Dynamic Systems and Applications 31 (2022) No.3, 133-172

STABILITY AND EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL SYSTEM WITH p-LAPLACIAN OPERATOR ON STAR GRAPHS

JUNPING NAN[a,b], WEIMIN HU[b,c], YOUHUI SU[a,*] , AND XIAOLING HAN[d]

[a]School of Mathematics and Statistics, Xuzhou University of Technology,
Xuzhou, Jiangsu, China
e-mail:suyh02@163.com

[b] School of Mathematics and Statistics, Yili Normal University, Yining,
Xinjiang, China
e-mail: njp7928@sina.com

[c] Institute of Applied Mathematic, Yili Normal University, Yining,
Xinjiang, China
e-mail: Hwm680702@163.com

[d] College of Mathematics and Statistics, Northwest Normal University, Lanzhou,
Gansu, China
e-mail: hanxiaoling9@163.com

ABSTRACT. As we all know, star graphs are very useful in physics, chemistry, biology and other fields, but few people have studied the existence of solutions to differential system on star graphs. In this paper, we investigate the existence of solutions to a fractional differential system on star graphs with p-Laplacian operator. Ulam's stability and existence of the solutions to the fractional differential system on star graphs are proved. In addition, two examples under different background graphs (star graphs and formaldehyde graphs) and the approximation graphics for the solutions are provided to illustrate the application of our main results. The interesting point of this article is that it not only studies the existence of solutions to fractional differential system on the star graphs, but also gives approximate graphs of solutions by using the iterative methods and numerical simulation.

AMS (MOS) Subject Classification. 34B15, 34B45.

Key Words and Phrases. Fractional differential system, star graphs, the fixed point theorems, Ulam's stability, numerical simulation.

Received August 8, 2022 ISSN 1056-2176(Print); ISSN 2693-5295 (online)
$15.00 © Dynamic Publishers, Inc.
https://doi.org/10.46719/dsa202231.03.01
www.dynamicpublishers.org.
CORRESPONDING AUTHOR: *

1. Introduction

Fractional calculus is the research of integrals and derivatives of any arbitrary real or complex order. In recent years, fractional differential equations have been acquired much attention due to its applications in a number of fields such as fluid flow, control theory of dynamical systems, chemistry, biology, optics and signal processing [1–11]. For example, Mohammadi et al [8] proposed a fractional differential equations model for hearing loss due to mumps virus:

$$\begin{cases} {}^{CF}D_0^{\vartheta}S(t) = \omega^{\vartheta} - \delta^{\vartheta}S(t)L(t) - \gamma^{\vartheta}S(t), \\ {}^{CF}D_0^{\vartheta}L(t) = \delta^{\vartheta}S(t)L(t) - (\gamma^{\vartheta} + \tau^{\vartheta})L(t), \\ {}^{CF}D_0^{\vartheta}R(t) = \tau^{\vartheta}L(t) - \gamma^{\vartheta}R(t), \end{cases}$$

where $S(0) = \check{S}_0$, $L(0) = \check{L}_0$ and $R(0) = \check{R}_0$, $\vartheta \in (0,1)$, ${}^{CF}D_0^{\vartheta}$ is the Caputo-Fabrizio fractional derivative operator, $S(t), L(t), R(t)$ denote, respectively, susceptible people (normal hearing), infected people (loss of hearing due to mumps virus) and recovered people (recovered hearing), ω denotes the recruitment rate of the population, δ shows the transmission rate of mumps virus, γ indicates the natural death rate of the population and the parameter τ represents the recovery rate of infective individuals.

It is novel to study the existence of solutions to differential system on star graphs which are concerned with networks of points connected by straight lines. This structure are common in all around us, such as water pipes, molecular structures in medicine and biology and so on [12–18]. The model described by differential equations on star graphs has applications in chemical engineering, biology, physics and other fields [19–23]. At present, a few researchers have studied the existence results of solutions of differential equations on star graphs [24–28].

A graph [23] $G = (V, E)$ consists of a finite set of nodes or vertices $V(G) = \{v_0, v_1, ..., v_k\}$ and a set of edges $E(G) = \{e_1 = \overrightarrow{v_1v_0}, e_2 = \overrightarrow{v_2v_0}, ..., e_k = \overrightarrow{v_kv_0}\}$ connecting these nodes, where v_0 is the joint point and e_i is the length of l_i the edge connecting the nodes v_i and v_0, i.e. $l_i = |\overrightarrow{v_iv_0}|$.

The origin of differential equations on star graphs is related to Lumer [24], who pioneered the application of differential equations to graph theory in the 1980s by exploring solutions of evolutionary equations on ramification spaces and under different operator rules. Nicaise [25] studied the propagation of nerve impulses. In 1989, Zavgorodnii and Pokornyi [28] considered linear differential equations on geometric graphs where the solutions of the differential equations are coordinated internally. In 2008, Gordeziani et al [19] solved the differential equations on graphs by the double-sweep method and proposed a numerical method.

In order to better explore the application of differential equations on star graphs, here we briefly review some related results in the existing literature [29–32]. In [30], Zuazua and Han discussed the asymptotic behaviour of the transmission problem on a star-shaped network consisting of elastic and thermoelastic rods and proved the exponential, approximate polynomial decay rate of the thermoelastic network system. Abdian and Behmaram et al [29] investigated signless Laplacian spectral characterization of graphs with independent edges and isolated vertices and proved that $G \cup \gamma K_1 \cup sK_2$ is DQS under certain conditions to obtain some DQS graphs(a graph is said to be DQS if there is no other non-isomorphic graph with the same signless Laplacian spectrum) with independent edges and isolated vertices. In [31], Pivovarchik analysed the direct and inverse Sturm-Liouville spectral of the problems on a star-shaped graphs and show that the spectrum of the problem on the graph uniquely determine the potentials on the edge. In addition, some scholars have studied the direction of chemical graphs theory, for example, Turab et al [32] studied the existence results of solutions for a class of nonlinear fractional boundary value problem on ethane graph.

There are relatively few studies on fractional differential equations on star graphs, the first appeared in 2004 when Graef et al [20] studied the fractional differential system on star graphs, a graph consisting $G = V \bigcup E$, v_0 is the connecting node and $\overrightarrow{v_i v_0}$ denotes the edge connecting v_i and v_0, $l_i = |\overrightarrow{v_i v_0}|$, $i = 1, 2$ coordinate system with v_i as the origin on each edge $\overrightarrow{v_i v_0}$, $i = 1, 2$ and $x \in (0, l_i)$. The authors researched questions as follows:

$$\begin{cases} -D^{\alpha}_{0+}\mathfrak{u}_i = \varpi_{\mathfrak{i}}\mathfrak{f}_i(x, \mathfrak{u}_i), 0 < x < l_i, i = 1, 2, \\ \mathfrak{u}_1(0) = \mathfrak{u}_2(0), \mathfrak{u}_1(l_1) = \mathfrak{u}_2(l_2), \\ D^{\beta}_{0+}\mathfrak{u}_1(l_1) + D^{\beta}_{0+}\mathfrak{u}_2(l_2) = 0, \end{cases}$$

where $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the Riemann-Liouville fractional derivative operator, $1 < \alpha \leq 2$, $0 < \beta < \alpha$, $\varpi_i \in C[0, 1]$, $i = 1, 2$ with $\varpi_i(x) \neq 0$ on $[0, l_i]$ and $\mathfrak{f}_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$. By a transformation, equivalent fractional differential system defined on $[0, 1]$ are obtained. The authors proved that the existence and uniqueness results by using Banach contraction principle and Schauder fixed point theorem.

Next, in 2019, Mehandiratta et al [23] explored the fractional differential system on star graphs with $n + 1$ nodes and n edges as follows.

$$\begin{cases} {}_CD^{\alpha}_{0,x}\mathfrak{u}_i(x) = \mathfrak{f}_i(x, \mathfrak{u}_i, {}_CD^{\beta}_{0,x}\mathfrak{u}_i(x)), 0 < x < l_i, i = 1, 2, ..., k, \\ \mathfrak{u}_i(0) = 0, i = 1, 2, ..., k, \\ \mathfrak{u}_i(l_i) = \mathfrak{u}_j(l_j), i, j = 1, 2, ..., k, i \neq j \\ \sum\limits_{i=1}^{k} \mathfrak{u}_i^{'} = 0, i = 1, 2, ..., k, \end{cases}$$

where ${}_CD^{\alpha}_{0,x}$, ${}_CD^{\beta}_{0,x}$ are the Caputo fractional derivative operator, $1 < \alpha \le 2$, $0 < \beta \le \alpha - 1$, $\mathfrak{f}_i, i = 1, 2, ..., k$ are continuous functions on $[0,1] \times \mathbb{R} \times \mathbb{R}$. By a transformation, the equivalent fractional differential system defined on $[0,1]$ is obtained. The author studied a nonlinear Caputo fractional boundary value problem on star graphs and established the existence and uniqueness results by fixed point theory.

Inspired by the above work and relevant literatures [20, 23, 27], we will devote to consider the Ulam's stability and existence of solutions to the following fractional differential system with p-Laplacian operator on star graphs

$$\begin{cases} \phi_p\left(D^{\alpha}_{0+}\mathfrak{u}_i(x)\right) = \mathfrak{f}_i\big(x, \mathfrak{u}_i(x), D^{\beta}_{0+}\mathfrak{u}_i(x)\big), \ t \in [0, l_i], \\ \mathfrak{u}_i(0) = \mathfrak{u}_i'(0) = 0, \\ \mathfrak{u}_i(l_i) = \mathfrak{u}_j(l_j), \ i, j = 1, 2, ..., k, i \neq j, \\ \sum_{i=1}^{k} D^{\beta}_{0+}\mathfrak{u}_i(l_i) = 0, \ i = 1, 2, ..., k, \end{cases} \tag{1.1}$$

where $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the Riemann-Liouville fractional derivative operator and $\phi_p(s)$ is the p-Laplacian operator, $\phi_p(s) = |s|^{p-2}s$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 < \alpha \le 3$, $0 < \beta \le 1$, $1 < p < 2$, $G = V \cup E$ with $V(G) = \{v_0, v_1, ..., v_k\}$ and $E(G) = \{\overrightarrow{v_iv_0}, i = 1, 2, ..., k\}$, $l_i = |\overrightarrow{v_iv_0}|$, $\mathfrak{f}_i \in C([0, l_i] \times \mathbb{R} \times \mathbb{R}), i = 1, 2, ..., k$, where $n = k$ represents the number of edges of the star graphs with $l_i = |\overrightarrow{v_iv_0}|$. For a more intuitive understanding, we establish a local coordinate system as shown on Figure 1 and Figure 2, where Figure 1 is the sketch of the star graphs and Figure 2 is the sketch of the directed star graphs. More specifically, using v_i as the origin and v_0 as the join point, $t \in [0, l_i]$, establish a coordinate system on each edge.

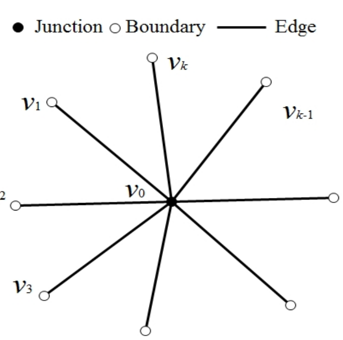

FIGURE 1. A sketch of the star graphs

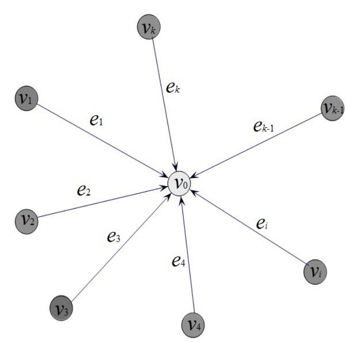

FIGURE 2. A sketch of the directed star graphs

Ulam's stability and existence results for the solutions to the fractional differential system (1.1) are proved by using fixed point theorems. In addition, two examples under different background graphs (star graphs and formaldehyde graphs) and the approximation graphics for the solutions are provided to illustrate the application of our main results. The interesting aspect of this paper is the use of iterative methods to give approximate graphs of solutions and connect fractional differential equations with graph theory.

Firstly, we extend the results of Graef et al [20] to the $k+1$ points and k edges. Secondly, the nonlinear term of fractional differential system not only depends on the unknown function but also on its fractional derivative term, which makes the study in this paper more general and difficulty. Nextly, we establish the modelling of differential equations on each edge of a star graphs which can be applied to different fields such as physics, chemical engineering etc. For example, in organic chemistry, each solution function $\mathfrak{u}_i$ on any edge can represent quantities of bond energy, bond strength, the bond polarity, etc. This will potentially have applications in chemical reaction theory. As far as we know, there is hardly any people consider the numerical simulations and existence of the solution to the fractional differential system with p-Laplacian on star graphs.

The outline of the paper is as follows, in Section 2, some basic definitions and related lemmas are given. The existence of uniqueness of solutions to the system of fractional differential system (1.1) under some assumptions are proved in Section 3. In Section 4 suitable conditions are constructed so that Ulam's stability is satisfied in system (2.1). Some examples and perform numerical simulations on the examples are given in the last section.

2. **Preliminaries**

Here we will show the basic definitions and lemmas for the fractional integrals and fractional derivatives which will be used in this paper later.

Definition 2.1. [3] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. [3] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$.

Lemma 2.3. *[3] If $\alpha > 0$, $\beta > 0$, $u \in L[0,1]$, Then*

(i) $D_{0+}^{\beta} I_{0+}^{\alpha} u(t) = I_{0+}^{\alpha-\beta} u(t), \ \alpha > \beta;$

(ii) $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$;

(iii) $I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + \sum_{i=0}^{n} C_i t^{\alpha-i}, \ n-1 < \alpha \leq n, \ C_i \in \mathbb{R}, \ i = 1,2,...,n, \ n = [\alpha]+1$;

(iv) $D_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \ \beta > \alpha - 1, \ \beta > -1, \ t > 0.$

Lemma 2.4. *[13] For $p > 2$, $| x |, \ | y | \leq M$, we have*

$$| \phi_p(x) - \phi_p(y) | \leq (p-1) M^{p-2} | x - y | .$$

Lemma 2.5. *(Scheafer's fixed point theorem) [20] Let X be a Banach space and let $T : X \to X$ be a completely continuous operator (i.e., an operator that restricted to any bounded set in X is compact). Then either*

(i) The set $\{x \in X : x = \mu T x$ for some $\mu \in (0,1)\}$ is unbounded, or
(ii) F has at least one fixed point in X.

Next, we will introduce an important Lemma, which helps to transform system (1.1) into a fractional boundary value system defined on [0,1].

Lemma 2.6. *[23] Let $\mathfrak{u}$ be a function defined on $[0,l]$ and $\alpha > 0$. Assume that $D_{0+}^{\alpha}\mathfrak{u}$ exists on $(0,l]$. Let $x \in [0,l]$, $t = \frac{x}{l} \in [0,1]$ and $v(t) = \mathfrak{u}(lt)$. Then*

$$D_{0+}^{\alpha} \mathfrak{u}(x) = l^{-\alpha} (D_{0+}^{\alpha})(v)(t).$$

By using Lemma 2.6, the fractional differential system (1.1) is equivalent to

$$\begin{cases} D_{0+}^{\alpha} v_i(t) = l_i^{\alpha} \phi_q(f_i(t, v_i(t), l_i^{-\beta} D_{0+}^{\beta} v_i(t))), \ t \in [0,1], \\ v_i(0) = v_i'(0) = 0, \\ v_i(1) = v_j(1), \ i,j = 1,2,...,k, \ i \neq j, \\ \sum_{i=1}^{k} l_i^{-\beta} D_{0+}^{\beta} v_i(1) = 0, \ i = 1,2,...,k. \end{cases} \tag{2.1}$$

where $\mathfrak{u}_i(l_i t) = v_i(t)$, $\mathfrak{f}_i(l_i t, x, y) = f_i(t,x,y)$, $i = 1,2,...,k$.

Lemma 2.7. *Let $h_i \in C[0,1]$, $i = 1,2,...,k$, then the solution of the fractional differential system*

$$\begin{cases} D_{0+}^{\alpha} v_i(t) = h_i(t), \\ v_i(0) = v_i'(0) = 0, \\ v_i(1) = v_j(1), \ i, \ j = 1,2,...,k, \ i \neq j, \\ \sum_{i=1}^{k} l_i^{-\beta} D_{0+}^{\beta} v_i(1) = 0, \ i = 1,2,...,k. \end{cases} \tag{2.2}$$

has the unique solution

$$\begin{aligned} v_i(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_i(s) ds + \sum_{\substack{j=1 \\ j \neq i}}^{k} \ell_j \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (h_j(s) \\ & - h_i(s)) ds - \sum_{j=1}^{k} \ell_j \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} h_j(s) ds. \end{aligned}$$

Proof. By using Lemma 2.3, we get

$$\begin{aligned} v_i(t) &= I_{0^+}^{\alpha} h_i(t) + c_i^{(1)} t^{\alpha-3} + c_i^{(2)} t^{\alpha-2} + c_i^{(3)} t^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_i(s) ds + c_i^{(1)} t^{\alpha-3} + c_i^{(2)} t^{\alpha-2} + c_i^{(3)} t^{\alpha-1}, \end{aligned}$$

noting that $v_i(0) = v_i'(0) = 0,\ i = 1, 2, ..., k$, it follows that

$$c_i^{(1)} = c_i^{(2)} = 0.$$

Since $v_i(1) = v_j(1),\ i,\ j = 1, 2, ..., k,\ i \neq j$, we obtain

$$I_{0^+}^{\alpha} h_i(t) \mid_{t=1} + c_i^{(3)} = I_{0^+}^{\alpha} h_j(t) \mid_{t=1} + c_j^{(3)},\ i, j = 1, 2, ..., k,\ i \neq j,$$

$$c_j^{(3)} = I_{0^+}^{\alpha} h_i(t) \mid_{t=1} - I_{0^+}^{\alpha} h_j(t) \mid_{t=1} + c_i^{(3)},$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{k} l_j^{-\beta} c_j^{(3)} = \sum_{\substack{j=1 \\ j \neq i}}^{k} l_j^{-\beta} \left(I_{0^+}^{\alpha} h_i(t) \mid_{t=1} - I_{0^+}^{\alpha} h_j(t) \mid_{t=1} + c_i^{(3)} \right). \tag{2.3}$$

$$D_{0^+}^{\beta} v_j(t) = I_{0^+}^{\alpha-\beta} h_j(t) + D_{0^+}^{\beta} c_j^{(3)} t^{\alpha-1} = I_{0^+}^{\alpha-\beta} h_j(t) + c_j^{(3)} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1},$$

$$\sum_{j=1}^{k} l_j^{-\beta} D_{0^+}^{\beta} v_j(1) = \sum_{j=1}^{k} l_j^{-\beta} \left(I_{0^+}^{\alpha-\beta} h_j(t) \mid_{t=1} + c_j^{(3)} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \right) = 0,$$

and

$$\sum_{j=1}^{k} l_j^{-\beta} c_j^{(3)} = - \sum_{j=1}^{k} l_j^{-\beta} \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} I_{0^+}^{\alpha-\beta} h_j(t) \mid_{t=1} = \sum_{\substack{j=1 \\ j \neq i}}^{k} l_j^{-\beta} c_j^{(3)} + l_i^{-\beta} c_i^{(3)}. \tag{2.4}$$

According to (2.3) and (2.4), there is

$$\begin{aligned} - \sum_{j=1}^{k} l_j^{-\beta} \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} I_{0^+}^{\alpha-\beta} h_j(t) \mid_{t=1} &= \sum_{\substack{j=1 \\ j \neq i}}^{k} l_j^{-\beta} I_{0^+}^{\alpha} h_i(t) \mid_{t=1} - \sum_{\substack{j=1 \\ j \neq i}}^{k} l_j^{-\beta} I_{0^+}^{\alpha} h_j(t) \mid_{t=1} \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{k} l_j^{-\beta} c_i^{(3)} + l_i^{-\beta} c_i^{(3)}, \end{aligned}$$

thus

$$
\begin{aligned}
c_i^{(3)} &= \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{-\beta}}{\sum_{j=1}^{k} l_j^{-\beta}} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h_j(s)ds \\
&\quad - \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{-\beta}}{\sum_{j=1}^{k} l_j^{-\beta}} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h_i(s)ds \\
&\quad - \sum_{j=1}^{k} \frac{l_j^{-\beta}}{\sum_{j=1}^{k} l_j^{-\beta}} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} h_j(s)ds.
\end{aligned}
$$

Let $\ell_j = \frac{l_j^{-\beta}}{\sum_{j=}^{k} l_j^{-\beta}}$, hence

$$
\begin{aligned}
v_i(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_i(s)ds \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{k} \ell_j \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (h_j(s) - h_i(s))ds \\
&\quad - \sum_{j=1}^{k} \ell_j \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} h_j(s)ds.
\end{aligned}
$$

3. **Main Results**

Let $X = \{v : v \in C([0,1]),\ D_{0^+}^{\beta} v \in C([0,1])\}$, then $(X, \|\cdot\|_X)$ is a Banach space endowed with norm

$$\|v\|_X = \|v\| + \|D_{0^+}^{\beta} v\|,$$

where

$$\|v\| = \max_{0\le t\le 1} |v(t)|,\ \|D_{0^+}^{\beta} v\| = \max_{0\le t\le 1} |{}_H^C D_{0^+}^{\beta} v(t)|.$$

Take $(X^k = X \times X \times ... \times X, \|\cdot\|_{X^k})$, and equipped with the norm

$$\|(v_1, v_2, ..., v_k)\|_{X^k} = \sum_{i=1}^{n} \|v_i\|_X,\ (v_1, v_2, ..., v_k) \in X^k,\ v_i \in X,\ i = 1, 2, ..., k.$$

According to the basic theory of functional analysis, we obtain that X^k is a Banach space.

Define an operator $T : X^k \to X^k$ as follows:

$$T(v_1, v_2, ..., v_k)(t) = (T_1(v_1, v_2, ..., v_k)(t), ..., T_k(v_1, v_2, ..., v_k)(t)),$$

$$\begin{aligned}
T_i(v_1,v_2,...,v_k)(t) =& \frac{l_i^\alpha}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0+}^\beta v_i(s))\right)ds \\
&+\sum_{\substack{j=1\\ j\neq i}}^k \frac{l_j^\alpha \ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0+}^\beta v_j(s))\right)ds \\
&-\sum_{\substack{j=1\\ j\neq i}}^k \frac{l_i^\alpha \ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0+}^\beta v_i(s))\right)ds \\
&-\sum_{j=1}^k \frac{l_j^\alpha \ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0+}^\beta v_j(s))\right)ds.
\end{aligned}$$

Assume that the following conditions hold:

$(H1)$ $f_i:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ i=1,2,...,k$ are continuous functions;

$(H2)$ There exist nonnegative functions on $b_i(t)\in C[0,1],\ i=1,2,...,k$, such that, for all $t\in[0,1],\ (u_1,v_1),\ (u_2,v_2)\in\mathbb{R}^2$,

$$| f_i(t,u_1,v_1)-f_i(t,u_2,v_2) |\leq b_i(t)(| u_1-u_2 | + | v_1-v_2 |),$$

where $a_i(t)=\sup_{0\leq t\leq 1} | b_i(t) |,\ i=1,2,...,k;$

$(H3)$ There exist $L_i>0$ such that

$$| f_i(t,u,v) |\leq L_i,\ t\in[0,1],\ (u,v)\in\mathbb{R},\ i=1,2,...,k;$$

$(H4)$ $\sup_{0\leq t\leq 1} | f_i(t,0,0) | = \lambda<\infty,\ i=1,2,...,k.$

In the following, the main results on the existence of solutions to the fractional differential system which we studied are listed.

Theorem 3.1. *Assume that* (H2) *and* (H3) *hold, then the fractional differential system* (2.1) *has a unique solution on* X^k *provided that*

$$\left(\sum_{i=1}^k N_i\right)\left(\sum_{i=1}^k a_i\right)<1,$$

where

$$\begin{aligned}
N_i \;=\; & (q-1)\left[\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)\right. \\
&\times(l_i^\alpha+l_i^{\alpha-\beta})L_i^{q-2}+\sum_{\substack{j=1\\ j\neq i}}^k\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right. \\
&\left.\left.+\frac{1}{\Gamma(\alpha-\beta+1)}\right)(l_j^\alpha+l_j^{\alpha-\beta})L_j^{q-2}\right].
\end{aligned}$$

Proof. For any $v=(v_1,v_2,...,v_k)(t),\ u=(u_1,u_2,...,u_k)(t)\in X^k,\ t\in[0,1]$, we have

$$\begin{aligned}
&\Big|T_iv(t)-T_iu(t)\Big|\\
=\ &\frac{l_i^\alpha}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\\
&-\phi_q\left(f_i(s,u_i(s),l_i^{-\beta}D_{0^+}^\beta u_i(s))\right)\Big|ds\\
&+\sum_{\substack{j=1\\j\neq i}}^k\frac{l_j^\alpha\ell_jt^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\Big|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^\beta v_j(s))\right)\\
&-\phi_q\left(f_j(s,u_j(s),l_j^{-\beta}D_{0^+}^\beta u_j(s))\right)\Big|ds\\
&-\sum_{\substack{j=1\\j\neq i}}^k\frac{l_i^\alpha\ell_jt^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\Big|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\\
&-\phi_q\left(f_i(s,u_i(s),l_i^{-\beta}D_{0^+}^\beta u_i(s))\right)\Big|ds\\
&-\sum_{j=1}^k\frac{l_j^\alpha\ell_jt^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}\Big|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^\beta v_j(s))\right)\\
&-\phi_q\left(f_j(s,u_j(s),l_j^{-\beta}D_{0^+}^\beta u_j(s))\right)\Big|ds.
\end{aligned}$$

It follows from Lemma 2.4 that

$$\begin{aligned}
&\Big|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)-\phi_q\left(f_i(s,u_i(s),l_i^{-\beta}D_{0^+}^\beta u_i(s))\right)\Big|\\
\leq\ &(q-1)L_i^{q-2}\Big|f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))-f_i(s,u_i(s),l_i^{-\beta}D_{0^+}^\beta u_i(s))\Big|\\
\leq\ &(q-1)L_i^{q-2}a_i(t)\left(|v_i(s)-u_i(s)|+\Big|l_i^{-\beta}D_{0^+}^\beta v_i(s)-l_i^{-\beta}D_{0^+}^\beta u_i(s)\Big|\right)\\
\leq\ &(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|+l_i^{-\beta}\|D_{0^+}^\beta v_i-D_{0^+}^\beta u_i\|).
\end{aligned}$$

Using $0<t\leq1$ and $0<\ell_j\leq1$ for $j=1,2,...,k$, we can write

$$\begin{aligned}
&\Big|T_iv(t)-T_iu(t)\Big|\\
\leq&\frac{l_i^\alpha}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\|v_i-u_i\|\\
&+\frac{l_i^\alpha}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}l_i^{-\beta}a_i(t)\|D_{0^+}^\beta v_i-D_{0^+}^\beta u_i\|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_j^{q-2}a_j(t)\|v_j-u_j\|\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_j^{q-2}l_j^{-\beta}a_j(t)\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}u_j\|\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\|v_i-u_i\|\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}l_i^{-\beta}a_i(t)\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}u_i\|\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}(q-1)L_j^{q-2}a_j(t)\|v_j-u_j\|\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}(q-1)L_j^{q-2}l_j^{-\beta}a_j(t)\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}u_j\|\\
&\leq\frac{2(l_i^{\alpha}+l_i^{\alpha-\beta})}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|+\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}u_i\|)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{\Gamma(\alpha+1)}(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}u_j\|)\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha)}(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}u_j\|)\\
&=\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|\\
&+\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}u_i\|)+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^{\alpha}+l_j^{\alpha-\beta})\\
&\times(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}u_j\|).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\| T_iv(t)-T_iu(t) \|\\
\leq&\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|\\
&+\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}u_i\|)+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^{\alpha}+l_j^{\alpha-\beta})
\end{aligned}
$$

$$\times (q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}u_j\|). \tag{3.1}$$

From Lemma 2.3, we have

$$\begin{aligned}
& D_{0^+}^{\beta}T_iv(t)\\
= & \frac{l_i^{\alpha}}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)ds\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)ds\\
& -\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)ds\\
& -\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)ds,
\end{aligned}$$

which reduces to

$$\begin{aligned}
& \left|D_{0^+}^{\beta}T_iv(t)-D_{0^+}^{\beta}T_iu(t)\right|\\
\leq & \frac{l_i^{\alpha}}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\Big|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\\
& -\phi_q\left(f_i(s,u_i(s),l_i^{-\beta}D_{0^+}^{\beta}u_i(s))\right)\Big|ds\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\Big|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\\
& -\phi_q\left(f_j(s,u_j(s),l_j^{-\beta}D_{0^+}^{\beta}u_j(s))\right)\Big|ds\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\Big|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\\
& -\phi_q\left(f_i(s,u_i(s),l_i^{-\beta}D_{0^+}^{\beta}u_i(s))\right)\Big|ds\\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}\Big|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\\
& -\phi_q\left(f_j(s,u_j(s),l_j^{-\beta}D_{0^+}^{\beta}u_j(s))\right)\Big|ds\\
\leq & \frac{l_i^{\alpha}t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}(q-1)L_i^{q-2}a_i(t)\left(\|v_i-u_i\|+l_i^{-\beta}\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}u_i\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_j^{q-2}a_j(t)\left(\|v_j-u_j\|+l_j^{-\beta}\|D_{0+}^{\beta}v_j-D_{0+}^{\beta}u_j\|\right)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_i^{q-2}a_i(t)\left(\|v_i-u_i\|+l_i^{-\beta}\|D_{0+}^{\beta}v_i-D_{0+}^{\beta}u_i\|\right)\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}(q-1)L_j^{q-2}a_j(t)\left(\|v_j-u_j\|+l_j^{-\beta}\|D_{0+}^{\beta}v_j-D_{0+}^{\beta}u_j\|\right)\\
\leq\;&\frac{l_i^{\alpha}+l_i^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|+\|D_{0+}^{\beta}v_i-D_{0+}^{\beta}u_i\|)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0+}^{\beta}v_j-D_{0+}^{\beta}u_j\|)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}+l_i^{\alpha-\beta}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|+\|D_{0+}^{\beta}v_i-D_{0+}^{\beta}u_i\|)\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0+}^{\beta}v_j-D_{0+}^{\beta}u_j\|)\\
=\;&\left(\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|\\
&+\|D_{0+}^{\beta}v_i-D_{0+}^{\beta}u_i\|)+\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^{\alpha}+l_j^{\alpha-\beta})\\
&\times(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0+}^{\beta}v_j-D_{0+}^{\beta}u_j\|).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\| D_{0+}^{\beta}T_iv(t)-D_{0+}^{\beta}T_iu(t)\|\\
&\leq\left(\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|\\
&+\|D_{0+}^{\beta}v_i-D_{0+}^{\beta}u_i\|)+\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^{\alpha}+l_j^{\alpha-\beta})\\
&\times(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0+}^{\beta}v_j-D_{0+}^{\beta}u_j\|). \qquad (3.2)
\end{aligned}
$$

It foollow from (3.1) and (3.2) that

$$
\| T_iv(t)-T_iu(t)\|+\| D_{0+}^{\beta}T_iv(t)-D_{0+}^{\beta}T_iu(t)\|
$$

$$\begin{aligned}
\leq & \left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right) \\
& \times(l_i^\alpha+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)(\|v_i-u_i\|+\|D_{0^+}^\beta v_i-D_{0^+}^\beta u_i\|) \\
& +\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right) \\
& \times\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^\alpha+l_j^{\alpha-\beta})(q-1)L_j^{q-2}a_j(t)(\|v_j-u_j\|+\|D_{0^+}^\beta v_j-D_{0^+}^\beta u_j\|).
\end{aligned}$$

We also get

$$\begin{aligned}
& \| T_iv(t)-T_iu(t) \|_X \\
\leq & \Bigg[\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)(l_i^\alpha+l_i^{\alpha-\beta})L_i^{q-2} \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)(l_j^\alpha+l_j^{\alpha-\beta})L_j^{q-2}\Bigg] \\
& \times\left(\sum_{i=1}^{k}a_i(t)\right)(q-1)\sum_{j=1}^{k}(\|v_j-u_j\|+\|D_{0^+}^\beta v_j-D_{0^+}^\beta u_j\|) \\
= & N_i\left(\sum_{i=1}^{k}a_i(t)\right)\| v-u \|_{X^k}.
\end{aligned}$$

Hence

$$\| Tv-Tu \|_{X^k}=\sum_{j=1}^{k}\| T_iv-T_iu \|_X\leq\left(\sum_{j=1}^{k}N_i\right)\sum_{i=1}^{k}a_i(t)\| v-u \|_{X^k}.$$

According to $\left(\sum_{i=1}^{k}N_i\right)\left(\sum_{i=1}^{k}a_i\right)<1$, we obtain that T is contraction operator. It follows from the Banach contraction principle that system (2.1) has a unique solution on [0,1], so the original system (1.1) has a unique solution.

Lemma 3.2. *Assume that* (H1)-(H4) *hold, then* $T:X^k\to X^k$ *is completely continuous.*

Proof. In view of continuity of the functions $f_i,\ i=1,2...k$, we see that the operator $T:X^k\to X^k$ is continuous. Let Ω be any bounded subset of X^k, for

$v=(v_1,v_2,...,v_k)\in\Omega,\ t\in[0,1]$, one can get

$$
\begin{aligned}
& |T_i v(t)| \\
\leq\ & \frac{l_i^{\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\right|ds \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\right|ds \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\right|ds \\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\right|ds \\
\leq\ & \frac{l_i^{\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)-\phi_q\left(f_i(s,0,0)\right)\right|ds \\
& +\frac{l_i^{\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\phi_q\left(f_i(s,0,0)\right)\right| \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)-\phi_q\left(f_j(s,0,0)\right)\right|ds \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_j(s,0,0)\right)\right|ds \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)ds-\phi_q\left(f_i(s,0,0)\right)\right|ds \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_i(s,0,0)\right)\right|ds \\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)-\phi_q\left(f_j(s,0,0)\right)\right|ds \\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\left|\phi_q\left(f_j(s,0,0)\right)\right|ds \\
\leq\ & \frac{l_i^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel+l_i^{-\beta}\parallel D_{0^+}^{\beta}v_i\parallel\Big) \\
& +\frac{l_i^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)|+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha+1)}|\phi_q(\lambda)|
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha+1)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j\parallel + l_j^{-\beta}\parallel D_{0^+}^{\beta}v_j\parallel\Big)\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel + l_i^{-\beta}\parallel D_{0^+}^{\beta}v_i\parallel\Big)\\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j\parallel + l_j^{-\beta}\parallel D_{0^+}^{\beta}v_j\parallel\Big)\\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}|\phi_q(\lambda)| + +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_j t^{\alpha-1}}{\Gamma(\alpha+1)}|\phi_q(\lambda)|\\
\leq\ & \frac{l_i^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel + l_i^{-\beta}\parallel D_{0^+}^{\beta}v_i\parallel\Big)\\
& +\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)| + \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)|\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j\parallel + l_j^{-\beta}\parallel D_{0^+}^{\beta}v_j\parallel\Big)\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel + l_i^{-\beta}\parallel D_{0^+}^{\beta}v_i\parallel\Big)\\
& +\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j\parallel + l_j^{-\beta}\parallel D_{0^+}^{\beta}v_j\parallel\Big)\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)| + \sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}|\phi_q(\lambda)|\\
\leq\ & \frac{l_i^{\alpha}+l_i^{\alpha-\beta}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel + \parallel D_{0^+}^{\beta}v_i\parallel\Big)\\
& +\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)| + \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)|\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{\Gamma(\alpha+1)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j\parallel + \parallel D_{0^+}^{\beta}v_j\parallel\Big)\\
& +\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}+l_i^{\alpha-\beta}}{\Gamma(\alpha+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel + \parallel D_{0^+}^{\beta}v_i\parallel\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j\parallel+\parallel D_{0^+}^{\beta}v_j\parallel\Big)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}}{\Gamma(\alpha+1)}|\phi_q(\lambda)|+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}|\phi_q(\lambda)|\\
= &\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\\
&+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X\\
&+\Bigg(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}\Bigg)|\phi_q(\lambda)|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|T_iv(t)\|\\
\leq &\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\\
&+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X\\
&+\Bigg(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}\Bigg)|\phi_q(\lambda)|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|D_{0^+}^{\beta}T_iv(t)\right|\\
\leq &\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\right|ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\right|ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\right|ds\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_jt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\right|ds\\
\leq &\frac{l_i^{\alpha}t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i\parallel+l_i^{-\beta}\parallel D_{0^+}^{\beta}v_i\parallel\Big)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{l_i^{\alpha} t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|\phi_q(\lambda)| + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}|\phi_q(\lambda)| \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j \parallel + l_j^{-\beta} \parallel D_{0^+}^{\beta} v_j \parallel \Big) \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_i^{q-2}a_i(t)\Big(\parallel v_i \parallel + l_i^{-\beta} \parallel D_{0^+}^{\beta} v_i \parallel \Big) \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}|\phi_q(\lambda)| + \sum_{j=1}^{k} \frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}|\phi_q(\lambda)| \\
& +\sum_{j=1}^{k} \frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}(q-1)L_j^{q-2}a_j(t)\Big(\parallel v_j \parallel + l_j^{-\beta} \parallel D_{0^+}^{\beta} v_j \parallel \Big) \\
\leq \; & \frac{l_i^{\alpha}+l_i^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}(q-1)L_i^{q-2}a_i(t) \parallel v_i \parallel_X + \frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}|\phi_q(\lambda)| \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_j^{q-2}a_j(t) \parallel v_j \parallel_X + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}|\phi_q(\lambda)| \\
& +\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_i^{\alpha}+l_i^{\alpha-\beta}}{\alpha\Gamma(\alpha-\beta)}(q-1)L_i^{q-2}a_i(t) \parallel v_i \parallel_X + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}|\phi_q(\lambda)| \\
& +\sum_{j=1}^{k} \frac{l_j^{\alpha}+l_j^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}(q-1)L_j^{q-2}a_j(t) \parallel v_j \parallel_X + \sum_{j=1}^{k} \frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}|\phi_q(\lambda)| \\
= \; & \left(\frac{2}{\Gamma(\alpha-\beta+1)} + \frac{1}{\alpha\Gamma(\alpha-\beta)} \right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X \\
& + \left(\frac{1}{\alpha\Gamma(\alpha-\beta)} + \frac{1}{\Gamma(\alpha-\beta+1)} \right) \sum_{\substack{j=1\\ j\neq i}}^{k} (q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X \\
& +\left(\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)} + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)} + \frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)} + \sum_{j=1}^{k} \frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)} \right)|\phi_q(\lambda)|,
\end{aligned}
$$

thus

$$
\begin{aligned}
& \|D_{0^+}^{\beta} T_i v(t)\| \\
\leq \; & \left(\frac{2}{\Gamma(\alpha-\beta+1)} + \frac{1}{\alpha\Gamma(\alpha-\beta)} \right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X
\end{aligned}
$$

$$+\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\sum_{\substack{j=1\\j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X$$

$$+\left(\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\right)|\phi_q(\lambda)|.$$

We obtain

$$\begin{aligned}
&\|T_iv\|_X\\
=\ &\|T_iv\|+\|D_{0^+}^{\beta}T_iv\|\\
\leq\ &(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-\beta+1)}\right.\\
&\left.+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\\
&\times\sum_{\substack{j=1\\j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X+\left(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}\right.\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}+\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}\\
&\left.+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\right)|\phi_q(\lambda)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|T_iv\|_{X^k}\\
=\ &\sum_{i=1}^{k}\|T_iv\|_X\\
\leq\ &(q-1)\sum_{i=1}^{k}L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right.\\
&\left.+\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right.\\
&\left.+\frac{1}{\Gamma(\alpha-\beta+1)}\right)(q-1)\sum_{\substack{j=1\\j\neq i}}^{k}L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X+\sum_{i=1}^{k}\left(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}\right.\\
&+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}+\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}\\
&\left.+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\right)|\phi_q(\lambda)|,
\end{aligned}$$

so, it follows that T is uniformly bounded.

Now it will prove that T is equi-continuous. For $v=(v_1,v_2,...,v_k)\in\Omega,\ t_1,\ t_2\in[0,1]$ with $t_1<t_2$, we have

$$\begin{aligned}
&|T_iv(t_2)-T_iv(t_1)|\\
\leq\ & \frac{l_i^\alpha}{\Gamma(\alpha)}\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\right|ds\\
&+\frac{l_i^\alpha}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\right|ds\\
&+\sum_{\substack{j=1\\j\neq i}}^k l_j^\alpha\ell_j\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^\beta v_j(s))\right)\right|ds\\
&+\sum_{\substack{j=1\\j\neq i}}^k l_i^\alpha\ell_j\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\right|ds\\
&+\sum_{j=1}^k l_j^\alpha\ell_j\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^\beta v_j(s))\right)\right|ds\\
\leq\ & l_i^\alpha\phi_q(L_i)\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\j\neq i}}^k l_j^\alpha\phi_q(L_j)\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha+1)}\\
&+\sum_{\substack{j=1\\j\neq i}}^k l_i^\alpha\phi_q(L_i)\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma(\alpha+1)}+\sum_{j=1}^k l_j^\alpha\phi_q(L_j)\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}\\
=\ & l_i^\alpha\phi_q(L_i)\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}+l_i^\alpha\phi_q(L_i)\left(\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}\right)(t_2^{\alpha-1}-t_1^{\alpha-1})\\
&+\sum_{\substack{j=1\\j\neq i}}^k l_j^\alpha\phi_q(L_j)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(t_2^{\alpha-1}-t_1^{\alpha-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0^+}^\beta T_iv(t_2)-D_{0^+}^\beta T_iv(t_1)\right|\\
\leq\ & \frac{l_i^\alpha}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left((t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right)\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\right|ds\\
&+\frac{l_i^\alpha}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^\beta v_i(s))\right)\right|ds\\
&+\sum_{\substack{j=1\\j\neq i}}^k l_j^\alpha\ell_j\frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^\beta v_j(s))\right)\right|ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{j=1\\ j\neq i}}^{k} l_i^{\alpha}\ell_j \frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(f_i(s,v_i(s),l_i^{-\beta}D_{0^+}^{\beta}v_i(s))\right)\right|ds\\
&+\sum_{j=1}^{k} l_j^{\alpha}\ell_j \frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\left|\phi_q\left(f_j(s,v_j(s),l_j^{-\beta}D_{0^+}^{\beta}v_j(s))\right)\right|ds\\
\leq\ & l_i^{\alpha}\phi_q(L_i)\frac{t_2^{\alpha-\beta}-t_1^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k} l_j^{\alpha}\phi_q(L_j)\frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\alpha\Gamma(\alpha-\beta)}\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k} l_i^{\alpha}\phi_q(L_i)\frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k} l_j^{\alpha}\phi_q(L_j)\frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\Gamma(\alpha-\beta+1)}\\
=\ & l_i^{\alpha}\phi_q(L_i)\frac{t_2^{\alpha-\beta}-t_1^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+l_i^{\alpha}\phi_q(L_i)(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1})\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k} l_j^{\alpha}\phi_q(L_j)(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1})\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|T_i v(t_2)-|T_i v(t_1)\|_X\\
\leq\ & l_i^{\alpha}\phi_q(L_i)\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}+l_i\phi_q(L_i)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(t_2^{\alpha-1}-t_1^{\alpha-1})\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k} l_j^{\alpha}\phi_q(L_j)\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(t_2^{\alpha-1}-t_1^{\alpha-1})+l_i^{\alpha}\phi_q(L_i)\frac{t_2^{\alpha-\beta}-t_1^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\
&+l_i^{\alpha}\phi_q(L_i)(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1})\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k} l_j^{\alpha}\phi_q(L_j)(t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1})\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right).
\end{aligned}
$$

In view of

$$\|T_i v(t_2)-T_i v(t_1)\|_X\to 0(t_2\to t_1),$$

there is

$$\|Tv(t_2)-Tv(t_1)\|_{X^k}\to 0(t_2\to t_1),$$

which implies that T of X is equicontinuous. It follows from the Arezela-Ascoli theorem that $T: X^k\to X^k$ is completely continuous.

Theorem 3.3. *Suppose that* (H1)-(H4) *hold, then fractional differential system* (2.1) *has at least one solution on* $[0,1]$.

Proof. Define

$$Q=\{(v_1,v_2,...,v_k)\in X^k:(v_1,v_2,...,v_k)=\mu T(v_1,v_2,...,v_k),\ 0<\mu<1\}.$$

Let $(v_1,v_2,...,v_k)\in Q$, then

$$(v_1,v_2,...,v_k)=\mu T(v_1,v_2,...,v_k),$$

and for each $t\in[0,1]$, we have

$$v_i(t)=\mu T_i(v_1,v_2,...,v_k),\ i=1,2,...,k.$$

It follow from (H3) that

$$\begin{aligned}
&|v_i(t)|\\
\le\ &\mu\Bigg(\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\\
&+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X\\
&+\left(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}\right)|\phi_q(\lambda)|\Bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|T_iv(t)\|\\
\le\ &\mu\Bigg(\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\\
&+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X\\
&+\left(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}\right)|\Phi_q(\lambda)|\Bigg).
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
&|D_{0^+}^{\beta}T_iv(t)|\\
\le\ &\mu\Bigg(\left(\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\\
&+\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\sum_{\substack{j=1\\j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X\\
&+\left(\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\right)|\phi_q(\lambda)|\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
&\|D^{\beta}_{0^+}T_iv(t)\| \\
\leq\ & \mu\Bigg(\left(\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)(q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X \\
&+\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(q-1)L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X \\
&+\Bigg(\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\Bigg)|\phi_q(\lambda)|\Bigg).
\end{aligned}$$

For all in all, we have

$$\begin{aligned}
&\|v_i\|+\|D^{\beta}_{0^+}v_i\| \\
\leq\ & \mu\Bigg((q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\Bigg(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)} \\
&+\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\Bigg)+(q-1)\sum_{\substack{j=1\\ j\neq i}}^{k}L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta}) \\
&\times\|v_j\|_X\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right) \\
&+\Bigg(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)}+\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)} \\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\Bigg)|\phi_q(\lambda)|\Bigg).
\end{aligned}$$

Let

$$\begin{aligned}
&P_i \\
=\ & (q-1)L_i^{q-2}a_i(t)(l_i^{\alpha}+l_i^{\alpha-\beta})\|v_i\|_X\Bigg(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-\beta+1)} \\
&+\frac{1}{\alpha\Gamma(\alpha-\beta)}\Bigg)+(q-1)\sum_{\substack{j=1\\ j\neq i}}^{k}L_j^{q-2}a_j(t)(l_j^{\alpha}+l_j^{\alpha-\beta})\|v_j\|_X\Bigg(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)} \\
&+\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\Bigg)+\Bigg(\frac{2l_i^{\alpha}}{\Gamma(\alpha+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{(\alpha-\beta)\Gamma(\alpha)} \\
&+\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta+1)}+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\frac{l_i^{\alpha}}{\alpha\Gamma(\alpha-\beta)}+\sum_{j=1}^{k}\frac{l_j^{\alpha}}{\Gamma(\alpha-\beta+1)}\Bigg)|\phi_q(\lambda)|.
\end{aligned}$$

Then, we obtain that

$$\|v\|_{X^k} = \sum_{i=1}^{k} \|v_i\|_X \leq \mu \left(\sum_{i=1}^{k} P_i \right) < \infty.$$

This show that the set Q is bounded. Hence, by Lemma 2.5, the operator T has at least one fixed point, which shows that the fractional differential system (2.1) has at least one solution on [0,1] and so fractional differential system (1.1) has at least one solution.

4. Ulam-Hyers Stability

Let $\varepsilon_i \leq 0$. Consider the following inequality

$$\left| D_{0+}^{\alpha} v_i(t) - l_i^{\alpha} \phi_q(f_i(t, v_i(t), l_i^{-\beta} D_{0+}^{\beta} v_i(t))) \right| \leq \varepsilon_i, \; t \in [0,1]. \tag{4.1}$$

Remark 4.1. Let function $v = (v_1, v_2, ..., v_k) \in X^k$ be the solution of system (4.1), If there are functions $\varphi_i : [0,1] \to \mathbb{R}^+$ dependent on v_i respectively, then

(i) $|\varphi_i(t)| \leq \varepsilon_i$, $t \in [0,1]$, $i = 1, 2, ..., k$;

(ii) $D_{0+}^{\alpha} v_i(t) = l_i^{\alpha} \phi_q(f_i(t, v_i(t), l_i^{-\beta} D_{0+}^{\beta} v_i(t))) + \varphi_i(t), t \in [0,1], i = 1, 2, ..., k.$

Definition 4.2. [18] The fractional differential system (2.1) is called Ulam-Hyers stable, if there is a constant $c_{f_1,f_2,...,f_n} > 0$ such that for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) > 0$ and for each solution $v = (v_1, v_2, ..., v_n) \in X$ of the inequality (4.1), there exists a solution $\bar{v} = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_n) \in X$ of (2.1) with

$$\|v - \bar{v}\|_X \leq c_{f_1,f_2,...,f_n} \varepsilon, \; t \in [0,1].$$

Definition 4.3. [18] The fractional differential system (2.1) is called generalized Ulam-Hyers stable, if there exists function $\psi_{f_1,f_2,...,f_n} \in \mathbb{C}(\mathbb{R}+, \mathbb{R}+)$ with $\psi_{f_1,f_2,...,f_n}(0) = 0$ such that for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) > 0$ and for each solution $v = (v_1, v_2, ..., v_k) \in X$ of the inequality (4.1), there exists a solution $\bar{v} = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_k) \in X$ of (2.1) with

$$\|v - \bar{v}\|_X \leq \psi_{f_1,f_2,...,f_n}(\varepsilon), \; t \in [0,1].$$

Lemma 4.4. *Suppose $v = (v_1, v_2, ..., v_k) \in X^k$ is the solution of inequality (4.1). Then, the following inequality holds:*

$$|v_i(t) - w_i(t)| \leq \frac{\varepsilon_i}{\Gamma(\alpha+1)} + \sum_{j=1}^{k} \frac{2\alpha - \beta}{(\alpha - \beta)\Gamma(\alpha+1)} \varepsilon_j,$$

$$|D^{\beta}_{0+}v_i(t) - D^{\beta}_{0+}w_i(t)| \leq \frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\varepsilon_i + \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\varepsilon_j,$$

where

$$\begin{aligned} & w_i(t) \\ = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h_i(s)ds + \sum_{\substack{j=1\\ j\neq i}}^{k}\ell_j\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h_j(s)ds \\ & -\sum_{\substack{j=1\\ j\neq i}}^{k}\ell_j\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h_i(s)ds - \sum_{j=1}^{k}\ell_j\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}h_j(s)ds, \end{aligned}$$

$$\begin{aligned} & D^{\beta}_{0+}w_i(t) \\ = & \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}h_i(s)ds + \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}h_j(s)ds \\ & -\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}h_i(s)ds - \sum_{j=1}^{k}\frac{\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}h_j(s)ds, \end{aligned}$$

and here

$$h_i(s) = l_i^{\alpha}\phi_q(f_i(t, v_i(t), l_i^{-\beta}D^{\beta}_{0+}v_i(t))), i = 1, 2, ..., k.$$

Proof. From Remark 1, we have

$$\left\{\begin{array}{l} D^{\alpha}_{0+}v_i(t) = l_i^{\alpha}\phi_q(f_i(t, v_i(t), l_i^{-\beta}D^{\beta}_{0+}v_i(t))) + \varphi_i(t),\ t\in[0,1], \\ v_i(0) = v_i'(0) = 0, \\ v_i(1) = v_j(1),\ i, j = 1, 2, ..., k,\ i\neq j, \\ \sum_{i=1}^{k} l_i^{-\beta}D^{\beta}_{0+}v_i(1) = 0,\ i = 1, 2, ..., k. \end{array}\right. \tag{4.2}$$

By Lemma 2.7, the solution of (4.2) can be given in the following form

$$\begin{aligned} v_i(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(h_i(s)+\varphi_i(s)\right)ds \\ & +\sum_{\substack{j=1\\ j\neq i}}^{k}\ell_j\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left(h_j(s)+\varphi_j(s)\right)ds \\ & -\sum_{\substack{j=1\\ j\neq i}}^{k}\ell_j\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left(h_i(s)+\varphi_i(s)\right)ds \end{aligned}$$

$$- \sum_{j=1}^{k} \ell_j \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \left(h_j(s) + \varphi_j(s)\right) ds,$$

and

$$\begin{aligned} D_{0^+}^{\beta} v_i(t) &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \left(h_i(s) + \varphi_i(s)\right) ds \\ &\quad + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} \left(h_j(s) + \varphi_j(s)\right) ds \\ &\quad - \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} \left(h_i(s) + \varphi_i(s)\right) ds \\ &\quad - \sum_{j=1}^{k} \frac{\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} \left(h_j(s) + \varphi_j(s)\right) ds. \end{aligned}$$

Then, we deduce that

$$\begin{aligned} |v_i(t) - w_i(t)| &\leq \frac{\varepsilon_i}{\Gamma(\alpha+1)} + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\ell_j \varepsilon_j}{\Gamma(\alpha+1)} + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\ell_j \varepsilon_i}{\Gamma(\alpha+1)} + \sum_{j=1}^{k} \frac{\ell_j \varepsilon_j}{(\alpha-\beta)\Gamma(\alpha)} \\ &\leq \frac{\varepsilon_i}{\Gamma(\alpha+1)} + \sum_{j=1}^{k} \frac{2\alpha-\beta}{(\alpha-\beta)\Gamma(\alpha+1)} \varepsilon_j, \end{aligned}$$

and

$$\begin{aligned} &\left| D_{0^+}^{\beta} v_i(t) - D_{0^+}^{\beta} w_i(t) \right| \\ &\leq \frac{\varepsilon_i}{\Gamma(\alpha-\beta+1)} + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\ell_j \varepsilon_j}{\alpha\Gamma(\alpha-\beta)} + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\ell_j \varepsilon_i}{\alpha\Gamma(\alpha-\beta)} + \sum_{j=1}^{k} \frac{\ell_j \varepsilon_j}{\Gamma(\alpha-\beta+1)} \\ &\leq \frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)} \varepsilon_i + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)} \varepsilon_j. \end{aligned}$$

Theorem 4.5. *Assume that Theorem* 3.1 *hold, then the fractional differential system* (2.1) *is Ulam-Hyers stable if the eigenvalues of matrix A are in the open unit disc, that is, $|\lambda| < 1$, for $\lambda \in \mathbb{C}$ with $det(\lambda I - A) = 0$, where*

$$A = \begin{pmatrix} \gamma_1(l_1^{\alpha} + l_1^{\alpha-\beta})(q-1)L_1^{q-2}a_1 & \gamma_2(l_2^{\alpha} + l_2^{\alpha-\beta})(q-1)L_2^{q-2}a_2 & \cdots & \gamma_2(l_k^{\alpha} + l_k^{\alpha-\beta})(q-1)L_k^{q-2}a_k \\ \gamma_2(l_1^{\alpha} + l_1^{\alpha-\beta})(q-1)L_1^{q-2}a_1 & \gamma_1(l_2^{\alpha} + l_2^{\alpha-\beta})(q-1)L_2^{q-2}a_2 & \cdots & \gamma_2(l_k^{\alpha} + l_k^{\alpha-\beta})(q-1)L_k^{q-2}a_k \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_2(l_1^{\alpha} + l_1^{\alpha-\beta})(q-1)L_1^{q-2}a_1 & \gamma_2(l_2^{\alpha} + l_2^{\alpha-\beta})(q-1)L_2^{q-2}a_2 & \cdots & \gamma_1(l_k^{\alpha} + l_k^{\alpha-\beta})(q-1)L_k^{q-2}a_k \end{pmatrix}.$$

Proof. Let $v = (v_1, v_2, ..., v_k) \in X^k$ be the solution of the inequality given by

$$\left| D^{\alpha}_{0+} v_i(t) - l_i^{\alpha} \phi_q(f_i(t, v_i(t), l_i^{-\beta} D^{\beta}_{0+} v_i(t))) \right| \le \varepsilon_i, \ t \in [0,1], i = 1, 2, ..., k,$$

and $\bar{v} = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_k) \in X^k$ be the solution of the following system

$$\begin{cases} D^{\alpha}_{0+} \bar{v}_i(t) = l_i^{\alpha} \phi_q(f_i(t, \bar{v}_i(t), l_i^{-\beta} D^{\beta}_{0+} \bar{v}_i(t))), \ t \in [0,1], \\ \bar{v}_i(0) = \bar{v}_i'(0) = 0, \\ \bar{v}_i(1) = \bar{v}_j(1), \ i, j = 1, 2, ..., k, \ i \neq j, \\ \sum_{i=1}^{k} l_i^{-\beta} D^{\beta}_{0+} \bar{v}_i(1) = 0, \ i = 1, 2, ..., k. \end{cases} \tag{4.3}$$

By Lemma 2.7, the solution of (4.3) can be given in the following form

$$\begin{aligned} \bar{v}_i(t) =& \frac{l_i^{\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(f_i(s, \bar{v}_i(s), l_i^{-\beta} D^{\beta}_{0+} \bar{v}_i(s)) \right) ds \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{l_j^{\alpha} \ell_j t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(f_j(s, \bar{v}_j(s), l_j^{-\beta} D^{\beta}_{0+} \bar{v}_j(s)) \right) ds \\ &- \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{l_i^{\alpha} \ell_j t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(f_i(s, \bar{v}_i(s), l_i^{-\beta} D^{\beta}_{0+} \bar{v}_i(s)) \right) ds \\ &- \sum_{j=1}^{k} \frac{l_j^{\alpha} \ell_j t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \phi_q \left(f_j(s, \bar{v}_j(s), l_j^{-\beta} D^{\beta}_{0+} \bar{v}_j(s)) \right) ds. \end{aligned}$$

For convenience, here we make

$$g_i(s) = \phi_q \left(f_i(s, v_i(s), l_i^{-\beta} D^{\beta}_{0+} v_i(s)) \right) - \phi_q \left(f_i(s, \bar{v}_i(s), l_i^{-\beta} D^{\beta}_{0+} \bar{v}_i(s)) \right).$$

Now, by Lemma 4.4, for $t \in [0,1]$, we can get

$$\begin{aligned} & |v_i(t) - \bar{v}_i(t)| \\ \le \ & |v_i(t) - w_i(t)| + |w_i(t) - \bar{v}_i(t)| \\ \le \ & \frac{\varepsilon_i}{\Gamma(\alpha+1)} + \sum_{j=1}^{k} \frac{2\alpha - \beta}{(\alpha-\beta)\Gamma(\alpha+1)} \varepsilon_j + \frac{l_i^{\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_i(s)| ds \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{l_j^{\alpha} \ell_j t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |g_j(s)| ds + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{l_i^{\alpha} \ell_j t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\ & \times |g_i(s)| ds + \sum_{j=1}^{k} \frac{l_j^{\alpha} \ell_j t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} |g_j(s)| ds \\ \le \ & \frac{\varepsilon_i}{\Gamma(\alpha+1)} + \sum_{j=1}^{k} \frac{2\alpha - \beta}{(\alpha-\beta)\Gamma(\alpha+1)} \varepsilon_j + \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{(\alpha-\beta)\Gamma(\alpha)} \right) \\ & \times (l_i^{\alpha} + l_i^{\alpha-\beta})(q-1) L_i^{q-2} a_i(t) (\|v_i - \bar{v}_i\| + \|D^{\beta}_{0+} v_i - D^{\beta}_{0+} \bar{v}_i\|) \end{aligned}$$

$$+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(\alpha-\beta)\Gamma(\alpha)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^{\alpha}+l_j^{\alpha-\beta})(q-1)L_j^{q-2}a_j(t)$$
$$\times(\|v_j-\bar{v}_j\|+\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}\bar{v}_j\|),$$

and

$$\begin{aligned}
&\left|D_{0^+}^{\beta}v_i(t)-D_{0^+}^{\beta}\bar{v}_i(t)\right|\\
\leq\ & \frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\varepsilon_i+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\varepsilon_j\\
&+\frac{l_i^{\alpha}}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}|g_i(s)|ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}|g_j(s)|ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{l_i^{\alpha}\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}|g_i(s)|ds\\
&+\sum_{j=1}^{k}\frac{l_j^{\alpha}\ell_j t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}|g_j(s)|ds\\
\leq\ & \frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\varepsilon_i+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\varepsilon_j\\
&+\left(\frac{2}{\Gamma(\alpha-\beta+1)}+\frac{1}{\alpha\Gamma(\alpha-\beta)}\right)\\
&\times(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)(\|v_i-\bar{v}_i\|+\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}\bar{v}_i\|)\\
&+\left(\frac{1}{\alpha\Gamma(\alpha-\beta)}+\frac{1}{\Gamma(\alpha-\beta+1)}\right)\sum_{\substack{j=1\\ j\neq i}}^{k}(l_j^{\alpha}+l_j^{\alpha-\beta})(q-1)L_j^{q-2}\\
&\times a_j(t)(\|v_j-\bar{v}_j\|+\|D_{0^+}^{\beta}v_j-D_{0^+}^{\beta}\bar{v}_j\|).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|v_i-\bar{v}_i\|_X\\
=\ & \|v_i-\bar{v}_i\|+\|D_{0^+}^{\beta}v_i-D_{0^+}^{\beta}\bar{v}_i\|\\
\leq\ & \left(\frac{3\alpha-2\beta}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\right)\varepsilon_i+\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{2\alpha-\beta}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\right)\varepsilon_j
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{3\alpha-2\beta}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\right)(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)\|v_i-\bar{v}_i\|_X\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\frac{2\alpha-\beta}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}\right)(l_j^{\alpha}+l_j^{\alpha-\beta})(q-1)L_j^{q-2}a_j(t)\|v_j-\bar{v}_j\|_X\\
=\;&\gamma_1\varepsilon_i+\sum_{\substack{j=1\\ j\neq i}}^{k}\gamma_2\varepsilon_j+\gamma_1(l_i^{\alpha}+l_i^{\alpha-\beta})(q-1)L_i^{q-2}a_i(t)\|v_i-\bar{v}_i\|_X\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\gamma_2(l_j^{\alpha}+l_j^{\alpha-\beta})(q-1)L_j^{q-2}a_j(t)\|v_j-\bar{v}_j\|_X,
\end{aligned}
$$

where

$$\gamma_1=\frac{3\alpha-2\beta}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{3\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)},\ \gamma_2=\frac{2\alpha-\beta}{(\alpha-\beta)\Gamma(\alpha+1)}+\frac{2\alpha-\beta}{\alpha\Gamma(\alpha-\beta+1)}.$$

Then we have

$$
\begin{aligned}
&(\|v_1-\bar{v}_1\|_X,\|v_2-\bar{v}_2\|_X,...,\|v_k-\bar{v}_k\|_X)^T\\
\leq\;&B(\varepsilon_1,\varepsilon_2,...,\varepsilon_k)^T+A(\|v_1-\bar{v}_1\|_X,\|v_2-\bar{v}_2\|_X,...,\|v_k-\bar{v}_k\|_X)^T,
\end{aligned}
$$

where

$$B_{k\times k}=\begin{pmatrix}\gamma_1&\gamma_2&\cdots&\gamma_2\\ \gamma_2&\gamma_1&\cdots&\gamma_2\\ \vdots&\vdots&\ddots&\vdots\\ \gamma_2&\gamma_2&\cdots&\gamma_1\end{pmatrix}.$$

Then, we can get

$$(\|v_1-\bar{v}_1\|_X,\|v_2-\bar{v}_2\|_X,...,\|v_k-\bar{v}_k\|_X)^T\leq(I-A)_{-1}B(\varepsilon_1,\varepsilon_2,...,\varepsilon_k)^T.$$

Let

$$D=(I-A)^{-1}B=\begin{pmatrix}d_{11}&d_{12}&\cdots&d_{1k}\\ d_{21}&d_{22}&\cdots&d_{2k}\\ \vdots&\vdots&\ddots&\vdots\\ d_{k1}&d_{k2}&\cdots&d_{kk}\end{pmatrix}.$$

Obviously, $d_{ij}>0$. Set $\varepsilon=max\{\varepsilon_1,\varepsilon_2,...,\varepsilon_k\}$, then wo can get

$$\|v-\bar{v}\|_X\leq\Big(\sum_{j=1}^{k}\sum_{i=1}^{k}d_{ij}\Big)\varepsilon \tag{4.4}$$

Thus, we have derived that system (2.1) is Ulam-Hyers stable.

Remark 4.6. Making $\psi_{f_1,f_2,...,f_k}(\varepsilon)$ in (4.4). We have $\psi_{f_1,f_2,...,f_k}(0)=0$. Then , by Definition 4.3, we deduce that the fractional differential system (2.1) is generalized Ulam-Hyers stable.

5. Numerical Simulation

In this section, the two examples are provided to show the flexibility of these criteria, in addition, the approximate graphs of solutions are presented by using iterative methods and numerical simulation.

Example 5.1. We investigate the existence and Ulam's stability of solutions to the differential equations on each edge of the star graphs as follows

$$
(5.1)\quad \begin{cases}
\phi_{\frac{3}{2}}\left(D_{0+}^{\frac{5}{2}}\mathfrak{u}_1(x)\right)=1+\frac{1}{(2x+8)^4}\left(sin(\mathfrak{u}_1(x))+\frac{|D_{0+}^{\frac{1}{2}}\mathfrak{u}_1(x)|}{1+|D_{0+}^{\frac{1}{2}}\mathfrak{u}_1(x)|}\right),0\le x\le \frac{1}{3},\\
\phi_{\frac{3}{2}}\left(D_{0+}^{\frac{5}{2}}\mathfrak{u}_2(x)\right)=\frac{1}{3(x^3+2)^5}\left(sin|\mathfrak{u}_2(x)|+\frac{|D_{0+}^{\frac{1}{2}}\mathfrak{u}_2(x)|}{1+|D_{0+}^{\frac{1}{2}}\mathfrak{u}_2(x)|}\right)+1,0\le x\le \frac{1}{4},\\
\phi_{\frac{3}{2}}\left(D_{0+}^{\frac{5}{2}}\mathfrak{u}_3(x)\right)=1+0.02x|arcsin(\mathfrak{u}_3(x))|+\frac{2x|D_{0+}^{\frac{1}{2}}\mathfrak{u}_3(x)|}{100+100|D_{0+}^{\frac{1}{2}}\mathfrak{u}_3(x)|},0\le x\le \frac{3}{4},\\
\mathfrak{u}_1(0)=\mathfrak{u}_1'(0)=\mathfrak{u}_2(0)=\mathfrak{u}_2'(0)=\mathfrak{u}_3(0)=\mathfrak{u}_3'(0)=0,\\
\mathfrak{u}_1(\frac{1}{3})=\mathfrak{u}_2(\frac{1}{4})=\mathfrak{u}_3(\frac{3}{4}),\\
D_{0+}^{\frac{1}{2}}\mathfrak{u}_1(\frac{1}{3})+D_{0+}^{\frac{1}{2}}\mathfrak{u}_2(\frac{1}{4})+D_{0+}^{\frac{1}{2}}\mathfrak{u}_3(\frac{3}{4})=0,
\end{cases}
$$

Corresponding to the system (1.1), we obtain

$$k=3,\alpha=\frac{5}{2},\beta=\frac{1}{2},p=\frac{3}{2},l_1=\frac{1}{3},l_2=\frac{1}{4},l_3=\frac{3}{4}.$$

We establish coordinate systems with v_1, v_2, and v_3 as coordinate origin respectively on the star graphs with 3 edges (Figure 4), where $\mathfrak{u}_1$ is the solution of the system (5.1) on $\overrightarrow{v_1v_0}$, $t\in[0,l_1]$; $\mathfrak{u}_2$ is the solution of the system (5.1) on $\overrightarrow{v_2v_0}$, $t\in[0,l_2]$; and $\mathfrak{u}_3$ is the solution of the system (5.1) on $\overrightarrow{v_2v_0}$, $t\in[0,l_3]$. Here,

$$l_1=|e_1|=|\overrightarrow{v_1v_0}|=\frac{1}{3},$$

$$l_2=|e_2|=|\overrightarrow{v_2v_0}|=\frac{1}{4},$$

and

$$l_3=|e_3|=|\overrightarrow{v_3v_0}|=\frac{3}{4}.$$

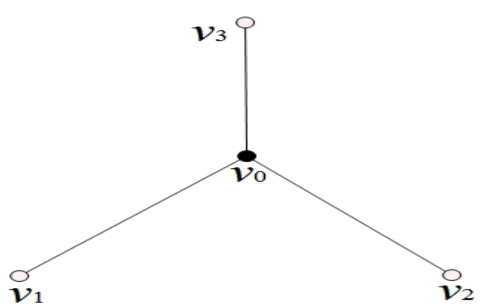

FIGURE 3. A sketch of the star graphs.

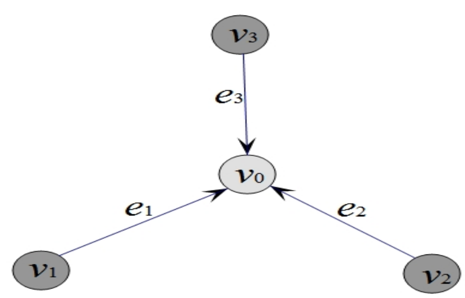

FIGURE 4. A sketch of the directed star graphs .

In the following, we will prove that system (5.1) has a unique solution on each edge and is Ulam stable.

It follows from Lemma 2.6 that we can get

$$(5.2)\quad \begin{cases} D_{0+}^{\frac{5}{2}} v_1(t) = (\frac{1}{3})^{\frac{5}{2}} \phi_3 \left(1 + \frac{1}{(2t+8)^4}\left(sin(v_1(t)) + \frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_1(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_1(t)|}\right)\right), \\ D_{0+}^{\frac{5}{2}} v_2(t) = (\frac{1}{4})^{\frac{5}{2}} \phi_3 \left(\frac{1}{3(t^3+2)^5}\left(sin|v_2(t)| + \frac{(\frac{1}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_2(t)|}{1+(\frac{1}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_2(t)|}\right) + 1\right), \\ D_{0+}^{\frac{5}{2}} v_3(t) = (\frac{1}{3})^{\frac{5}{2}} \phi_3 \left(1 + 0.02t|arcsin(v_3(t))| + \frac{2t\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_3(t)|}{100+100\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_3(t)|}\right), \\ v_1(0) = v_1'(0) = v_2(0) = v_2'(0) = v_3(0) = v_3'(0), \\ v_1(1) = v_2(1) = v_3(1), \\ (\frac{1}{3})^{-\frac{1}{2}} D_{0+}^{\frac{1}{2}} v_1(1) + (\frac{1}{4})^{-\frac{1}{2}} D_{0+}^{\frac{1}{2}} v_2(1) + (\frac{3}{4})^{-\frac{1}{2}} D_{0+}^{\frac{1}{2}} v_3(1) = 0 \end{cases}$$

where $t \in [0,1]$, and

$$f_1(t, v_1(t), l_1 D_{0+}^{\beta} v_1(t)) = 1 + \frac{1}{(2t+8)^4}\left(sin(v_1(t)) + \frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_1(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_1(t)|}\right),$$

$$f_2(t, v_2(t), l_2 D_{0+}^{\beta} v_2(t)) = \frac{1}{3(t^3+2)^5}\left(sin|v_2(t)| + \frac{(\frac{1}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_2(t)|}{1+(\frac{1}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_2(t)|}\right) + 1,$$

$$f_3(t, v_3(t), l_3 D_{0+}^{\beta} v_3(t)) = 1 + 0.02t|arcsin(v_3(t))| + \frac{2t\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_3(t)|}{100+100\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0+}^{\frac{1}{2}} v_3(t)|}.$$

For $t \in [0,1]$ and $u, v, u_1, v_1 \in \mathbb{R}$, it is clear that

$$f_1(t,u,v) - f(t,u_1,v_1) \le \frac{1}{(2t+8)^4}\left(|u-u_1| + |v-v_1|\right),$$

$$f_2(t,u,v) - f(t,u_2,v_1) \le \frac{1}{3(t^3+2)^5}\left(|u-u_2| + |v-v_2|\right),$$

and

$$f_3(t,u,v) - f(t,u_3,v_3) \le \frac{2t}{100}\left(|u-u_3| + |v-v_3|\right).$$

So we have

$$b_1(t) = \frac{1}{(2t+8)^4}, b_2(t) = \frac{1}{3(t^3+2)^5}, b_3(t) = \frac{2t}{100}.$$

Hence

$$a_1 = \sup_{t\in[0,1]} |b_1(t)| = \frac{1}{4096}, a_2 = \sup_{t\in[0,1]} |b_2(t)| = \frac{1}{96}, a_3 = \sup_{t\in[0,1]} |b_3(t)| = \frac{1}{50},$$

$$L_1 = 1.005, L_2 = 1.0208, L_3 = 2,$$

$$N_1 = 7.7570, N_2 = 7.6294, N_3 = 10.8388,$$

and

$$(N_1 + N_2 + N_3)(a_1 + a_2 + a_3) = 0.8041 < 1.$$

Therefore, It follows from Theorem 3.1 that system (5.2) has a unique solution on $[0, 1]$ and so system (5.1) has a unique solution.

$$\gamma_1 = 2.3779,\ \ \gamma_2 = 1.5770.$$

$$A = \begin{pmatrix} 2.0451e-04 & 0.0031 & 0.1324 \\ 1.3563e-04 & 0.0047 & 0.1324 \\ 1.3563e-04 & 0.0031 & 0.1997 \end{pmatrix}.$$

Let

$$0 = det(\lambda I - A) = (\lambda - 0.2019)(\lambda - 0.0001)(\lambda - 0.0026),$$

so we have

$$\lambda_1 = 0.2019 < 1,\ \ \lambda_2 = 0.0001 < 1,\ \ \lambda_3 = 0.0026.$$

It follows from Theorem 4.5 that system (5.2) is Ulam-Hyers stable, and by Remark 4.6, it will be generalized Ulam-Hyers stable.

Finally, the simulate iterative process curve and approximate solution to the fractional differential system (5.2) are given by using the iterative method and numerical simulation. Let $u_i(t) = l_i^{-\beta} D_{0+}^{\beta} v_i(t))$, where $v_{i,0} = u_{i,0} = 0$, the iteration sequence is as follows,

$$\begin{aligned}
& v_{1,n+1}(t) \\
= & \frac{(\frac{1}{3})^{\frac{5}{2}}}{\Gamma(\frac{5}{2})} \int_0^t (t-s)^{\frac{3}{2}} \phi_3\Bigg(1 + \frac{1}{(2t+8)^4}\Bigg(sin(v_{1,n}(t)) + \frac{(\frac{1}{3})^{-\frac{1}{2}} |D_{0+}^{\frac{1}{2}} v_{1,n}(t)|}{1 + (\frac{1}{3})^{-\frac{1}{2}} |D_{0+}^{\frac{1}{2}} v_{1,n}(t)|}\Bigg)\Bigg) ds \\
& + \frac{\frac{1}{16} t^{\frac{3}{2}}}{\Big((\frac{1}{3})^{-\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}} + (\frac{3}{4})^{-\frac{1}{2}}\Big)\Gamma(\frac{5}{2})} \int_0^1 (1-s)^{\frac{3}{2}} \phi_3\Bigg(\frac{1}{3(t^3+2)^5}\, (sin|v_{2,n}(t)| \\
& + \frac{(\frac{1}{4})^{-\frac{1}{2}} |D_{0+}^{\frac{1}{2}} v_{2,n}(t)|}{1 + (\frac{1}{4})^{-\frac{1}{2}} |D_{0+}^{\frac{1}{2}} v_{2,n}(t)|}\Bigg) + 1\Bigg) ds + \frac{\frac{9}{16} t^{\frac{3}{2}}}{\Big((\frac{1}{3})^{-\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}} + (\frac{3}{4})^{-\frac{1}{2}}\Big)\Gamma(\frac{5}{2})} \\
& \times \int_0^1 (1-s)^{\frac{3}{2}} \phi_3\Bigg(1 + 0.02t|arcsin(v_{3,n}(t))| + \frac{2t \times (\frac{3}{4})^{-\frac{1}{2}} |D_{0+}^{\frac{1}{2}} v_{3,n}(t)|}{100 + 100 \times (\frac{3}{4})^{-\frac{1}{2}} |D_{0+}^{\frac{1}{2}} v_{3,n}(t)|}\Bigg) ds \\
& - \frac{(\frac{1}{4})^{-\frac{1}{2}} (\frac{1}{3})^{\frac{5}{2}} t^{\frac{3}{2}}}{\Big((\frac{1}{3})^{-\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}} + (\frac{3}{4})^{-\frac{1}{2}}\Big)\Gamma(\frac{5}{2})} \int_0^1 (1-s)^{\frac{3}{2}} \phi_3\Bigg(1 + \frac{1}{(2t+8)^4}\Bigg(sin(v_{1,n}(t))
\end{aligned}$$

$$+\frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}\Bigg)\Bigg)ds-\frac{(\frac{1}{3})^{\frac{5}{2}}(\frac{3}{4})^{-\frac{1}{2}}t^{\frac{3}{2}}}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(\frac{5}{2})}$$

$$\times\int_0^1(1-s)^{\frac{3}{2}}\phi_3\Bigg(1+\frac{1}{(2t+8)^4}\Bigg(sin(v_{1,n}(t))+\frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}\Bigg)\Bigg)ds$$

$$-\frac{\frac{1}{9}t^{\frac{3}{2}}}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(\frac{5}{2})}\int_0^1(1-s)\phi_3\Bigg(1+\frac{1}{(2t+8)^4}\Bigg(sin(v_{1,n}(t))$$

$$+\frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}\Bigg)\Bigg)ds-\frac{\frac{1}{16}t^{\frac{3}{2}}}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(\frac{5}{2})}\int_0^1(1-s)$$

$$\times\phi_3\Bigg(\frac{1}{3(t^3+2)^5}\Bigg(sin|v_{2,n}(t)|+\frac{(\frac{1}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{2,n}(t)|}{1+(\frac{1}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{2,n}(t)|}\Bigg)+1\Bigg)ds$$

$$-\frac{\frac{9}{16}t^{\frac{3}{2}}}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(\frac{5}{2})}\int_0^1(1-s)\phi_3\Bigg(1+0.02t|arcsin(v_{3,n}(t))|$$

$$+\frac{2t\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{3,n}(t)|}{100+100\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{3,n}(t)|}\Bigg)ds.$$

$$u_{1,n+1}(t)$$

$$=\left(\frac{1}{3}\right)^{-\frac{1}{2}}\times\Bigg(\frac{(\frac{1}{3})^{\frac{5}{2}}}{\Gamma(2)}\int_0^t(t-s)\phi_3\Bigg(1+\frac{1}{(2t+8)^4}\Bigg(sin(v_{1,n}(t))+\frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}\Bigg)\Bigg)ds$$

$$+\frac{\frac{1}{16}t}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(2)}\int_0^1(1-s)^{\frac{3}{2}}\phi_3\Bigg(\frac{1}{3(t^3+2)^5}\Bigg(sin|v_{2,n}(t)|$$

$$+\frac{(\frac{1}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{2,n}(t)|}{1+(\frac{1}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{2,n}(t)|}\Bigg)+1\Bigg)ds+\frac{\frac{9}{16}t}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(2)}\int_0^1(1-s)^{\frac{3}{2}}$$

$$\times\phi_3\Bigg(1+0.02t|arcsin(v_{3,n}(t))|+\frac{2t\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{3,n}(t)|}{100+100\times(\frac{3}{4})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{3,n}(t)|}\Bigg)ds$$

$$-\frac{(\frac{1}{4})^{-\frac{1}{2}}(\frac{1}{3})^{\frac{5}{2}}t}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(2)}\int_0^1(1-s)^{\frac{3}{2}}\phi_3\Bigg(1+\frac{1}{(2t+8)^4}\Bigg(sin(v_{1,n}(t))$$

$$+\frac{(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}{1+(\frac{1}{3})^{-\frac{1}{2}}|D_{0^+}^{\frac{1}{2}}v_{1,n}(t)|}\Bigg)\Bigg)ds-\frac{(\frac{1}{3})^{\frac{5}{2}}(\frac{3}{4})^{-\frac{1}{2}}t}{\Bigg((\frac{1}{3})^{-\frac{1}{2}}+(\frac{1}{4})^{-\frac{1}{2}}+(\frac{3}{4})^{-\frac{1}{2}}\Bigg)\Gamma(2)}$$

$$\times \int_0^1 (1-s)^{\frac{3}{2}} \phi_3 \left(1 + \frac{1}{(2t+8)^4}\left(sin(v_{1,n}(t)) + \frac{(\frac{1}{3})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{1,n}(t)|}{1 + (\frac{1}{3})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{1,n}(t)|}\right)\right) ds$$

$$- \frac{\frac{1}{9}t}{\left((\frac{1}{3})^{-\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}} + (\frac{3}{4})^{-\frac{1}{2}}\right)\Gamma(2)} \int_0^1 (1-s)\phi_3 \left(1 + \frac{1}{(2t+8)^4}\left(sin(v_{1,n}(t))\right.\right.$$

$$\left.\left. + \frac{(\frac{1}{3})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{1,n}(t)|}{1 + (\frac{1}{3})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{1,n}(t)|}\right)\right) ds - \frac{\frac{1}{16}t}{\left((\frac{1}{3})^{-\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}} + (\frac{3}{4})^{-\frac{1}{2}}\right)\Gamma(2)} \int_0^1 (1-s)$$

$$\times \phi_3 \left(\frac{1}{3(t^3+2)^5}\left(sin|v_{2,n}(t)| + \frac{(\frac{1}{4})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{2,n}(t)|}{1 + (\frac{1}{4})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{2,n}(t)|}\right) + 1\right) ds$$

$$- \frac{\frac{9}{16}t}{\left((\frac{1}{3})^{-\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}} + (\frac{3}{4})^{-\frac{1}{2}}\right)\Gamma(2)} \int_0^1 (1-s)\phi_3 \left(1 + 0.02t|arcsin(v_{3,n}(t))|\right.$$

$$\left. + \frac{2t \times (\frac{3}{4})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{3,n}(t)|}{100 + 100 \times (\frac{3}{4})^{-\frac{1}{2}} |D_{0^+}^{\frac{1}{2}} v_{3,n}(t)|}\right) ds \Bigg).$$

The iterative sequence of $v_{2,n+1}$ and $v_{3,n+1}$ is similar to $v_{1,n+1}$, and the iterative sequence of $u_{2,n+1}$ and $u_{3,n+1}$ is similar to $u_{1,n+1}$. After several iterations, the approximate solution of fractional differential system (5.2) can be obtained by using the numerical simulation. The absolute errors for the iterative approach to system (5.2) are shown in Table 1, which demonstrates the applicability of the iterative approach, where $E_k = |v_{i,k+1} - v_{i,k}|$. Figure 5 is the approximate graph of the solution of $\overrightarrow{v_1 v_0}$ after 10 iterations. Figure 6 is the approximate graph of the solution of $\overrightarrow{v_2 v_0}$ after 10 iterations. Figure 7 is the approximate graph of the solution of $\overrightarrow{v_3 v_0}$ after 10 iterations.

TABLE 1. The absolute errors in Example 5.1

k	$E(k)$ for v_1	$E(k)$ for v_2	$E(k)$ for v_3
0	1.7208439898e-03	8.3918382213e-04	1.7451763972e-04
1	2.3766095090e-07	2.1145425774e-06	1.3741570428e-05
2	6.1177701793e-11	3.1584249972e-11	1.7010818504e-08
3	7.0013439490e-15	2.2973983826e-13	2.1037602216e-11
4	7.0013422247e-17	1.0928757899e-16	2.6700863742e-14

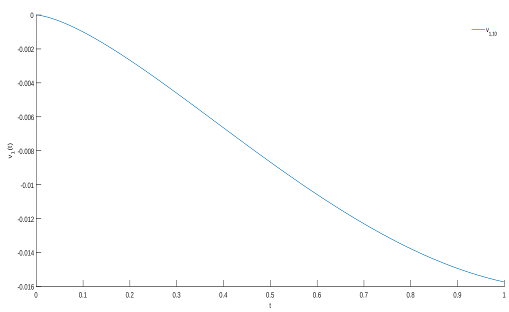

FIGURE 5. Approximate solution of v_1

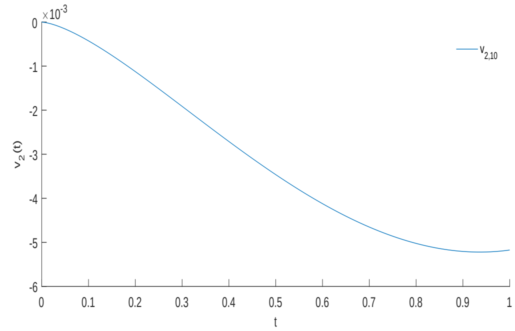

FIGURE 6. Approximate solution of v_2

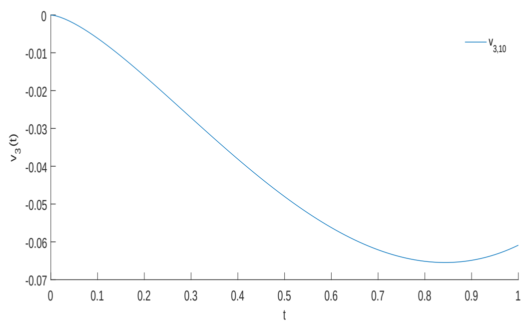

FIGURE 7. Approximate solution of v_3

Example 5.2. The star graphs we studied in system (1.1) can be extended to other types of graphs. For example, chordal bipartite graphs , ethane graphs, etc. provide theoretical basis for computer network, biology, chemical engineering and other fields. Here we only deal with boundary value problem of fractional differential system on formaldehyde graphs (Figure 9) as follows

$$(5.3)\quad \begin{cases} \phi_{\frac{4}{3}}\left(D_{0^+}^{\frac{7}{3}}\mathfrak{u}_1(x)\right) = 1+\frac{1}{2(x+3)^3}\left(sin(\mathfrak{u}_1(x))+|D_{0^+}^{\frac{1}{3}}\mathfrak{u}_1(x)|\right), 0\le x\le 1,\\ \phi_{\frac{4}{3}}\left(D_{0^+}^{\frac{7}{3}}\mathfrak{u}_2(x)\right) = 1+\frac{1}{(2x+4)^3}|\mathfrak{u}_2(x)|+\frac{t}{12}sin\left(D_{0^+}^{\frac{1}{3}}\mathfrak{u}_2(x)\right), 0\le x\le 2,\\ \phi_{\frac{4}{3}}\left(D_{0^+}^{\frac{7}{3}}\mathfrak{u}_3(x)\right) = 1+\frac{1}{6\sqrt{3}(x+2)^2}|\mathfrak{u}_3(x)|+\frac{3^{\frac{1}{3}}x}{12\sqrt{3}(1+x)}|D_{0^+}^{\frac{1}{3}}\mathfrak{u}_3(x)|, 0\le x\le 3,\\ \mathfrak{u}_1(0)=\mathfrak{u}_1'(0), \mathfrak{u}_2(0)=\mathfrak{u}_2'(0), \mathfrak{u}_3(0)=\mathfrak{u}_3'(0),\\ \mathfrak{u}_1(1)=\mathfrak{u}_2(2)=\mathfrak{u}_3(1),\\ D_{0^+}^{\frac{1}{3}}\mathfrak{u}_1(1)+D_{0^+}^{\frac{1}{3}}\mathfrak{u}_2(2)=D_{0^+}^{\frac{1}{3}}\mathfrak{u}_3(1)=0, \end{cases}$$

From the system (1.1), we have

$$k=3, \alpha=\frac{7}{3}, \beta=\frac{1}{3}, l_1=1, l_2=2, l_3=1, p=\frac{4}{3}.$$

The molecular structure of formaldehyde graphs is composed of one carbon atom, two hydrogen atoms and one oxygen atom. We regard the carbon atom as the vertex of the graph and the chemical bond between atoms as the edge of the graph, see the Figure 9. The coordinate system is established with the origin of coordinates $v_1, v_2, and\ v_3$ respectively, where $\mathfrak{u}_1$ is the solutions of the system (5.3) on $\overrightarrow{v_1v_0}$, $t \in [0, l_1]$; $\mathfrak{u}_2$ is the solution of the system (5.3) on $\overrightarrow{v_2v_0}$, $t \in [0, l_2]$; and $\mathfrak{u}_3$ is the solution of the system (5.3) on $\overrightarrow{v_2v_0}$, $t \in [0, l_3]$.Here

$$l_1 = |e_1| = |\overrightarrow{v_1v_0}| = 1,$$

$$l_2 = |e_2| = |\overrightarrow{v_2v_0}| = 2,$$

and

$$l_3 = |e_3| = |\overrightarrow{v_3v_0}| = 1.$$

In the following, we will prove that system (5.3) has a solutions on each edge of the formaldehyde graphs.

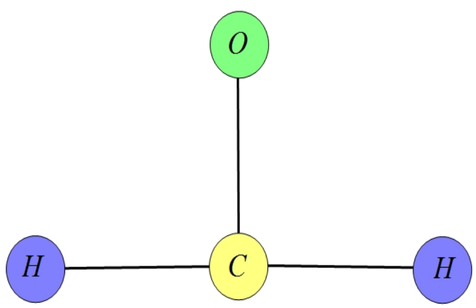

FIGURE 8. A sketch of the graphs representation of Formaldehyde (CH_2O)

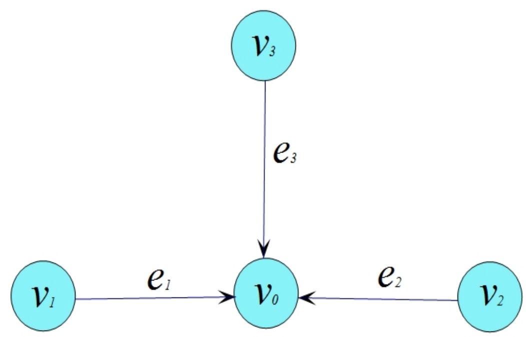

FIGURE 9. A sketch of the graphs representation of CH_2O with labeled vertices by v_0 or v_i

By Lemma 2.6 , we can get

$$(5.4)\quad \begin{cases} D_{0^+}^{\frac{7}{3}} v_1(t) = \phi_4 \left(1 + \frac{1}{2(t+3)^3} \left(sin(v_1(t)) + |D_{0^+}^{\frac{1}{3}} v_1(t)|\right)\right), \\ D_{0^+}^{\frac{7}{3}} v_2(t) = 2^{\frac{7}{3}} \phi_4 \left(1 + \frac{1}{(2t+4)^3}|v_2(t)| + \frac{t}{12} sin\left(2^{-\frac{1}{3}}|D_{0^+}^{\frac{1}{3}} v_2(t)|\right)\right), \\ D_{0^+}^{\frac{7}{3}} v_3(t) = \phi_4 \left(1 + \frac{1}{6\sqrt{3}(t+2)^2}|v_3(t)| + \frac{t}{12\sqrt{3}(1+t)}|D_{0^+}^{\frac{1}{3}} v_3(t)|\right), \\ v_1(0) = v_1'(0) = v_2(0) = v_2'(0) = v_3(0) = v_3'(0), \\ v_1(1) = v_2(1) = v_3(1), \\ D_{0^+}^{\frac{1}{2}} v_1(1) + 2^{-\frac{1}{3}} D_{0^+}^{\frac{1}{2}} v_2(1) + D_{0^+}^{\frac{1}{2}} v_3(1) = 0, \end{cases}$$

where $t \in [0, 1]$. Clearly, if f_1, f_2 and f_3 are continuous function then there exist

$$L_i > 0,\ i = 1, 2, 3$$

such that

$$f_i \leq L_i.$$

It follows from Theorem 3.3 that fractional differential system (5.4) has at least one solution on the interval $[0,1]$ and fractional differential system (5.3) also has at least one solution.

Referring to the iterative sequence of Example (5.1), we can also obtain the iterative process and approximate solution of the fractional differential system (5.4). The absolute errors for the iterative approach to system (5.4) are shown in Table 2, which demonstrates the applicability of the iterative approach, where $E_k = |v_{i,k+1} - v_{i,k}|$. The iterative process of the solution of the fractional differential system (5.4) on $\overrightarrow{v_1 v_0}$ is shown on Figure 10. Figure 11 is the approximate graph of the solution after 10 iterations on $\overrightarrow{v_1 v_0}$. The process of 12 and Figure 14 are similar to that of Figure 10. Figure 13 and Figure 15 are also similar to picture 11.

TABLE 2. The absolute errors in Example 5.2

k	$E(k)$ for v_1	$E(k)$ for v_2	$E(k)$ for v_3
0	9.3283339675e-02	2.3550104749e-01	3.9340242269e-01
1	4.4097046511e-01	6.3499709317e-01	6.7401336342e-01
2	7.4871879811e-02	2.4142506783e-01	2.3935139057e-01
3	5.1435673237e-02	8.3425148505e-02	6.0165876256e-02
4	2.4592020315e-02	2.9729890329e-02	1.5064530257e-02
5	7.5386799387e-03	1.0477975759e-02	3.7487899110e-03
6	1.2261031457e-03	3.7064401143e-03	9.3480617003e-04
7	6.6044736960e-04	1.3093656601e-03	2.3309516122e-04
8	3.8898730337e-04	4.6276925173e-04	5.8117726304e-05
9	1.3469130928e-04	1.6353024857e-04	1.4490467280e-05

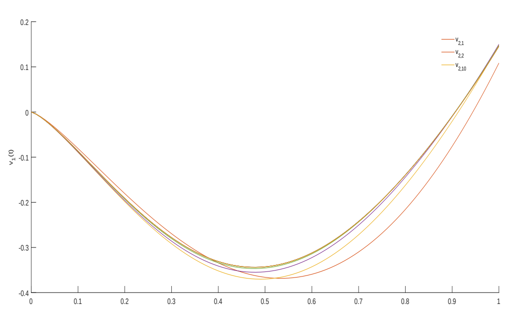

FIGURE 10. Iterative process of v_1

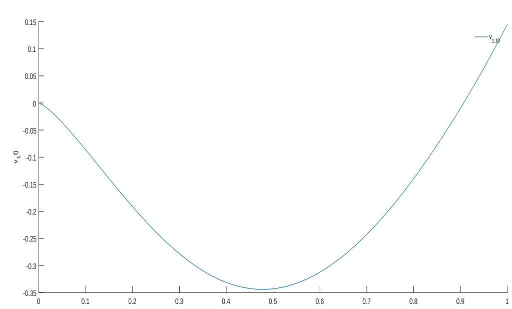

FIGURE 11. Approximate solution of v_1

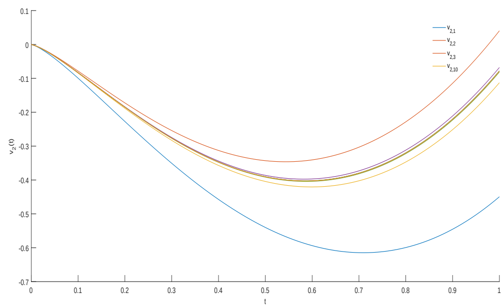

FIGURE 12. Iterative process of v_2

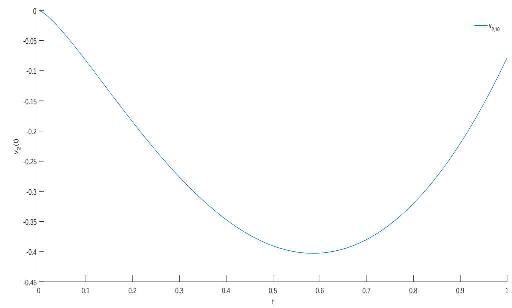

FIGURE 13. Approximate solution of v_2

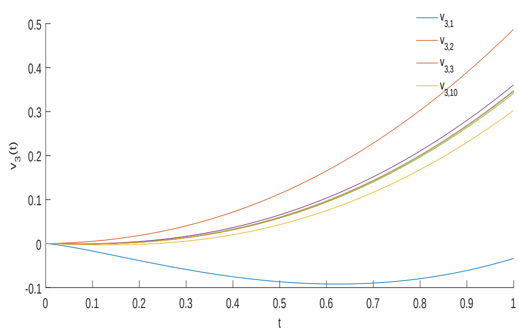

FIGURE 14. Iterative process of v_3

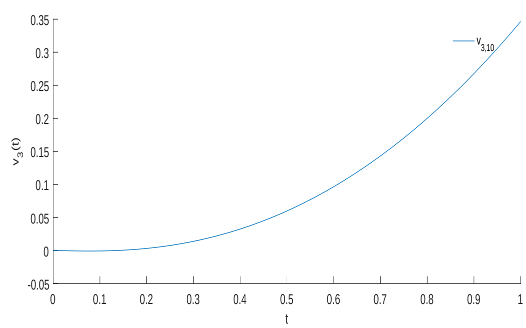

FIGURE 15. Approximate solution of v_3

Funding: This work is partially supported by the National Natural Science Foundation of China (12161079), (12126427).

REFERENCES

[1] R. Hilfer. *Applications of Fractional Calculus in Physics*. World Scientific, Singapore, 2000.

[2] N. Heymans, J.C. Bauwens. Fractal rheological models and fractional differential equations for viscoelastic behavior. *Rheol. Acta*. **33** (3) (1994), 210-219.

[3] F. Liu, O.P. Agrawal, S. Momani, et al. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[4] R. Metzler, T.F. Nonnenmacher. Fractional relaxation processes and fractional rheological models for the description of a class of viscoelastic materials. *Int. J. Plast.* **19** (7) (2003), 941-959.

[5] Y.H. Su, Y.Z. Yun, D. Wang, et al. Existence of solutions to nonlinear p-Laplacian fractional differential equations with higher-order derivative terms. *Electron. J. Differ. Eq.* **2018** (2018), 1-24.

[6] M. Tetsuya, N. Masahiro. A chaos neuron model with fractional differential equation. *J. Phys. Soc. Jpn.* **72** (10) (2003), 2678-2684.

[7] Y.H. Su, Z. Feng, Existence theory for an arbitrary order fractional differential equation with deviating argument, *Acta Appl. Math.* **118** (1) (2012), 81-105.

[8] H. Mohammadi, S. Kumar, S. Rezapour, et al. A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. *Chaos. Solitons Fractals.* **144** (2021), 110668.

[9] N.H. Sweilam, M.M. Khader, R.F. Al-Bar. Numerical studies for a multi-order fractional differential equation. *Phys. Lett. A.* **371** (1-2) (2007), 26-33.

[10] A.D. Gaetano, S. Sakulrang, A. Borri. Modeling continuous glucose monitoring with fractional differential equations subject to shocks. *J. Theor. Biol.* 2021, **526** (2021), 110776.

[11] M. Benchohra, S. Bouriah, J.R. Graef. Boundary value problems for nonlinear implicit Caputo-Hadamard-type fractional differential equations with impulses. *Mediterr. J. Math.* **14** (5) (2017), 206.

[12] B. Ahmad, J.J. Nieto. Anti-periodic fractional boundary value problems. *Comput. Math. Appl.* **62** (3) (2011), 1150-1156.

[13] H. Aktuğlu, Özarslan M. Solvability of differential equations of order $1 < 2 \leq 3$ involving the p-Laplacian operator with boundary conditions. *Adv. Differ. Equ-ny.* **2013** (1) (2013), 1-13.

[14] A. Alsaedi, B. Ahmad, M. Alghanmi. Extremal solutions for generalized Caputo fractional differential equations with Steiltjes-type fractional integro-initial conditions. *Appl. Math. Lett.* **91** (2019), 113-120.

[15] A. Boutiara, M.S. Abdo, M. Benbachir. Existence results for nonlinear neutral generalized Caputo fractional differential equations. *J. Pseudo-Differ. Oper.* **12** (2) (2021), 25-42.

[16] D. Kai, N.J. Ford. Analysis of fractional differential equations. *J. Math. Anal. Appl.* **265** (2) (2002), 229-248.

[17] R.C. Kulaev. On the solvability of a boundary value problem for a fourth-order equation on a graph. *Diff. Equ.* **50** (1) (2014), 25-32.

[18] W. Zhang, W. Liu. Existence and Ulam's type stability results for a class of fractional boundary value problems on a star graphs. *Math. Method. Appl. Sci.* **43** (2020), 8568-8594.

[19] D. Gordeziani, M. Kupreishvili, H. Meladze, et al. On the solution of boundary value problem for differential equations given in graphs. *Appl. Math. Mech-Engl.* **13** (2008), 80-91.

[20] J.R. Graef, L. Kong, M. Wang. Existence and uniqueness of solutions for a fractional boundary value problem on a graph. *Fract. Calc. Appl. Anal.* **17** (2) (2014), 499-510.

[21] Y. Li, A. Qi. Positive solutions for multi-point boundary value problems of fractional differential equations with p-Laplacian. *Math. Method. Appl. Sci.* **39** (6) (2016), 1425-1434.

[22] D. Li, F. Chen, Y. Wu, et al. Variational formulation for nonlinear impulsive fractional differential equations with (p, q)-Laplacian operator. *Math. Method. Appl. Sci.* **45** (1) (2022), 515-531.

[23] V. Mehandiratta, M. Mehra, G. Leugering. Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graphs. *J. Math. Anal. Appl.* **477** (2) (2019), 1243-1264.

[24] G. Lumer. Connecting of local operators and evolution equations on networks. *Lect. Notes. Math.* **787** (19980), 219-234.

[25] S. Nicaise. Some results on spectral theory over networks, applied to nerve impulse transmission. *Notes. Math. Vol.* **1171** (1985), 532-541.

[26] W.C. Sun, Y.H. Su, A. Sun, Q. Zhu. Existence and simulation of positive solutions for m-point fractional differential equations with derivative terms, *Open. Math.* **19** (2021), 1820-1846.

[27] A. Sun, Y.H. Su, Q. Yuan, et al. Existence of solutions to fractional differential equations with fractional-order derivative terms. *J. Comput. Appl. Anal.* **11** (1) (2021), 486-520.

[28] M.G. Zavgorodnii, Y.V. Pokornyi, On the spectrum of second-order boundary value problems on spatial networks. *Usp. Mat. Nauk.* **44** (1989), 220-221.

[29] A.Z. Abdian, A. Behmaram, G.H. Fath-Tabar. Graphs determined by signless Laplacian Spectra. *Akce. Int. J. Graphs Co.* **17** (1) (2020), 45-50.

[30] Z. Han, E. Zuazua. Decay rates for elastic-thermoelastic star-shaped networks. *Netw. Heterog. Media.* **12** (3) (2017), 461-488.

[31] V. Pivovarchik. Inverse problem for the Sturm-Liouville equation on a starshaped graph. *Math. Nachr.* **280** (13) (2007), 1595-1619.

[32] A. Turab, W. Sintunavarat. The novel existence results of solutions for a nonlinear fractional boundary value problem on the ethane graph. *Alex. Eng. J.* **60** (6) (2021), 5365-5374.

[33] Y. Xi, X. Zhang, Y.H. Su. Uniqueness results of fuzzy fractional differential equations under krasnoselskii-krein conditions. *Dyn. Syst. Appl.* **30**(12) (2021), 1792-1801.

[34] X. Zhang, Y.H. Su, Y. Kong. Positive solutions for boundary value problem of fractional differential equation in banach spaces, *Dyn. Syst. Appl.* **30** (9) (2021), 1449-1461.