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# REMARKS ON THE QUASI-METRIC EXTENSION OF THE MEIR-KEELER FIXED POINT THEOREM WITH AN APPLICATION TO $D^3$-SYSTEMS

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**ABSTRACT.** In this paper we first observe that the classical Meir-Keeler fixed point theorem admits a full extension to complete $T_1$ quasi-metric spaces. Related to this fact we show that the key example of the paper "On the MeirKeeler theorem in quasi-metric spaces" (J. Fixed Theory Appl. 23:37 (2021)) is not correct. We present a quasi-metric version of a fixed point theorem due to B. Samet, C. Vetro and H. Yazidi, that involves a Meir-Keeler type contraction and, finally, connections between our quasi-metric version of the Meir-Keeler theorem and discrete disperse dynamical systems ($D^3$-systems in short) are discussed.

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## 1. **Introduction and preliminaries**

We start by reminding some notions and properties which will be essential throughout this paper. Our basic reference for quasi-metric spaces is [3].

Let us recall that a quasi-metric on a set $X$ is a function $d$ from $X \times X$ to $\mathbb{R}^+$ (the set of non-negative real numbers) such that for every $x, y, z \in X$ the following conditions are fulfilled:

(qm1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;

(qm2) $d(x, z) \leq d(x, y) + d(y, z)$.

By a quasi-metric space we mean a pair $(X, d)$ where $X$ is a set and $d$ is a quasi-metric on $X$.

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Any quasi-metric $d$ on a set $X$ induces a $T_0$ topology $\tau_d$ on $X$ which has as a base the family of open balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-metric space $(X,d)$ is said to be a $T_1$ quasi-metric space provided that $\tau_d$ is a $T_1$ topology on $X$. Consequently, a quasi-metric space $(X,d)$ is a $T_1$ quasi-metric space if and only if the following condition holds: $d(x,y) = 0 \Leftrightarrow x = y$.

According to [5] a sequence $\{x_n\}$ in a quasi-metric space $(X,d)$ is called a Cauchy sequence if for each $\varepsilon > 0$ there is a natural number $n_\varepsilon$ such that $d(x_n,x_m)$ whenever $n,m \geq n_\varepsilon$, and $(X,d)$ is called complete (bicomplete in the classical terminology) if for each Cauchy sequence $\{x_n\}$ in $(X,d)$ there is an $x \in X$ such that $d(x,x_n) \to 0$ and $d(x_n,x) \to 0$ as $n \to \infty$.

There are abundant examples of complete quasi-metric spaces in the literature (see e.g. [3,8]).

**Remark 1.** It is well known, and easy to see, that given a quasi-metric $d$ on a set $X$, the function $d^s$ defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$, is a metric on $X$. Thus, a sequence in a quasi-metric space $(X,d)$ is a Cauchy sequence if and only if it is a Cauchy sequence in $(X,d^s)$, and, hence, a quasi-metric space $(X,d)$ is complete if and only if the metric space $(X,d^s)$ is complete.

The problem of extending the celebrated Meir-Keeler fixed point theorem to quasi-metric spaces has been recently discussed by Rachid, Mitrović, Parvaneh and Bagheri [5]. This problem was previously studied in [1] for $T_1$ quasi-metric spaces and in [8] for the general case. In particular, it was given in [8] an easy example of a Meir-Keeler map on a complete non-$T_1$ quasi-metric space that has no fixed points and also was obtained a fixed point theorem from which we immediately deduce that every Meir-Keeler map on a complete $T_1$ quasi-metric space has a unique fixed point (see Corollary 1 below). Related to this result we shall show that both the key example of [1] and a natural modification of it are not valid. We shall extend to the quasi-metric framework a fixed point theorem obtained by Samet, Vetro and Yazidi in [12], which involves a Meir-Keeler type contraction and, finally, connections between our quasi-metric version of the Meir-Keeler theorem and discrete disperse dynamical systems ($D^3$-systems in short) will be discussed.

## 2. The Meir-Keeler fixed point theorem in quasi-metric spaces

Meir and Keeler obtained in [4] their renowned fixed point theorem which is established in the next.

**Theorem 1** ( [4]). *Let $T$ be a self map of a complete metric space $(X,d)$. If for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x,y \in X$,*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow d(Tx,Ty) < \varepsilon,$$

*then $T$ has a unique fixed point.*

Theorem 1 suggests the following concept.

**Definition 1**. A self map $T$ of a quasi-metric space $(X,d)$ is called a Meir-Keeler map on $(X,d)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in X$,

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow d(Tx,Ty) < \varepsilon.$$

As we indicated above, it was given in [8, page 2] an easy example of a Meir-Keeler map on a complete non-$T_1$ quasi-metric space that has no fixed points. Here we reproduce it for the sake of completeness.

**Example 1** ( [8]). Let $(X,d)$ be the complete quasi-metric space such that $X = \{0,1\}$, and $d(0,0) = d(1,1) = d(0,1) = 0$, $d(1,0) = 1$. Define $T : X \to X$ by $T0 = 1$ and $T1 = 0$. Now, given $\varepsilon > 0$ choose $\delta = \varepsilon$ and let $x, y \in X$ such that $\varepsilon \leq d(x,y) < 2\varepsilon$. Then $x = 1$, $y = 0$. Therefore $d(Tx,Ty) = d(T1,T0) = d(0,1) = 0 < \varepsilon$, so $T$ is a Meir-Keeler map on $(X,d)$.

It is well known that every quasi-metric $d$ on a set $X$ induces a partial order $\leq_d$ on $X$ (the so-called specialization order) defined by

$$x \leq_d y \Leftrightarrow d(x,y) = 0.$$

The specialization order will play a central role in the last section of this paper.

Notice that a self map $T$ of a quasi-metric space $(X,d)$ is $\leq_d$-increasing if and only if $d(Tx,Ty) = 0$ whenever $d(x,y) = 0$ (see [8, page 2]).

Thus, the fact that the self map of Example 1 is not $\leq_d$-increasing, suggests the following notion.

**Definition 2** ( [8]). A Meir-Keeler map on a quasi-metric space $(X,d)$ is said to be a $d$-Meir-Keeler map provided that it is $\leq_d$-increasing.

Then, it was proved in [8, Theorem 1] the following result.

**Theorem 2** ( [8]). *Every $d$-Meir-Keeler map on a complete quasi-metric space $(X,d)$ has a unique fixed point.*

**Corollary 1.** *Every Meir-Keeler map on a complete $T_1$ quasi-metric space has a unique fixed point.*

**Proof.** Let $T$ be a Meir-Keeler map on a complete $T_1$ quasi-metric space $(X, d)$. Let $x, y \in X$ such that $d(x, y) = 0$. Then $x = y$, and, hence, $d(Tx, Ty) = d(Tx, Tx) = 0$. We deduce that $T$ is a $d$-Meir Keeler map on $(X, d)$. Theorem 2 concludes the proof.

Rachid, Mitrović, Parvaneh and Bagheri asserted in [5] that the Meir-Keeler fixed point theorem can not be extended to the setting of quasi-metric spaces (compare Example 1 above). To this end, they make the following construction of a self map of a complete $T_1$ quasi-metric space, claiming that it is a Meir-Keeler map free of fixed points (see [5, Proposition 2.1]).

Let $X = [2, 3]$, let $T$ be the self map of $X$ given by $T\theta = 2 + \theta/3$ for all $\theta \in X$, and let $d$: $X \times X \to \mathbb{R}^+$ defined as $d(\theta, \vartheta) = \theta - \vartheta$ if $\theta \geq \vartheta$, and $d(\theta, \vartheta) = (1/\theta)^{n_\theta}$ if $\theta < \vartheta$, where:

$n_\theta = E(\ln 3 \ \ln^{-1}(6T\theta/5\theta)) + 1$ for all $\theta \in [2, 3)$, and $E(\cdot)$ is the integer part of function.

Then, the authors of [5] proved the next properties:

(p1) $n_\theta \leq n_\vartheta$ whenever $\theta \leq \vartheta$;

(p2) $d$ is a quasi-metric on $X$ (actually, it is a $T_1$ quasi-metric on $X$);

(p3) the Cauchy sequences in $(X, d)$ are those that are eventually constant, and, thus, $(X, d)$ is complete.

They also asserted that $T$ is a Meir-Keeler map on $(X, d)$.

Note that $\theta = 3$ is a fixed point of $T$, so this example, as written, is not valid for the goals of the authors.

For this reason we proceed to modify that example, in a natural way, as follows.

Let $X_0 = [2, 3)$, let $T$ be the self map of $X_0$ given by $T\theta = 2+\theta/3$ for all $\theta \in X_0$, and denote also by $d$ be the restriction of $d$ to $X_0$.

Then $T$ has no fixed points in $X_0$. Moreover, it is obvious that the properties (p1), (p2) and (p3) remain true, so $(X_0, d)$ is a complete $T_1$ quasi-metric space. However, by Corollary 1, $T$ is not a Meir-Keeler map on $(X_0, d)$.

Indeed, the authors claim (see [5, page 6]) that $d(\theta, \vartheta) > 3d(T\theta, T\vartheta)$ whenever $\theta < \vartheta$. Since we also have that $d(\theta, \vartheta) \geq 3d(T\theta, T\vartheta)$ whenever $\theta \geq \vartheta$, we would get $d^s(T\theta, T\vartheta) \leq d^s(\theta, \vartheta)/3$ for all $\theta, \vartheta \in X_0$ (recall that by $d^s$ we denote the metric defined in Remark 1). Consequently, by Remark 1, again, and the Banach contraction principle, $T$ should have a fixed point, which yields a contradiction.

In fact, the error occurs on lines 11-12 of page 6 of [5] because the inequality

$$\ln \frac{(1/5\theta)^{n_\theta}}{(1/6T\theta)^{n_\theta}} < \ln \frac{(1/\theta)^{n_\theta}}{(1/T\theta)^{n_{T\theta}}},$$

is not true in general. Indeed, routine calculations show that $n_\theta = 6$ for all $\theta \in [2.9, 3)$. Since $\theta < T\theta < 3$, we deduce that $n_\theta = n_{T\theta} = 6$ for all $\theta \in [2.9, 3)$. So we come to the contradiction that $6 < 5$.

## 3. **A quasi-metric extension of a theorem by Samet, Vetro and Yazidi**

Samet, Vetro and Yazidi proved in [12, Theorem 2.1] the following theorem for a Meir-Keeler type contraction.

**Theorem 3** ( [12]). *Let $T$ be a self map of a complete metric space $(X, d)$. If for each $\varepsilon$ >0 there exists $\delta > 0$ such that for any $x, y \in X$,*

$$2\varepsilon \le d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)} + d(x, y) < 2\varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon,$$

*then $T$ has a unique fixed point.*

Next, we apply Theorem 3 to deduce a quasi-metric version of it.

**Theorem 4.** *Let $T$ be a $\le_d$-increasing self map of a complete quasi-metric space $(X, d)$. If for each $\varepsilon$ >0 there exists $\delta > 0$ such that for any $x, y \in X$,*

$$2\varepsilon \le d(y, Ty)\frac{1 + d(x, Tx)}{1 + d^s(x, y)} + d(x, y) < 2\varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon,$$

*then $T$ has a unique fixed point.*

**Proof.** For each $x, y \in X$, put

$$M_d(x, y) = \frac{1}{2}\left(d(y, Ty)\frac{1 + d(x, Tx)}{1 + d^s(x, y)} + d(x, y))\right),$$

and

$$M_{d^s}(x, y) = \frac{1}{2}\left(d^s(y, Ty)\frac{1 + d^s(x, Tx)}{1 + d^s(x, y)} + d^s(x, y)\right).$$

Since $d(x, y) \le d^s(x, y)$ for all $x, y \in X$, we deduce that $M_d(x, y) \le M_{d^s}(x, y)$ for all $x, y \in X$.

Fix $\varepsilon > 0$. By hypothesis, there exists $\delta := \delta(\varepsilon, d)$ such that for any $x, y \in X$,

$$\varepsilon \le M_d(x, y) < \varepsilon + \frac{\delta}{2} \Rightarrow d(Tx, Ty) < \varepsilon.$$

We now show that for any $x, y \in X$,

$$\varepsilon \leq M_{d^s}(x,y) < \varepsilon + \frac{\delta}{2} \Rightarrow d^s(Tx,Ty) < \varepsilon.$$

Indeed, let $x, y \in X$ such that $\varepsilon \leq M_{d^s}(x,y) < \varepsilon + \delta/2$.

Suppose, without loss of generality, that $d^s(Tx,Ty) = d(Tx,Ty)$.

- If $\varepsilon \leq M_d(x,y) < \varepsilon + \delta/2$, then $d(Tx,Ty) < \varepsilon$, i.e., $d^s(Tx,Ty) < \varepsilon$.
- If $M_d(x,y) < \varepsilon$ we distinguish two cases.

  Case 1. $d(x,y) = 0$. Then $d(Tx,Ty) = 0$ because $T$ is $\leq_d$-increasing, so $d^s(Tx,Ty) = 0 < \varepsilon$.

  Case 2. $d(x,y) > 0$. Then $M_d(x,y) > 0$, so $d(Tx,Ty) < M_d(x,y)$, i.e., $d^s(Tx,Ty) < M_d(x,y) < \varepsilon$.
- If $M_d(x,y) \geq \varepsilon + \delta/2$, we deduce that $M_d(x,y) > M_{d^s}(x,y)$ which contradicts the fact that $M_d(x,y) \leq M_{d^s}(x,y)$ for all $x, y \in X$.

Since, by Remark 1, the metric space $(X, d^s)$ is complete we can apply Theorem 3 to deduce that $T$ has a unique fixed point. This finishes the proof.

The following is an example where we can apply Theorem 4 but not Theorem 2.

**Example 2.** Let $X = \{0\} \cup [1,3]$ and let $d$ be the quasi-metric on $X$ given by $d(x,y) = \max\{x - y, 0\}$ for all $x, y \in X$.

Since $d^s(x,y) = |x - y|$ for all $x, y \in X$, we have that $(X, d^s)$ is a compact metric space and, thus, a complete metric space. Hence $(X, d)$ is a complete quasi-metric space by Remark 1.

Let $T : X \to X$ defined as $T3 = 1$ and $Tx = 0$ for all $x \in X \backslash \{3\}$.

It is clear that $T$ is $\leq_d$-increasing because if $d(x,y) = 0$, then $x \leq y$, so $Tx \leq Ty$, and thus $d(Tx,Ty) = 0$.

Observe that $T$ is not a $d$-Meir-Keeler map because $d(3,2) = d(T3,T2)$. Hence we can not apply Theorem 2.

Now fix $\varepsilon > 0$ and put $\delta = \varepsilon$. We shall show that for any $x, y \in X$, the contraction condition in the statement of Theorem 4 is fulfilled.

To this end, define $M_d(x,y)$ as in the proof of Theorem 4 and suppose that $\varepsilon \leq M_d(x,y) < \varepsilon + \delta/2$.

- If $x, y \in X \backslash \{3\}$ or $y = 3$, we get $d(Tx,Ty) = 0 < \varepsilon$.

- If $x = 3$ and $y = 0$ we get $M_d(x, y) = M_d(3, 0) = 3/2$. Thus, we have $\varepsilon \leq 3/2 < 3\varepsilon/2$. So $\varepsilon > 1$, and, consequently, $d(Tx, Ty) = d(T3, T0) = d(1, 0) = 1 < \varepsilon$.
- If $x = 3$ and $y \in X \backslash \{0, 3\}$ we get
  $$M_d(x, y) = M_d(3, y) = \frac{1}{2}\left(y\frac{1+2}{1+3-y} + 3 - y\right) = \frac{1}{2}\left(\frac{y^2 - 4y + 12}{4 - y}\right).$$
  Therefore
  $$\varepsilon \leq \frac{1}{2}\left(\frac{y^2 - 4y + 12}{4 - y}\right) < \frac{3\varepsilon}{2}.$$
  Since $y \geq 1$, we deduce that $3 \leq (y^2 - 4y + 12)/(4 - y)$, and hence $3/2 < 3\varepsilon/2$, which implies that $\varepsilon > 1$. Consequently, we have $d(Tx, Ty) = d(T3, Ty) = d(1, 0) = 1 < \varepsilon$.

We have shown that the contraction condition in Theorem 4 is satisfied. Hence $T$ has a unique fixed point (in fact $x = 0$ is the unique fixed point of $T$).

We conclude this section with the following open question: Can be replaced "$d^s(x, y)$" with "$d(x, y)$" in the statement of Theorem 4 ?

## 4. $D^3$-systems on quasi-metric spaces and an application

Rubinov introduced in [10] the so-called discrete disperse dynamical systems ($D^3$-systems in short) defined on compact metric spaces, which provides abstract models of economic dynamics [11]. Since then some authors have investigated this kind of dynamical systems (see e.g. [17] and the references therein).

We generalize the notion of a $D^3$-system to the quasi-metric framework (note that our approach is more general to the one proposed in [6, Definition 1]).

**Definition 1.** Let $(X, d)$ be a quasi-metric space. A set-valued map $D : X \rightrightarrows \mathcal{P}_0(X)$ (the family of non-empty subsets of $X$) is called a $D^3$-system on $(X, d)$ if for any $x \in X$, $D(x)$ is a compact subset of the topological space $(X, \tau_d)$.

The following result shows that it is easy to obtain suitable $D^3$-systems for non-$T_1$ quasi-metric spaces with the help of the specialization order.

**Proposition 1.** *Let $(X, d)$ be a quasi-metric space and let $D_{\leq_d} : X \rightrightarrows \mathcal{P}_0(X)$ be the set-valued map defined as $D_{\leq_d}(x) = \{y \in X : x \leq_d y\}$ for all $x \in X$. Then, every set-valued map $D : X \rightrightarrows \mathcal{P}_0(X)$ such that $x \in D(x) \subseteq D_{\leq_d}(x)$ for all $x \in X$, is a $D^3$-system on $(X, d)$.*

**Proof.** Fix $x \in X$ and let $\mathcal{V}$ be a $\tau_d$-open cover of $D(x)$. There are $V \in \mathcal{V}$ and $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq V$. Since $D_{\leq_d}(x) \subseteq B_d(x, \varepsilon)$ we conclude that $D(x) \subseteq V$.

Schellekens introduced in [13, Definition 2.6] the notion of an improver in the study of the complexity analysis of programs and algorithms. We adapt this concept to a more general context as follows.

**Definition 2.** Let $(X, d)$ be a quasi-metric space. We say that a self map $T$ of $X$ is an improver of a point $x_0 \in X$ if $x_0 \leq_d Tx_0$.

Intuitively, if $T$ is an improver of $x_0$, with $x_0 \neq Tx_0$, the element $Tx_0$ contains more information than $x_0$ in the computational process represented by $T$. In other words, $Tx_0$ is 'more efficient' than $x_0$.

By $\mathcal{I}_T(X, d)$ we shall denote the set of all points of $X$ for which $T$ is an improver. Notice that $\mathcal{I}_T(X, d)$ could be the empty set.

**Definition 3.** Let $T$ be a self map of a quasi-metric space such that $\mathcal{I}_T(X, d) \neq \varnothing$. We say that $z \in X$ is an optimal point of $T$ with respect to the $D^3$-system $D_{\leq_d}$ if it satisfies the following two conditions:

(op1) $z \in \bigcap_{x \leq_d Tx} D_{\leq_d}(x)$;

(op2) $z \leq_d y$ for every $y \in \bigcap_{x \leq_d Tx} D_{\leq_d}(x)$.

Note that, by (op2), if $T$ has an optimal point $z \in X$ then $z$ is its unique optimal point.

**Remark 2.** Intuitively, if $z$ is the optimal point of $T$ then, by (op1), $z$ is more efficient than any element in $\mathcal{I}_T(X, d)$, and by (op2), $z$ contains all amount of information provided by $\mathcal{I}_T(X, d)$ but not more of the necessary. Thus, condition (op2) provides a suitable caution which is especially interesting in the case of those quasi-metric spaces $(X, d)$ for which there is a top element $\top$ for $\leq_d$ (this situation occurs, for instance, in the case of the complexity quasi-metric space [7, 13]).

Now denote by $\omega$ the set of non-negative integer numbers and suppose that $R : \omega \to \mathbb{R}^+$ is a recurrence equation for which it is possible to associate a suitable self map (functional) $\Phi$ of $(\mathbb{R}^+)^\omega$. If $(\mathbb{R}^+)^\omega$ is endowed with a complete quasi-metric $d$ and $\mathcal{S}$ is a closed subset of the metric space $((\mathbb{R}^+)^\omega, d^s)$ such that the restriction of $\Phi$ to $\mathcal{S}$ is a self map of $\mathcal{S}$, it is natural to ask if there exists an $f_0 \in \mathcal{S}$ for which $\Phi$ is an improver. In that case, we would obtain that $\Phi f_0$ is at least as efficient as $f_0$ on all inputs. In many cases it suffices to apply the quasi-metric version of the Banach contraction principle ( [8, Corollary 1], [13, Theorem 3.4]), to obtain an affirmative answer to this requirement (see e.g. [2, 7, 9, 13]).

In the sequel we present an instance where Banach's contraction principle does not work and yet we can apply Theorem 2 to show that the self map $\Phi$ has a unique fixed point $f_0 \in \mathcal{S}$ and thus $\Phi f_0 \in D_{\leq_d}(f_0)$ obviously. Moreover, we shall see that $f_0$ is the (unique) optimal point of $\Phi$ with respect to the $D^3$ -system $D_{\leq_d}$.

Motivated by the study of certain nonlinear difference equations [14–16], it was analyzed in [8, Section 3] the recurrence equation $R : \omega \to [0,1]$ defined by $R(0) = c$ $(0 < c \leq 1)$, and

$$R(n) = \frac{R(n-1)}{1 + R(n-1)},$$

for all $n \in \omega \backslash \{0\}$.

Denote by $\mathcal{S}$ the set of all functions from $\omega$ to $[0,1]$, and in the light of the recurrence equation $R$ defined above construct a map $\Phi : \mathcal{S} \to \mathcal{S}$ as follows:

For each $f \in \mathcal{S}$, $\Phi f(0) = c$, and

$$\Phi(f)(n) = \frac{f(n-1)}{1 + f(n-1)},$$

whenever $n \in \omega \backslash \{0\}$.

Let $d$ be the quasi-metric on $\mathcal{S}$ given by

$$d(f,g) = \sup_{n \in \omega} \max(g(n) - f(n), 0),$$

for all $f, g \in \mathcal{S}$.

Since $d^s$ is the supremum metric on $\mathcal{S}$, $(\mathcal{S}, d^s)$ is a complete metric space, so $(\mathcal{S}, d)$ is a complete quasi-metric space.

Next we show that $\Phi$ is not a Banach contraction on $(\mathcal{S}, d)$.

Fix an $r \in (0,1)$. Choose $\varepsilon \in (0,1)$ such that $1/(1+\varepsilon) > r$. Let $f, g \in \mathcal{S}$ such that $f(n) = 0$ for all $n \in \omega$, $g(1) = \varepsilon$, and $g(n) = 0$ for all $n \in \omega \backslash \{1\}$.

Then we have

$$d(\Phi f, \Phi g) = \Phi g(2) = \frac{g(1)}{1 + g(1)} = \frac{\varepsilon}{1+\varepsilon} = \frac{g(1)}{1+\varepsilon} = \frac{d(f,g)}{1+\varepsilon} > rd(f,g).$$

We are going to prove that, however, $\Phi$ is a $d$-Meir Keeler map on $(\mathcal{S}, d)$.

- $\Phi$ is $\leq_d$-increasing.

  Indeed, if $d(f,g) = 0$, we get $g(n) \leq f(n)$ for all $n \in \omega$, so $\Phi g(n) \leq \Phi f(n)$ for all $n \in \omega$, which implies that $d(\Phi f, \Phi g) = 0$.

- Fix $\varepsilon > 0$. Put $\delta = \min\{1, \varepsilon^2\}$, and let $f, g \in \mathcal{S}$ such that $\varepsilon \leq d(f, g) < \varepsilon + \delta$.

  Analogously to [8, Section 3], we get

$$\max\{\Phi g(n) - \Phi f(n), 0\} \leq \frac{d(f,g)}{1 + d(f,g)},$$

for all $n \in \omega$, so

$$d(\Phi f, \Phi g) \leq \frac{d(f,g)}{1 + d(f,g)} < \frac{\varepsilon + \delta}{1 + \varepsilon} \leq \frac{\varepsilon + \varepsilon^2}{1 + \varepsilon} = \varepsilon.$$

Therefore, we can apply Theorem 2 to conclude that there is a unique $f_0 \in \mathcal{S}$ such that $\Phi f_0 = f_0$. So $f_0$ is the unique solution of the recurrence equation $R$.

It remains to check that $f_0$ is the (unique) optimal point of $\Phi$ with respect to the $D^3$-system $D_{\leq_d}$.

We first observe that $f_0 \in \mathcal{I}_\Phi(\mathcal{S}, d)$.

Now let $g \in \mathcal{S}$ such that $g \leq_d \Phi g$. Then

$$d(g, f_0) \leq d(g, \Phi g) + d(\Phi g, \Phi f_0) + d(\Phi f_0, f_0) = d(\Phi g, \Phi f_0) \leq \frac{d(g, f_0)}{1 + d(g, f_0)},$$

which implies that $d(g, f_0) = 0$. Hence $f_0$ satisfies condition (op1).

Finally, if $f \in \bigcap_{g \leq_d \Phi g} D_{\leq_d}(g)$, we get, in particular, that $f \in D_{\leq_d}(f_0)$, so $d(f_0, f) = 0$, i.e., $f_0 \leq_d f$. Thus, condition (op2) is also fulfilled. Consequently $f_0$ is the (unique) optimal point of $\Phi$ with respect to the $D^3$-system $D_{\leq_d}$.

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