Dynamic Systems and Applications 31 (2022) No.3, 181-193

A CARISTI AND MEIR-KEELER TYPE CHARACTERIZATION OF COMPLETENESS FOR MENGER PM SPACES

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ABSTRACT. In this paper the Caristi fixed point theorem and a Meir-Keeler type fixed point theorem for Menger PM-spaces is generalized by employing a weaker form of continuity. We prove that these theorems characterize completeness of the Menger PM space as well as Cantor's intersection property.

AMS (MOS) Subject Classification. 47H10, 54H25.

Key Words and Phrases. Menger PM-spaces; Fixed point; Weak orbital continuity; Contractive mapping.

1. Introduction

In 1971, Ćirić [2] introduced the notion of orbital continuity.

Definition 1.1. [2] If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_{i\to\infty} f^{m_i}x$ implies $fu = \lim_{i\to\infty} ff^{m_i}x$.

Every continuous self-mapping is orbitally continuous but not conversely. Pant and Pant [7] introduced another weaker form of continuity.

Definition 1.2. [7] A self-mapping f of a metric space X is called a k-continuous, $k = 1, 2, 3, \ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $f^{k-1}x_n \to t$.

Received March 12, 2022 ISSN 1056-2176(Print); ISSN 2693-5295 (online)
$15.00 © Dynamic Publishers, Inc. https://doi.org/10.46719/dsa202231.03.03
www.dynamicpublishers.org.
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It was shown in [7] that continuity of f^k and k-continuity of f are independent conditions when $k > 1$. It is easy to see that 1-continuity is equivalent to continuity. Also, it can be proved that continuity implies k-continuity implies $(k+1)$-continuity, for $k = 2, 3, 4, \ldots$, but converse is not true. A k-continuous mapping is obviously orbitally continuous. Recently, Pant et al. [9] introduced a weaker form of the above definitions.

Definition 1.3. [9] A self-mapping f of a metric space (X, d) will be called a weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i} y = u \Rightarrow \lim_i f f^{m_i} y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i} x = u\}$ is nonempty.

Using the notion of a weak orbital continuity, Pant et al. [9] generalized Caristi fixed point theorem [1] omitting the assumption that the function ϕ is a lower semi-continuous.

Theorem 1.4. *[9] Let f be a self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x, y in X we have*

$$d(x, fx) \leq \phi(x) - \phi(fx).$$

If f is a weakly orbitally continuous or f^k is continuous or f is a k-continuous for some $k \geq 1$, then f has a fixed point.

Also, using the notion of a weak orbital continuity, Pant et al. [9] obtained a Meir-Keeler type solution of the problem of continuity at the fixed point (see [5], [8] and [11]). Actually, this notion is a necessary and sufficient condition for the existence of fixed points of Meir-Keeler type mappings. Furthermore, these results characterize completeness of the metric spaces as well as Cantor's intersection property. In this sense, our main results extends to the framework of Menger PM spaces results obtained in [9]. Also, the aim of this paper is to instigate the study of characterizing complete Menger PM spaces by means of fixed point results. Recently, one characterization in this sense is obtained in [10].

2. **Preliminaries**

The idea of probabilistic metric space (briefly PM space) was introduced by Menger in [6] in 1942. The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. Schweizer and Sklar [14, 15] studied the properties of spaces introduced by Menger and and have developed their theory in depth. The first result from the fixed point theory in probabilistic metric spaces was obtain by Sehgal and Bharucha–Reid [16] as a generalization of the classical Banach Contraction Mapping Principle.

Let Δ^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$ such that F is a nondecreasing, left-continuous mapping, which satisfies $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1. [15] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $([0,1], T(\cdot,\cdot))$ is a topological monoid with unit 1 such that $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

Some examples of t-norm are minimum $T(a,b) = \min\{a,b\}$, product $T(a,b) = a \cdot b$ and *Łukasievicz* t-norm $\mathcal{L}(a,b) = \max\{a+b-1, 0\}$ and they satisfy $\min\{a,b\} \geq a \cdot b \geq \mathcal{L}(a,b)$.

Definition 2.2. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$ where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t+s) \geq T\big(F_{x,y}(t), F_{y,z}(s)\big)$ for all $x,y,z \in X$ and $s,t \geq 0$.

Remark 2.3. [16] Every metric space is a PM-space. Let (X,d) be a metric space and $T(a,b) = \min\{a,b\}$ is a continuous t-norm. Define $F_{x,y}(t) = \varepsilon_0(t - d(x,y))$ for all $x, y \in X$ and every $t > 0$. The triple $(X, \mathcal{F}, T)$ is a PM-space induced by the metric d.

Definition 2.4. Let $(X, \mathcal{F}, T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$.
(3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The (ε, λ)-topology (see [15]) in a Menger PM-space $(X, \mathcal{F}, T)$ is introduced by the family of neighbourhoods $\mathcal{N}_x$ of a point $x \in X$ given by

$$\mathcal{N}_x = \big\{N_x(\varepsilon, \lambda) \mid \varepsilon > 0, \lambda \in (0,1)\big\},$$

where

$$N_x(\varepsilon, \lambda) = \{y \in X \mid F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

The (ε, λ)-topology is a Hausdorff topology. In this topology the function f is continuous in $x_0 \in X$ if and only if for every sequence $x_n \to x_0$ it holds that $f(x_n) \to f(x_0)$.

The following Lemma is proved by B. Schweizer and A. Sklar.

Lemma 2.5. *[15] Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $\mathcal{F}$ is a lower semi-continuous for every fixed $t > 0$, i.e. for every fixed $t > 0$ and every two convergent sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to x, y_n \to y$, as $n \to \infty$ it follows that*

$$\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Definition 2.6. Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The closure of the set A is the smallest closed set containing A, denoted by $\overline{A}$.

Obviously, keeping in mind the Hausdorff topology, and the definition of converging sequences we note that the next remark holds.

Remark 2.7. $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A such that $x_n \to x$, as $n \to \infty$.

The concept of a probabilistic boundedness was introduced by Egbert [3]. A version on this definition follows.

Definition 2.8. [3] Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The probabilistic diameter of set A is given by

$$\delta_A(t) = \sup_{\varepsilon < t} \inf_{x,y\in A} F_{x,y}(\varepsilon).$$

The diameter of the set A is defined by

$$\delta_A = \sup_{t>0} \delta_A(t).$$

If there exists $\lambda \in (0, 1)$ such that $\delta_A = 1-\lambda$ the set A will be called a probabilistically semi-bounded. If $\delta_A = 1$ the set A will be called a probabilistically bounded.

Lemma 2.9. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A set $A \subseteq X$ is a probabilistically bounded if and only if for every $\lambda \in (0, 1)$ there exists $t > 0$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in A$.*

Proof. The proof follows from the definitions of $\sup A$ and $\inf A$ of nonempty sets. $\square$

Remark 2.10. It is not difficult to see that every metrically bounded set is also probabilistically bounded if it is considered in the induced PM-space.

Sherwood has proved the following theorem.

Theorem 2.11. *[12] Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $\{F_n\}_{n\in\mathbb{N}}$ a nested sequence of nonempty, closed subsets of X such that $\delta_{F_n} \to \varepsilon_0$, as $n \to \infty$. Then there is exactly one point $x_0 \in F_n$, for every $n \in \mathbb{N}$.*

It is easy to show that the following lemma holds.

Lemma 2.12. *[12] Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Let $\{F_n\}_{n\in\mathbb{N}}$ be a nested sequence of nonempty, closed subsets of X. The sequence $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero i.e. for every $\lambda \in (0,1)$ and every $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in F_{n_0}$ if and only if $\delta_{F_n} \to \varepsilon_0$, as $n \to \infty$.*

3. **Main results**

The following theorem is proved by Pant et al. [10] (see Theorem 3.1, p. 4).

Theorem 3.1. *[10] Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be self-mapping of X such that $F_{fx,fy}(t) \geq \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\}$ and*

$$\begin{aligned}&\text{given } \epsilon \in (0,1) \text{ there exists } \delta \in (0,\epsilon] \text{ such that}\\ &\epsilon - \delta < \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\} < \epsilon \;\;\Rightarrow\;\; F_{fx,fy}(t) \geq \epsilon,\end{aligned} \tag{3.1}$$

hold for all $x, y \in X$, and every $t > 0$. Then there exists a point z in X such that the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every x in X, and $\lim\limits_{n\to\infty} f^n x = z$.

The contractive condition employed in the previous theorem ensures the convergence of sequence of iterates but does not ensure the existence of a fixed point.

Now, we will prove Meir-Keeler type fixed point theorem using the notion of weak orbital continuity. Actually, we will prove that if all assertions of Theorem 3.1 hold and additionally f is a weakly orbitally continuous mapping, then it is a necessary and sufficient condition for the existence of fixed points of Meir-Keeler type mappings.

Theorem 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space a and let f be a self-mapping of X which satisfies the conditions of the previous theorem. Then f possesses a fixed point if and only if f is a weakly orbitally continuous mapping. Moreover, the fixed point is unique and f is continuous at the fixed point, say z, if and only if $\liminf\limits_{x\to z} \left\{F_{fx,fz}(t)\right\} = 1$, for every $t > 0$.*

Proof. From Theorem 3.1 we have that there exists a point z in X such that the sequence of iterates $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence and $\lim\limits_{n\to\infty} f^n x = z$, for every x in X.

If we assume that f is weakly orbitally continuous mapping, then since $f^n x_0 \to z$ for every x_0 we obtain that $f^n y_0 \to z$ and $f^{n+1} y_0 \to fz$ for some y_0 in X. From previous, we have $z = fz$ since $f^{n+1}y_0 \to z$. Hence z is a fixed point of f. Uniqueness of the fixed point follows easily.

Conversely, if we assume that the mapping f possesses a fixed point, say z, then $\{f^n z = z\}_{n\in\mathbb{N}}$ is a constant sequence such that $\lim\limits_{n\to\infty} f^n z = z$ and $\lim\limits_{n\to\infty} f^{n+1} z = z = fz$. Hence, f is a weakly orbitally continuous mapping. It is also easy to verify that f is continuous at z if and only if $\liminf\limits_{x\to z} F_{fz,fx}(t) = 1$.

This completes the proof. □

Remark 3.3. From Theorem 3.2 we have that a mappings f is continuous at z if and only if $\liminf\limits_{x\to z} F_{fz,fx}(t) = 1$ or equivalently $\lim\limits_{x\to z} \min\left\{F_{fx,x}(t), F_{fz,z}(t)\right\} = 1$, for every $t > 0$. This can alternatively be stated as

$$f \text{ is discontinuous at } z \text{ if and only if } \liminf_{x\to z} F_{fz,fx}(t) < 1.$$

Pant et al. [9] generalized Caristi fixed point theorem (see Theorem 1.4 from Introduction) omitting the assumption that the function ϕ is a lower semi-continuous. In next theorem we extends this theorem to the framework of Menger PM spaces. Among the first extensions of Caristi fixed point theorems to the framework of Menger PM spaces were obtained by Hadžić and Ovcin [4] (Theorem 9, p. 207) and Shisheng et al. [13] (Theorem 3, p. 221). The next theorem generalizes mentioned results.

Theorem 3.4. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space. Let f be a weakly orbitally continuous or f^k is continuous or f is k-continuous mapping, for some $k \geq 1$, satisfying*

$$F_{x,fx}(t) \geq \varepsilon_0\left(t - \left(\phi(x) - \phi(fx)\right)\right), \tag{3.2}$$

for every $x \in X$ and $t > 0$, where $\phi : X \mapsto [0, \infty)$. Then f has a fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1 = fx_0$, $x_2 = fx_1$, $\ldots$, $x_n = fx_{n-1}$, i.e. $x_n = f^n x_0, n \in \mathbb{N}$. If there exists some point x_i, $i \in \mathbb{N}$ such that $\phi(x_i) - \phi(fx_i) \leq 0$ applying condition (3.2) we obtain that $x_i = fx_i$, i.e. x_i is a fixed point of f. Hence, we will assume that

$$\phi(x_i) - \phi(fx_i) = \phi(x_i) - \phi(x_{i+1}) > 0,$$

holds for every $i \in \mathbb{N}$, i.e. $x_i \neq x_{i+1}$, for every $i \in \mathbb{N}$. It is obvious that $\{\phi(x_n)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence.

Now we will prove that from

$$F_{x_0,x_1}(t) \geq \varepsilon_0\left(t - \left(\phi(x_0) - \phi(x_1)\right)\right)$$

and

$$F_{x_1,x_2}(t) \geq \varepsilon_0\left(t - \left(\phi(x_1) - \phi(x_2)\right)\right)$$

it follows

$$F_{x_0,x_2}(t) \geq \varepsilon_0 \left(t - \left(\phi(x_0) - \phi(x_2)\right)\right) \tag{3.3}$$

for every $t > 0$.

When $t \leq \phi(x_0) - \phi(x_2)$, it is obvious that (3.3) is satisfied, for every $t > 0$. On the other hand, if $t > \phi(x_0) - \phi(x_2)$, then exist $t_1, t_2 > 0$ such that $t_1 + t_2 = t$, $t_1 > \phi(x_0) - \phi(x_1)$ and $t_2 > \phi(x_1) - \phi(x_2)$. Let us assume that condition (3.3) is not true. Then, there exists $t^* > 0$, such that $t^* > \phi(x_0) - \phi(x_2)$, for which it is satisfied

$$F_{x_0,x_2}(t^*) < \varepsilon_0 \left(t - \left(\phi(x_0) - \phi(x_2)\right)\right)$$

i.e.

$$F_{x_0,x_2}(t^*) < 1. \tag{3.4}$$

Also, for such $t^* > 0$ then exist $t_1^*, t_2^* > 0$, such that $t_1^* + t_2^* = t^*$, $t_1^* > \phi(x_0) - \phi(x_1)$ and $t_2^* > \phi(x_1) - \phi(x_2)$. Now, using condition (PM3) from Definition 2.2 we obtain

$$\begin{aligned} F_{x_0,x_2}(t^*) &\geq T\big(F_{x_0,x_1}(t_1^*), F_{x_1,x_2}(t_2^*)\big) = \\ &= T\left(\varepsilon_0\left(t_1^* - \left(\phi(x_0) - \phi(x_1)\right)\right), \varepsilon_0\left(t_2^* - \left(\phi(x_1) - \phi(x_2)\right)\right)\right) = \\ &= T(1,1) = 1. \end{aligned}$$

This is contradiction with (3.4). Hence, condition (3.3) holds for every $t > 0$. In general, we have

$$F_{x_0,x_n}(t) \geq \varepsilon_0 \left(t - \left(\phi(x_0) - \phi(x_n)\right)\right), \tag{3.5}$$

for every $n \in \mathbb{N}$.

Since, $\{\phi(x_n)\}_{n\in\mathbb{N}} \subset [0, +\infty)$ is a strictly decreasing sequence, there exists $s \geq 0$, such that $\lim\limits_{n\to\infty} \phi(x_n) = s$. Hence, for any given $\varepsilon, \lambda > 0$ such that $\varepsilon > \lambda$, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$s \leq \phi(x_n) < s + \lambda. \tag{3.6}$$

From (3.6) we get that $\phi(x_n) - \phi(x_m) \leq \lambda$ holds, for every $m \in \mathbb{N}$ such that $n_0 \leq n < m$. Now, having in mind inequality (3.5) we obtain

$$F_{x_n,x_m}(\varepsilon) \geq \varepsilon_0 \left(\varepsilon - \left(\phi(x_n) - \phi(x_m)\right)\right) \geq \varepsilon_0(\varepsilon - \lambda) = 1.$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is a complete, there exists a point z in X such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} f^p x_n = z$, for every $p \geq 1$. Suppose that f is a weakly orbitally continuous mapping. Since $\{f^n x_0\}_{n\in\mathbb{N}}$ converges for every x_0 in X, weak orbital continuity implies that there exists $y_0 \in X$ such that $f^n y_0 \to u$, and $f^{n+1} y_0 \to fu$ for some u in X. This implies $u = fu$, that is u is a fixed point of f. If f is k-continuous or f^k is continuous for some $k \geq 1$, then f is a weakly orbitally continuous. This completes the proof. □

Remark 3.5. The condition (3.2) is equivalent to the following condition

$$(\forall \alpha \in [0,1]) \ \ \sup\left\{t : F_{x,fx}(t) \leq 1-\alpha\right\} \leq \phi(x) - \phi(fx).$$

Example 3.6. In accordance with Remark 2.3 we have that the triple $(X, \mathcal{F}, T_{min})$ is a complete Menger PM-space, for $X \subseteq \mathbb{R}$. Hence, let $X = [1,3]$ equipped with the Euclidean metric. Define $f : X \mapsto X$ by

$$fx = \begin{cases} \frac{x+2}{2}, & \text{if } 1 \leq x < 2, \\ 1, & \text{if } 2 \leq x < 3, \\ 3, & \text{if } x = 3. \end{cases}$$

It is easy to see that mapping f is a weakly orbitally continuous. Let us define $\phi : X \to [0,\infty)$ by

$$\phi(x) = \begin{cases} -x+2, & \text{if } x < 2, \\ x+2, & \text{if } x \geq 2. \end{cases}$$

Now, we will observe next three cases:

1. if $x \in [1,2)$ then we have

$$F_{x,fx}(t) = \varepsilon_0\left(t - \left|x - \frac{x+2}{2}\right|\right) = \varepsilon_0\left(t - \left|\frac{x-2}{2}\right|\right),$$

and

$$\begin{aligned} \varepsilon_0\left(t - \left(\phi(x) - \phi(fx)\right)\right) &= \varepsilon_0\left(t - \left(-x+2-\phi\left(\frac{x+2}{2}\right)\right)\right) \\ &= \varepsilon_0\left(t - \left(-x+2-\left(-\frac{x+2}{2}+2\right)\right)\right) \\ &= \varepsilon_0\left(t - \frac{2-x}{2}\right), \end{aligned}$$

for every $t > 0$;
2. if $x \in [2,3)$ then we have

$$F_{x,fx}(t) = \varepsilon_0\left(t - |x-1|\right),$$

and

$$\varepsilon_0\left(t - \left(\phi(x) - \phi(fx)\right)\right) = \varepsilon_0\left(t - \left(x+1\right)\right),$$

for every $t > 0$;
3. if $x = 3$ then we have

$$F_{3,f3}(t) = \varepsilon_0\left(t\right),$$

and

$$\varepsilon_0\left(t - \left(\phi(3) - \phi(f3)\right)\right) = \varepsilon_0\left(t\right),$$

for every $t > 0$.

For all three cases condition (3.2) is satisfied, for every $x \in [1,3]$ and $t > 0$. Hence, all the conditions of Theorem 3.4 are satisfied and mapping f possess a fixed point $x = 3$.

Remark 3.7. Mapping ϕ from previous example is not lower semi-continuous because $\liminf_{x\to 2} \phi(x) = 0 < \phi(2)$ and we can't apply mentioned theorems (Theorem 9 from [4] and Theorem 3 from [13]) for obtaining a fixed point in this example. On the other hand, if a mapping f satisfies the conditions of any of these two theorems then it has a fixed point and it can be proven that mapping f is a weakly orbitally continuous (see Remark 2.12 in [9]). Therefore, Theorem 3.4 contains these theorems as a particular case. This also holds for Theorem 1.4.

Next Theorem is a corollary of Theorem 3.4.

Theorem 3.8. *Let f be a contractive type self-mapping of a complete Menger PM space $(X, \mathcal{F}, T)$. Suppose $\phi : X \mapsto [0, \infty)$ is a function such that*

$$F_{x,fx}(t) \geq \varepsilon_0 \left(t - \left(\phi(x) - \phi(fx)\right)\right),$$

holds for every $x \in X$ and $t > 0$. If f is a weakly orbitally continuous or f^k is continuous or f is k-continuous mapping, for some $k \geq 1$, then f has a unique fixed point.

We now show that Theorem 3.2 as well as Theorem 3.4 characterizes probabilistic metric completeness.

Theorem 3.9. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. If every k-continuous or weakly orbitally continuous self-mappings of X satisfying the condition (3.1) of Theorem 3.2 or condition (3.2) of Theorem 3.4 has a fixed point, then X is complete.*

Proof. Let us assume that every k-continuous self-mapping of X satisfying condition (3.1) of Theorem 3.2 or condition (3.2) of Theorem 3.4 possesses a fixed point. We will prove that X is a complete Menger PM-space. If we suppose opposite, i.e that X is not complete, then there exists a Cauchy sequence in X, say $\{a_n\}_{n\in\mathbb{N}}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$, there exists a least positive integer $N(x)$ such that $x \neq a_{N(x)}$ and

$$F_{a_{N(x)},a_m}(t) > \frac{1}{2}\left(1 + F_{x,a_{N(x)}}(t)\right) \tag{3.7}$$

is satisfied, for every $m \geq N(x)$ and $t > 0$. Let us define a mapping $f : X \mapsto X$ by $f(x) = a_{N(x)}$.

Then, $f(x) \neq x$ for every x in X. Using (3.7) for all x, y in X and every $t > 0$, we get

$$F_{fx,fy}(t) = F_{a_{N(x)},a_{N(y)}}(t) > \frac{1}{2}\Big(1 + F_{x,a_{N(x)}}(t)\Big) = \frac{1}{2}\big(1 + F_{x,fx}(t)\big) \tag{3.8}$$

if $N(x) \leq N(y)$, or

$$F_{fx,fy}(t) = F_{a_{N(x)},a_{N(y)}}(t) > \frac{1}{2}\Big(1 + F_{y,u_{N(y)}}(t)\Big) = \frac{1}{2}\big(1 + F_{y,fy}(t)\big)$$

if $N(x) > N(y)$. This implies that

$$F_{fx,fy}(t) > \frac{1}{2}\min\Big\{1 + F_{x,fx}(t), 1 + F_{y,fy}(t)\Big\} > \min\Big\{F_{x,fx}(t), F_{y,fy}(t)\Big\},$$

holds for all x, y in X and every $t > 0$. In other words, we can choose $0 < \delta(\epsilon) \leq \min\{\varepsilon, 1 - \epsilon\}$, for given $\epsilon \in (0, 1)$ such that

$$\epsilon - \delta < \min\big\{F_{x,fx}(t), F_{y,fy}(t)\big\} < \epsilon \Rightarrow F_{fx,fy}(t) \geq \epsilon,$$

is satisfied for all x, y in X and every $t > 0$. Hence, it is clear that the mapping f satisfies condition (3.1) of Theorem 3.2.

On the other hand, taking $y = fx$ in (3.8) we get $N(fx) > N(x)$ and

$$F_{fx,f^2x}(t) = F_{a_{N(x)},a_{N(fx)}}(t) > \frac{1}{2}\Big(1 + F_{x,a_{N(x)}}(t)\Big) = \frac{1}{2}\big(1 + F_{x,fx}(t)\big),$$

for every $t > 0$. From the previous inequality we have that

$$\sup\big\{t : F_{fx,f^2x}(t) \leq 1 - \alpha\big\} \leq \frac{1}{2}\sup\big\{t : F_{x,fx}(t) \leq 1 - \alpha\big\}, \tag{3.9}$$

holds for every $\alpha \in [0, 1]$. Now, let us define a function $\phi : X \mapsto [0, \infty)$ by $\phi(x) = 2\sup\big\{t : F_{x,fx}(t) \leq 1 - \alpha_0\big\}$, for arbitrary fixed $\alpha_0 \in [0, 1]$. Then using (3.9) we get that

$$\begin{aligned} \phi(x) - \phi(fx) &= 2\sup\big\{t : F_{x,fx}(t) \leq 1 - \alpha_0\big\} - 2\sup\big\{t : F_{fx,f^2x}(t) \leq 1 - \alpha_0\big\} \\ &\geq 2\sup\big\{t : F_{x,fx}(t) \leq 1 - \alpha_0\big\} - \sup\big\{t : F_{x,fx}(t) \leq 1 - \alpha_0\big\} \\ &= \sup\big\{t : F_{x,fx}(t) \leq 1 - \alpha_0\big\}, \end{aligned} \tag{3.10}$$

for arbitrary fixed $\alpha_0 \in [0, 1]$. From (3.10) and Remark 3.5 it is clear that f satisfies condition (3.2) of Theorem 3.4.

Since the range of f is contained in the non-convergent Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $\{fx_n\}_{n\in\mathbb{N}}$ converges for which the condition $fx_n \to t \Rightarrow f^2x_n \to ft$ is violated. Therefore, f is a 2-continuous mapping. In a similar manner can be proved that f is a weakly orbitally continuous. Thus, f is a 2-continuous as well as weak orbitally continuous self-mapping of X which satisfy the condition (3.1) of Theorem 3.2 and condition (3.2) of Theorem 3.4, but does not possess a fixed point. This contradicts the hypothesis of the theorem. Hence X is complete. This completes the proof. □

Definition 3.10. A Menger PM-space $(X, \mathcal{F}, T)$ is said to satisfy Cantor's intersection property if every nested sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty, closed subsets of X such that $\delta_{F_n} \to \varepsilon_0$, as $n \to \infty$ has a nonempty intersection.

Having in mind Theorem 2.11 it follows that intersection from previous definition consists of exactly one point. Recently, Tafti and Azhini [17] proved that $(X, \mathcal{F}, T)$ is a complete probabilistic metric space if and only if Cantor's intersection property holds in X.

Now, we will show that Theorems 3.2 and 3.4 characterize Cantor's intersection property.

Theorem 3.11. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let f be a self-mapping of X satisfying condition* (3.1) *of Theorem 3.2 or condition* (3.2) *of Theorem 3.4. If X satisfies Cantor's intersection property and f^k is continuous or f is k-continuous, for some $k \geq 1$, or f is a weakly orbitally continuous, then f has a fixed point and f is continuous at z if and only if $\lim\limits_{x\to z} \min\left\{F_{x,fx}(t), F_{z,fz}(t)\right\} = 1$, for every $t > 0$.*

Proof. Let x_0 be arbitrary point in X. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X recursively by $x_n = fx_{n-1}$, $n \in \mathbb{N}$. Then from Theorem 3.1 it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let us consider nested sequence of nonempty closed sets defined by

$$G_n = \{x_n, x_{n+1}, \dots\} \quad \text{and} \quad F_n = \overline{G_n},\ n \in \mathbb{N}.$$

It is obvious that $\delta_{F_n} \to \varepsilon_0$, as $n \to \infty$. Since Cantor's intersection property holds in X, in accordance with Theorem 2.11 there is exactly one point $z \in F_n$, for every $n \in \mathbb{N}$, which is obviously the limit of the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$. Now, following the proof of Theorem 3.2 or Theorem 3.4 it is easy to prove that z is the fixed point of f and f is continuous at z if and only if $\lim\limits_{x\to z} \min\{F_{x,fx}(t), F_{z,fz}(t)\} = 1$, for every $t > 0$. This completes the proof. □

Theorem 3.12. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. If every k-continuous or weakly orbitally continuous self-mappings of X satisfying the condition* (3.1) *of Theorem 3.2 or condition* (3.2) *of Theorem 3.4 has a fixed point, then X satisfies Cantor's intersection property.*

Proof. Let us assume that every k-continuous or weakly orbitally continuous self-mapping of X satisfying condition (3.1) or condition (3.2) possesses a fixed point. We assert that X satisfies Definition 3.10. For, if not, then there exists a nested sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty, closed subsets of X such that $\delta_{F_n} \to \varepsilon_0$, as $n \to \infty$ which has empty intersection. Construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_i \in F_i$, for every $i \in \mathbb{N}$. Since $\delta_{F_n} \to \varepsilon_0$, as $n \to \infty$, for every $\lambda \in (0, 1)$, and every $t > 0$ there exists a positive integer n_0 such that $F_{x_n,x_m}(t) > 1 - \lambda$, for every $n, m \geq n_0$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, but non-convergent Cauchy sequence since the sequence $\{F_n\}_{n\in\mathbb{N}}$ has empty intersection. As done in the proof of Theorem 3.9 we can now define a k-continuous as well as weakly orbitally continuous self-mapping of X satisfying (3.1) and (3.2) which does not possess a fixed point. This contradicts our hypothesis. Therefore, Cantor's intersection theorem holds in X. $\square$

Remark 3.13. Theorems 3.11 and 3.12 show that fixed point property for k-continuous or weakly orbitally continuous self-mappings of a Menger PM-space $(X, \mathcal{F}, T)$ satisfying (3.1) and (3.2) characterize Cantor's intersection property.

If we combine Theorem 3.9 and Theorem 3.12 then we get the following theorem.

Theorem 3.14. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. The following assertions are equivalent:*

(a) *X is complete;*
(b) *X satisfies Cantor's intersection property;*
(c) *every k-continuous or weakly orbitally continuous self-mapping f of X satisfying the conditions $F_{fx,fy}(t) \geq \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\}$ and*

$$\text{given } \epsilon \in (0,1) \text{ there exists } \delta \in (0,\epsilon] \text{ such that}$$
$$\epsilon - \delta < \min\left\{F_{x,fx}(t), F_{y,fy}(t)\right\} < \epsilon \;\Rightarrow\; F_{fx,fy}(t) \geq \epsilon,$$

has a fixed point;
(d) *every k-continuous or weakly orbitally continuous self-mapping f of X such that there exists a function $\phi : X \to [0,\infty)$ satisfying the condition*

$$F_{x,fx}(t) \geq \varepsilon_0\left(t - \left(\phi(x) - \phi(fx)\right)\right).$$

for every $x \in X$ and $t > 0$, has a fixed point.

Acknowledgement: The second listed author acknowledge the support of the Ministry of Education, Science and Technological Development of the Republic of Serbia, institutionally funded through Military Academy, University of Defence.

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