$.

Let $y(t)=y(t;t_0,y_0)$ be any solution of FrDE (3.1) with $||y_0||<\lambda$. Thus, $||y_0||<A$.

We will prove that
$$\|y(t)\|<A \quad \text{for } t\ge t_0. \tag{4.8}$$
Assume the opposite, i.e., there exists a point $T>t_0$ such that
$$||y(s)||<A,\quad s\in[t_0,T),\qquad ||y(T)||=A.$$
From the choice of the point T it follows that $||y(t)||\le A$ for $t\in[t_0,T]$, i.e., $y(t)\in S_A$ on $[t_0,T]$ and according to condition 2(ii) with $\tau=T$ the inequality
$$({}_{t_0}^{C}\mathcal{D}^{q,\rho}V(.,y(.)))(t)\le g(t,V(t,y(t)))\quad t\in(t_0,T] \tag{4.9}$$
holds.

Let $\tilde{u}_0 = V(t_0, y_0)$. Consider the maximal solution $u^*(t) = u(t; t_0, \tilde{u}_0)$ of SFrDE (3.2). Thus, the conditions of Lemma 3.12 are satisfied with $\Delta = S_A$. According to Lemma 3.12, the inequality

$$V(t, y(t)) \leq u^*(t) \qquad \text{for } t \in [t_0, T] \tag{4.10}$$

holds.

From the choice of $\tilde{u}_0$ and condition 2(i) we have $|\tilde{u}_0| = V(t_0, y_0) \leq \alpha_2(||y_0||) < \alpha_2(\lambda)$, and therefore the inequality (4.7) holds for $u^*(t)$.

Then, from condition 2 (i) and inequalities (4.7), (4.10) we obtain

$$\alpha_1(\|y(T)\|) \leq V(T, y(T)) \leq u^*(T) < \alpha_1(A),$$

which implies that

$$A = \|y(T)\| < A.$$

The obtained contradiction proves the inequality (4.8). Therefore, according to Definition 3.4, the zero solution is exponentially practically stable w.r.t. (λ, A). □

According to Corollary 4.2 in the case when the generalized proportional Caputo fractional derivative of Lyapunov function is non-positive we obtain the exponential practical stability of the zero solution of (3.1).

We could obtain sufficient conditions for practical stability for non-positive generalized proportional Caputo fractional derivative of Lyapunov function of any solution without the restriction of the solution being on a ball.

Theorem 4.4. *Let condition (A1) be satisfied and there exists a continuously differentiable Lyapunov function $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ with $V(t, 0) = 0$, such that*

(i) *the inequalities*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \text{ for } t \geq t_0, x \in \mathbb{R}^n,$$

hold, where $\alpha_i \in \mathcal{K}$, $i = 1, 2$, and $\lambda > 0$ is a given number.

(ii) *for any $t > t_0$ and for any solution $y(t) = y(t; t_0, y_0)$ of (3.1) such that $||y_0|| < \lambda$ the inequality*

$$({}_{t_0}^{C}\mathcal{D}_t^{q,\rho} V(., y(.)))(t) \leq 0,$$

holds.

Then, the zero solution of (3.1) is practically stable w.r.t. $(\lambda, \alpha_1^{-1}(\alpha_2(\lambda)))$.

Proof. Let $y(t) = y(t; t_0, x_0)$ be a solution of FrDE (3.1) with $||x_0|| < \lambda$. Define the function

$$v(t) = \sup_{s \in [t_0, t]} V(s, y(s)), \ t \geq t_0.$$

Obviously, the function v is nondecreasing. We will prove that

$$v(t) = v(t_0) \quad \text{for } t \geq t_0. \tag{4.11}$$

Assume that (4.11) is not true, i.e., there exists a point $T > t_0$ such that $v(t) = v(t_0)$, for $t \in [t_0, T]$, but $v(t) > v(t_0)$ and v is strictly increasing for $t \in (T, T+\varepsilon]$, $\varepsilon > 0$ is a small enough number. Then $v(s) = v(t_0) \geq V(s, y(s))$, for $s \in [t_0, T]$, and $v(t) = V(t, y(t))$, for $t \in (T, T+\varepsilon]$.

Then, for any fixed $t \in (T, T+\varepsilon]$ from the inequality $v(s) \geq V(s, y(s))$ for $s \in [t_0, t]$ we get

$$\begin{aligned}
({}^{C}_{t_0}\mathcal{D}^{q,\rho} v)(t) &= \frac{1}{\rho^{1-q}\Gamma(1-q)} \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-q} (\mathcal{D}^{\rho} v)(s)\, ds \\
&= \frac{1}{\rho^{1-q}\Gamma(1-q)} \Big((1-\rho) \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-q} v(s)\, ds \\
&\quad + \rho \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-q} (v'(s)\, ds \Big) \\
&= \frac{1}{\rho^{1-q}\Gamma(1-q)} \Big((1-\rho) \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{V(s, y(s))}{(t-s)^{-q}}\, ds \\
&\quad + \rho \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{v'(s)}{(t-s)^{-q}}\, ds \Big)
\end{aligned} \tag{4.12}$$

where $(\mathcal{D}^{\rho} u)(t) = (1-\rho) u(t) + \rho u'(t)$.

Define the function $g(s) := \frac{v(s) - V(s, y(s))}{(t-s)^q}$, $\quad s \in [t_0, t)$. Note $g(t_0) = 0$ and $g(s) > 0$ because $v(s) \geq V(s, y(s))$ for $s \in [t_0, t]$. Thus, integrating by parts we get

(4.13)

$$\begin{aligned}
&\int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{v'(s)}{(t-s)^q} ds = \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{v'(s) - V'(s, y(s))}{(t-s)^q} ds + \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{V'(s, y(s))}{(t-s)^q} ds \\
&= \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \Big[g'(s) - q \frac{v(s) - V(s, y(s))}{(t-s)^{q+1}} \Big] ds + \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{V'(s, y(s))}{(t-s)^q} ds \\
&= \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} g'(s) ds - q \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{v(s) - V(s, y(s))}{(t-s)^{q+1}} ds + \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{V'(s, y(s))}{(t-s)^q} ds \\
&\leq e^{\frac{\rho-1}{\rho}(t-s)} g(s)\Big|_{s=t_0}^{s=t} - \frac{1-\rho}{\rho} \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} g(s) ds + \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{V'(s, y(s))}{(t-s)^q} ds \\
&\leq \lim_{s \to t^-} e^{\frac{\rho-1}{\rho}(t-s)} g(s) + \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} \frac{V'(s, y(s))}{(t-s)^q} ds.
\end{aligned}$$

Use L'Hôpital's rule to obtain

$$\begin{aligned}(4.14)\qquad \lim_{s\to t^-} e^{\frac{\rho-1}{\rho}(t-s)}g(s) &= \lim_{s\to t^-}\frac{e^{\frac{\rho-1}{\rho}(t-s)}h(s)}{(t-s)^q}\\ &= \lim_{s\to t^-}\frac{e^{\frac{\rho-1}{\rho}(t-s)}h'(s)+\frac{1-\rho}{\rho}e^{\frac{\rho-1}{\rho}(t-s)}h(s)}{q(t-s)^{q-1}} = 0,\end{aligned}$$

where $h(s) = v(s) - V(s, y(s)),\ s \in [t_0, t]$.

From (4.12), (4.13) and (4.14) we obtain

$$(4.15)\qquad ({}_{t_0}^{C}\mathcal{D}^{q,\rho}v)(t) \leq ({}_{t_0}^{C}\mathcal{D}^{q,\rho}V(.,y(.)))(t) \leq 0.$$

According to the assumption, we get $v'(t) = 0$, for $t \in [t_0, T]$, and $v'(t) > 0$, for $t \in (T, T+\varepsilon]$. Then, for any $t \in (T, T+\varepsilon]$, we obtain $(\mathcal{D}^\rho v)(t) = (1-\rho)v(t) + \rho v'(t)$

$$\begin{aligned}{}_{t_0}^{c}\mathcal{D}^{q,\rho}v(t) &= \frac{1}{\rho^{1-q}\Gamma(1-q)}\int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)}\,(t-s)^{-q}\,(\mathcal{D}^\rho v)(s)\ ds\\ &= \frac{1}{\rho^{1-q}\Gamma(1-q)}\Big(\int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(t-s)}\frac{(1-\rho)v(t_0)}{(t-s)^q}ds\\ (4.16)\qquad &\quad + \int_{T}^{t}\frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^q}((1-\rho)v(s)+\rho v'(s))\ ds\Big)\\ &> \frac{1}{\rho^{1-q}\Gamma(1-q)}\Big(\int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)}\frac{(1-\rho)v(t_0)}{(t-s)^q}ds + \int_{T}^{t}\frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^q}\rho v'(s)\ ds\Big)\\ &\geq \frac{1}{\rho^{1-q}\Gamma(1-q)}\int_{T}^{t}\frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^q}\rho v'(s)\ ds > 0.\end{aligned}$$

Inequality (4.16) contradicts inequality (4.15). The contradiction proves the inequality (4.11).

From condition (i), we get

$$\alpha_1(\|y(t)\|) \leq V(t, y(t)) \leq v(t) = v(t_0) = V(t_0, y_0) \leq \alpha_2(||y_0||) \leq \alpha_2(\lambda).$$

□

In the case when the Lyapunov function is equal to the norm in $\mathbb{R}^n$ we obtain the following sufficient conditions for exponential practical stability:

Theorem 4.5. *Let the condition (A1) be satisfied and there exists a constant $\lambda > 0$ such that for any solution of (3.1) with $y(t) \in S_\lambda$ on an interval $[t_0, \tau]$ the inequality*

$$(4.17)\qquad (\ {}_{t_0}^{C}\mathcal{D}^{q,\rho}||y(.)||)(t) \leq 0, \quad t \in [t_0, \tau]$$

holds.

Then, the zero solution of FrDE (3.1) *is exponentially practically stable w.r.t.* (λ, λ).

Proof. Let $y(t) = y(t; t_0, y_0)$ be any solution of FrDE (3.1) with $||y_0|| < \lambda$, i.e., $y_0 \in S_\lambda$.

We will prove that

$$\|y(t)\| < \lambda e^{\frac{\rho-1}{\rho}(t-t_0)} \quad \text{for } t \geq t_0. \tag{4.18}$$

Assume the opposite, i.e., there exists a point $T > t_0$ such that

$$||y(s)|| < \lambda e^{\frac{\rho-1}{\rho}(s-t_0)}, \quad s \in [t_0, T), \qquad ||y(T)|| = \lambda e^{\frac{\rho-1}{\rho}(T-t_0)}.$$

Thus, the function $h(s) = ||y(s)|| \in S_\lambda,\ s \in [t_0, T]$ and according to condition 2 the inequality (4.1) holds for $\tau = T$.

Consider the function $v(t) = ||y(t)|| - \lambda e^{\frac{\rho-1}{\rho}(t-t_0)}$ defined on $[t_0, T]$, $v(t) < 0$ on $[t_0, t)$ and $v(T) = 0$. According to Lemma 2.7 the inequality

$$({}^{c}_{t_0}\mathcal{D}^{q,\rho} v(.)(t)|_{t=T} > 0$$

holds.

According to Remark 2.3 we have $({}^{c}_{t_0}\mathcal{D}^{q,\rho} v(.))(t)|_{t=T} = ({}^{c}_{t_0}\mathcal{D}^{q,\rho} ||y(.)||)(t)|_{t=T} > 0$ which contradicts (4.17) for $t = T = \tau$. The obtained contradiction proves the claim.

□

5. **Applications**

Example 5.1. Consider the following system of fractional differential equations with generalized proportional Caputo type derivative

$$\begin{aligned} &({}^{C}_{0}\mathcal{D}^{q,\rho} y_1)(t) = g_1(t)y_1 - g_2(t)y_2, \\ &({}^{C}_{0}\mathcal{D}^{q,\rho} y)(t) = g_1(t)y_2 + g_2(t)y_1, \text{ for } t > 0,\ q \in (0,1),\ \rho \in (0,1), \end{aligned} \tag{5.1}$$

with initial conditions

$$y_1(0) = y_{1,0}, \qquad y_2(0) = y_{2,0},$$

where $y_{1,0}, y_{2,0} \in \mathbb{R}$, $g_1(t) = 0.5\frac{(1-\rho)^q}{\Gamma(1-q)}\gamma(1-q, \frac{1-\rho}{\rho}t)$ for $t \geq 0$ and $g_2 \in C(\mathbb{R}_+, \mathbb{R})$ is an arbitrary function.

Note (5.1) is equivalent to (3.1) with $y = (y_1, y_2),\ f = (f_1, f_2)$ where $f_1(t, y) = g_1(t)y_1 - g_2(t)y_2$ and $f_2(t, y) = g_1(t)y_2 + g_2(t)y_1$.

Consider $V(t, y) = y_1^2 + y_2^2$ for $t \in \mathbb{R}_+,\ y = (y_1, y_2)$ with $\alpha_1(s) = \alpha_2(s) = s^2 \in \mathcal{K}$ and $A = \lambda$ be any positive number.

Let $y(t) = (y_1(t), y_2(t)),\ t \geq 0$, be any solution of FrDE(5.1). Apply Lemma 2.6 and obtain

$$\begin{aligned} ({}^{C}_{0}\mathcal{D}^{q,\rho} V(., y(.))(t) = &\ ({}^{C}_{0}\mathcal{D}^{q,\rho} y_1^2(.))(t) + ({}^{C}_{0}\mathcal{D}^{q,\rho} y_2^2(.))(t) \\ \leq &\ 2y_1(t)({}^{C}_{0}\mathcal{D}^{q,\rho} y_1(.))(t) + 2y_2(t)({}^{C}_{0}\mathcal{D}^{q,\rho} y_2(.))(t) \\ = &\ 2g_1(t)V(t, y(t)) = \frac{(1-\rho)^q}{\Gamma(1-q)}\gamma(1-q, \frac{1-\rho}{\rho}t)V(t, y(t)). \end{aligned} \tag{5.2}$$

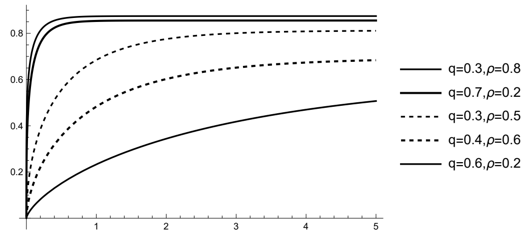

Figure 1. Graph of the function $g(t)$ for various values of $\rho \in (0,1]$ and $q \in (0,1)$.

Consider the SFrDE (3.2) with $t_0 = 0$, $g(t,u) = \frac{(1-\rho)^q}{\Gamma(1-q)}\gamma(1-q, \frac{1-\rho}{\rho}t)u$, $u \in \mathbb{R}$, $u_0 = u(0) = K$ whose solution is the constant function $u(t) \equiv K$, (see Example 3.8) and it is practically stable w.r.t. (λ^2, λ^2). According to Theorem 4.3 the zero solution of (5.1) is practically stable w.r.t. (λ, λ).

Since the function $g(t) \geq 0$, $t \geq 0$ (see Figure 1 for various values of ρ and q), we are not able to apply Theorem 4.4 to conclude the exponential practical stability of the zero solution of (5.1).

Example 5.2. Consider the following system of nonlinear generalized proportional Caputo fractional differential equations

$$\begin{aligned} &({}_0^C D^{q,\rho} y_1)(t) = (e^{y_1(t)} - e)\sin(y_1(t)) - g(t)y_2(t), \\ &({}_0^C D^{q,\rho} y_2)(t) = g(t)y_1(t), \quad \text{for } \ t > 0, \ q \in (0,1), \ \rho \in (0,1), \end{aligned} \tag{5.3}$$

with initial conditions

$$y_1(0) = y_{1,0}, \qquad y_2(0) = y_{2,0},$$

where $y_{1,0}, y_{2,0} \in \mathbb{R}$, $g \in C([0,\infty), \mathbb{R})$ is an arbitrary function.

Consider $V(t,y) = y_1^2 + y_2^2$ for $t \in \mathbb{R}_+$, $y = (y_1, y_2)$ with $\alpha_1(s) = \alpha_2(s) = s^2$, $\alpha_1, \alpha_2 \in \mathcal{K}$. Also, $\alpha_1(Ce^{\frac{\rho-1}{\rho}s}) = C^2 e^{2\frac{\rho-1}{\rho}s} \leq C^2 e^{\frac{\rho-1}{\rho}s}$, i.e. $a_1 \in \mathcal{M}$ with $a_1(s) = b(s) = s^2$, i.e. condition 2(i) of Theorem 4.1 is satisfied.

Let $y(t) = (y_1(t), y_2(t))$, $t \geq 0$, be any solution of FrDE(5.3) such that $||y_0|| < 1$, Because of the continuity of the solution $y(t)$ there exists a number $\tau > 0$ such that $||y(t)|| < 1$, $t \in [0,\tau]$. Then $|y_1(t)| \leq 1$ and $e^{y_1(t)} \leq e$ for $t \in [0,\tau]$. Also, $u\ \sin(u) \geq 0$ for $u \in [-1,1]$. Thus, by Lemma 2.6 we obtain

$$\begin{aligned} &({}_0^C D^{q,\rho} V(.,y(.)))(t) \leq 2y_1(t)({}_0^C D^{q,\rho} y_1(.))(t) + 2y_2(t)({}_0^C D^{q,\rho} y_2(.))(t) \\ &= 2(e^{y_1(t)} - e)y_1(t)\sin(y_1(t)) - 2g(t)y_1(t)y_2(t) + 2y_2(t)g(t)y_1(t) \\ &\leq 0, \qquad t \in [0,\tau]. \end{aligned} \tag{5.4}$$

Therefore, condition 2(ii) of Theorem 4.1 is satisfied with $\lambda = 1$.

According to Corollary 4.2 the zero solution of FrDE (5.3) is exponentially practically stable w.r.t. $(1,1)$.

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