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# A FIXED POINT THEOREM IN LENGTH SPACES

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**ABSTRACT.** Let $K$ be a closed subset of a complete metric space $X$ and let $CB(X)$ denote the family of all nonempty, closed and bounded subsets of $X$. Assuming further that $X$ is a length space, we establish a fixed point theorem for those strictly contractive set-valued mappings $F : K \to CB(X)$ for which the image of each point on the boundary of $K$ is a subset of $K$.

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**Key Words and Phrases.** Fixed point, length space, metric space, set-valued mapping, strict contraction.

## 1. Introduction

Fixed point theory is known to be useful in the study of existence of solutions to many real life problems which can be modeled as either a differential equation, an integral equation or an integro-differential equation. In particular, fixed point theorems for contractive mappings find applications in several fields of study such as mathematics, physics, engineering and economics. This theory covers the search for fixed points of both single-valued and set-valued mappings. Quite a few articles on contractive set-valued mappings followed the seminal work of Nadler [5]. For example, Assad and Kirk [1] considered the fixed point problem for non-self contractive set-valued mappings in a (metrically) convex metric space $X$. Such results are of interest because the condition $F(\partial K) \subset 2^K$, where $K$ is a closed subset of $X$, $2^K$ is its power set, $F : K \to 2^X$ and $\partial K$ is the boundary of $K$, is less restrictive than the usual requirement that $F : K \to 2^K$. For some of the results in this direction and their applications, we refer the interested reader to, for instance, [1, 3, 6, 8–12] and to references therein.

The recent research interest in length spaces leads us to ask the following question:

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**Question:** Can we extend the fixed point theorem of Assad and Kirk [1] to length spaces?

For the sake of completeness and the reader's convenience, we now state the theorem established in [1]. We denote by $CB(X)$ the set of all nonempty, closed and bounded subsets of a metric space $X$ and the boundary of a set $K \subset X$ by $\partial K$.

**Theorem 1.1.** *Let $X$ be a complete and convex metric space, $K$ a nonempty closed subset of $X$ and $F$ a strict contraction mapping $K$ into $CB(X)$. If $F(x) \subset K$ for each $x \in \partial K$, then there exists a point $x_0 \in K$ such that $x_0 \in F(x_0)$ (that is, $F$ has a fixed point in $K$).*

It is our aim in the present paper to provide an affirmative answer to the above question. In other words, our aim is to prove an extension of Theorem 1.1 to all length spaces.

Indeed we prove (see Theorem 3.1 below) that if $K$ be a nonempty closed subset of a complete metric space $X$, $F : K \to CB(X)$ is a strict contraction, $F$ maps the boundary of $K$ to subsets of $K$, and $X$ is a length space, then there exists a point $x \in K$ such that $x \in F(x)$. In other words, $F$ has a fixed point in $K$.

## 2. **Preliminaries**

We start this section by recalling a few important concepts and definitions regarding the geometry of length spaces (see [2, 7] for more information).

**Definition 2.1.** Let $(X, d)$ be a metric space. A path in $X$ is a continuous map $\gamma : [a, b] \to X$, where $a$ and $b$ are arbitrary real numbers satisfying $a \le b$. If $\gamma(a) = x$ and $\gamma(b) = y$, then $x$ and $y$ are called the endpoints of $\gamma$ and $\gamma$ is said to join $x$ to $y$.

**Definition 2.2.** The length of a path $\gamma : [a, b] \to X$ is the quantity given by

$$L(\gamma) = \sup_{\sigma} \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over the set of partitions $\sigma = \{t_i\}_{i=0}^{n}$ of $[a, b]$. A path $\gamma$ is said to be rectifiable if $L(\gamma) < \infty$. The length of a path $L$ is additive in the sense that for any $c \in [a, b]$, we have

$$L(\gamma[a, b]) = L(\gamma[a, c]) + L(\gamma[c, b]).$$

A metric space $(X, d)$ is said to be connected by rectifiable paths if for every two points $x$ and $y$ in $X$, there exists a rectifiable path $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. If $X$ is a metric space connected by rectifiable paths, then there is a natural metric $d$ on $X$ which is called the length metric.

**Definition 2.3.** A metric space $(X, d)$ is said to be a length space if for every $x$ and $y$ in $X$,

$$d(x, y) = \inf_{\gamma} L(\gamma),$$

where the infimum is taken over the set of paths $\gamma$ joining $x$ to $y$. The metric $d$ of a length space is called a length metric or an inner metric. In particular, a length space is a metric space connected by rectifiable paths. Thus, $(X, d)$ is a length space if and only for any two points $x, y \in X$ and any $\epsilon > 0$, there exists a curve $\gamma$ joining $x$ and $y$ such that

$$L(\gamma) < d(x, y) + \epsilon.$$

A length space is said to be complete if for all $x, y \in X$, there is a length minimizing rectifiable path $\gamma$ from $x$ to $y$ so that $d(x, y) = L(\gamma([a, b]))$.

At this point we recall the notion of betweenness defined by Karl Menger. Given three points $x$, $y$ and $z$ in a metric space $X$, the point $z$ is said to lie between $x$ and $y$ if these three points are pairwise distinct (that is, $x \neq y \neq z$) and

$$d(x, z) + d(z, y) = d(x, y).$$

A metric space endowed with this property is said to be (metrically) convex. Menger showed that any complete and convex metric space is a length space. However, there exist length spaces which are not convex.

We now make the following remark regarding length spaces.

**Remark 2.4.** Let $K$ be a nonempty closed subset of a length space $X$. If $x \in K$ and $y \notin K$, then for each $\epsilon > 0$, there exists a point $z$ on the boundary of $K$ such that

$$d(x, z) + d(z, y) \leq d(x, y) + \epsilon.$$

Indeed, given $\epsilon > 0$, let $\gamma : [a, b] \to X$ be a path joining $x$ to $y$ and satisfying $L(\gamma) \leq d(x, y) + \epsilon$. It is not difficult to see that there is $c \in [a, b]$ such that $z = \gamma(c)$ belongs to the boundary of $K$. There are paths $\gamma' : [a, c] \to X$ and $\gamma'' : [c, b] \to X$ joining $x$ to $z$ and $z$ to $y$, respectively. Using the additive property of $L$, we have

$$\begin{aligned} L(\gamma') + L(\gamma'') &= L(\gamma) \\ &\leq d(x, y) + \epsilon. \end{aligned}$$

It follows that

$$d(x, z) + d(z, y) \leq d(x, y) + \epsilon.$$

**Definition 2.5.** Let $(X, d)$ be a metric space and let $CB(X)$ denote the family of all nonempty, closed and bounded subsets of $X$. For $A, B \in CB(X)$ and $x \in X$, let $d(x, A) = \inf\{d(x, a) : a \in A\}$ and

$$\mathcal{H}(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\}.$$

It is not difficult to verify that $\mathcal{H}$ is a metric on $CB(X)$. This metric is called the Hausdorff metric induced by $d$.

Throughout this article, unless otherwise stated, we denote a metric space by $(X, d)$, $CB(X)$ stands for the family of nonempty, closed and bounded subsets of $X$ and $\partial K$ is the boundary of a set $K$. In the proof of our main result, we use the following two lemmata, which can be found in Nadler [5].

**Lemma 2.6.** *If $A, B \in CB(X)$ and $x \in A$, then for each positive number $\alpha$, there exists a point $y \in B$ such that*

$$d(x, y) \leq \mathcal{H}(A, B) + \alpha.$$

**Lemma 2.7.** *Let $\{X_n\}$ be a sequence of sets in $CB(X)$ such that $\mathcal{H}(X_n, X_0) \to 0$ as $n \to \infty$ with $X_0 \in CB(X)$. If $x_n \in X_n$ for each $n = 1, 2, \cdots$ and $x_n \to x_0$ as $n \to \infty$, then $x_0 \in X_0$.*

**Definition 2.8.** Let $K$ be a nonempty closed subset of a metric space $(X, d)$ and let $F : K \to CB(X)$ be a mapping. Then $F$ is said to be a strict contraction if there exists a number $k \in [0, 1)$ such that

$$\mathcal{H}(F(x), F(y)) \leq kd(x, y).$$

for all $x, y \in$.

**Definition 2.9.** [4]

(i) A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to converge or to be convergent if there exists a point $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

(ii) A sequence $\{x_n\}$ is said to be Cauchy if for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for every $m, n > N$.

(iii) A metric space $X$ is said to be complete if every Cauchy sequence in $X$ converges to an element of $X$.

## 3. **Main result**

In this section we state and prove our main result.

**Theorem 3.1.** *Let $K$ be a nonempty closed subset of a complete metric space $(X, d)$ and let $F$ be a strict contraction mapping $K$ into $CB(X)$. If $(X, d)$ is a length space and $F(x) \subset K$ for each $x \in \partial K$, then $F$ has a fixed point. That is, there exists a point $x_0 \in K$ such that $x_0 \in F(x_0)$.*

*Proof:* Let $\alpha \in [0, 1)$ be the contraction constant of $F$. Choose arbitrary points $x_0 \in K$ and $y_1$ in $F(x_0)$. If $y_1 \in K$, then set $x_1 = y_1$; otherwise, choose $x_1 \in \partial K$ such that

$$d(x_0, x_1) + d(x_1, y_1) \leq d(x_0, y_1) + 1.$$

By Lemma 2.6, we may choose $y_2 \in F(x_1)$ such that

$$d(y_1, y_2) \leq \mathcal{H}(F(x_0), F(x_1)) + \alpha.$$

If $y_2 \in K$, then set $x_2 = y_2$; otherwise, choose $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, y_2) \leq d(x_1, y_2) + \alpha.$$

Continuing this process inductively, we obtain two sequences, $\{x_n\}$ and $\{y_n\}$, $n = 1, 2, \dots$, such that

(i) $y_{n+1} \in F(x_n)$;
(ii) $d(y_n, y_{n+1}) \leq \mathcal{H}(F(x_{n-1}), F(x_n)) + \alpha^n$;
(iii) $y_{n+1} \in K \Rightarrow x_{n+1} = y_{n+1}$;
(iv) $y_{n+1} \notin K \Rightarrow x_{n+1} \in \partial K$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \leq d(x_n, y_{n+1}) + \alpha^n.$$

Define

$$P = \{x_j \in \{x_n\} : x_j = y_j,\ \ j = 1, 2, \cdots\};$$
$$Q = \{x_j \in \{x_n\} : x_j \neq y_j,\ \ j = 1, 2, \cdots\}.$$

It is clear that if $x_n \in Q$ for some $n$, then $x_{n+1} \in P$, that is, there can be no consecutive terms in $Q$. We now consider the following three possible cases:

**Case 1:** Let $x_n, x_{n+1} \in P$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(y_n, y_{n+1}) \\ &\leq \mathcal{H}(F(x_{n-1}), F(x_n)) + \alpha^n \\ &\leq \alpha d(x_{n-1}, x_n) + \alpha^n \\ &\leq \alpha d(x_{n-1}, x_n) + 2\alpha^n. \end{aligned}$$

**Case 2:** Let $x_n \in P$ and $x_{n+1} \in Q$. By using (iv), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_{n+1}) + \alpha^n \\ &= d(y_n, y_{n+1}) + \alpha^n \\ &\leq \mathcal{H}(F(x_{n-1}), F(x_n)) + \alpha^n + \alpha^n \\ &\leq \alpha d(x_{n-1}, x_n) + 2\alpha^n. \end{aligned}$$

**Case 3:** Let $x_n \in Q$ and $x_{n+1} \in P$. It is clear that two consecutive terms cannot both lie in $Q$. Thus $x_{n-1} \in P$ and we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq d(x_n, y_n) + d(y_n, x_{n+1}) \\
&= d(x_n, y_n) + d(y_n, y_{n+1}) \\
&\leq d(x_n, y_n) + \mathcal{H}(F(x_{n-1}), F(x_n)) + \alpha^n \\
&\leq d(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \alpha^n \\
&\leq d(x_{n-1}, y_n) + \alpha^n + \alpha^n \\
&= d(y_{n-1}, y_n) + 2\alpha^n \\
&\leq \mathcal{H}(F(x_{n-2}), F(x_{n-1})) + \alpha^{n-1} + 2\alpha^n \\
&\leq \alpha d(x_{n-2}, x_{n-1}) + \alpha^{n-1} + 2\alpha^n.
\end{aligned}$$

As we have already mentioned, the case $x_n, x_{n+1} \in Q$ is not possible. Therefore, for $n \geq 2$, we have the following two cases:

$$d(x_n, x_{n+1}) \leq \begin{cases} \alpha d(x_{n-1}, x_n) + 2\alpha^n, \\ \alpha d(x_{n-2}, x_{n-1}) + \alpha^{n-1} + 2\alpha^n. \end{cases} \tag{3.1}$$

**Claim:** For $n \geq 1$, $d(x_n, x_{n+1}) \leq \alpha^{\frac{n}{2}}(\beta+2n)$ where $\beta = \alpha^{-\frac{1}{2}} \max\{d(x_0, x_1), d(x_1, x_2)\}$. It is not difficult to see that the claim holds for $n = 1$. Indeed,

$$d(x_1, x_2) \leq \alpha^{\frac{1}{2}}\beta \leq \alpha^{\frac{1}{2}}(\beta + 2).$$

Now for $n = 2$, we consider the two possibilities for $d(x_n, x_{n+1})$ given by (3.1).

$$\begin{aligned}
d(x_2, x_3) &\leq \alpha d(x_1, x_2) + 2\alpha^2 \\
&= \alpha(d(x_1, x_2) + 2\alpha) \\
&\leq \alpha(\alpha^{\frac{1}{2}}\beta + 2\alpha) \\
&\leq \alpha(\beta + 4)
\end{aligned}$$

and

$$\begin{aligned}
d(x_2, x_3) &\leq \alpha d(x_0, x_1) + \alpha + 2\alpha^2 \\
&= \alpha(d(x_0, x_1) + 1 + 2\alpha) \\
&\leq \alpha(\alpha^{\frac{1}{2}}\beta + 1 + 2\alpha) \\
&\leq \alpha(\beta + 4).
\end{aligned}$$

Also for $n = 3$, we have

$$\begin{aligned} d(x_3, x_4) &\leq \alpha d(x_2, x_3) + 2\alpha^3 \\ &= \alpha(d(x_2, x_3) + 2\alpha^2) \\ &\leq \alpha(\alpha(\beta + 4) + 2\alpha^2) \\ &\leq \alpha^{\frac{3}{2}}(\beta + 6) \end{aligned}$$

and

$$\begin{aligned} d(x_3, x_4) &\leq \alpha d(x_1, x_2) + \alpha^2 + 2\alpha^3 \\ &= \alpha(d(x_1, x_2) + \alpha + 2\alpha^2) \\ &\leq \alpha(\alpha^{\frac{1}{2}}\beta + \alpha + 2\alpha^2) \\ &\leq \alpha^{\frac{3}{2}}(\beta + 6). \end{aligned}$$

Now suppose the claim holds for $1 \leq n \leq k$, then for $k \geq 3$ we again consider the two possibilities regarding $d(x_n, x_{n+1})$ as before.

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &\leq \alpha d(x_k, x_{k+1}) + 2\alpha^{k+1} \\ &\leq \alpha\left(\alpha^{\frac{k}{2}}(\beta + 2k)\right) + 2\alpha^{k+1} \\ &\leq \alpha^{\frac{k+1}{2}}\beta + (2k+2)\alpha^{\frac{k+2}{2}} \\ &\leq \alpha^{\frac{k+1}{2}}(\beta + 2(k+1)) \end{aligned}$$

and

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &\leq \alpha d(x_{k-1}, x_k) + \alpha^k + 2\alpha^{k+1} \\ &\leq \alpha\left(\alpha^{\frac{k-1}{2}}(\beta + (k-1))\right) + \alpha^k + 2\alpha^{k+1} \\ &\leq \alpha^{\frac{k+1}{2}}\beta + (k-1)\alpha^{\frac{k+1}{2}} + \alpha^k + 2\alpha^{k+1} \\ &\leq \alpha^{\frac{k+1}{2}}(\beta + 2(k+1)). \end{aligned}$$

Hence the **claim** is true. Thus we obtain that

$$d(x_n, x_m) \leq \beta \sum_{l=m}^{\infty} (\alpha^{\frac{1}{2}})^l + 2\sum_{l=m}^{\infty} l(\alpha^{\frac{1}{2}})^l, \ n > m \geq 1.$$

This implies that the sequence $\{x_n\}$ is Cauchy. Since the space $X$ is complete and the set $K$ is closed in $X$, it follows that the sequence $\{x_n\}$ converges to a point $x_0 \in K$. There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ each term of which belongs to the set $P$. That is, $\{x_{n_j} = y_{n_j},\ j = 1, 2, \cdots\}$. Therefore by (i), $y_{n_j} \in F(x_{n_j-1})$ for $j = 1, 2, \cdots$. Since $x_{n_j-1} \to x_0$ as $j \to \infty$, we obtain that $F(x_{n_j-1}) \to F(x_0)$ as $j \to \infty$ in the Hausdorff metric. Lemma 2.7 now implies that $x_0 \in F(x_0)$. This completes the proof of our theorem.

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