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SHOWING THE EXISTENCE AND UNIQUENESS OF SOLUTION FOR CERTAIN RATIONAL DIFFERENCE EQUATIONS BY MEANS OF A FIXED POINT THEOREM OF BROWDER'S TYPE

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ABSTRACT. By applying a quasi-metric version of the famous Browder's fixed point theorem, we prove the existence and uniqueness of solution of certain rational difference equations. Several special cases of well-known and relevant rational difference equations are contained in our approach.

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1. INTRODUCTION AND PRELIMINARIES

For a long time the study of several aspects of rational difference equations attracted the attention of numerous researchers, producing a wide literature on this topic. The excellent book [4] contains an extensive list of well-known and distinguished instances of this kind of difference equations (see also the recent and interesting articles [11, 12] and the references there in).

The main objective of this paper is to show, in a straightforward manner, the existence and uniqueness of solution for certain rational difference equations by means of fixed point techniques in the frame of quasi-metric spaces. Several special cases of typical rational difference equations are contained in our approach.

In the sequel, by $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ and ω we will denote the sets of real numbers, the set of non-negative real numbers, the set of positive integer numbers and the set of non-negative integer numbers, respectively. Our basic reference for quasi-metric spaces will be [5]. A quasi-metric on a set X is a function $d : X \times X \to \mathbb{R}^+$ verifying the following conditions for all $x, y, z \in X$:

(qm1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;

(qm2) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) where X is a set and d is a quasi-metric on X.

If (X, d) is a quasi-metric space, the function $d^s : X \times X \to \mathbb{R}^+$ given by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

A quasi-metric space (X, d) is called bicomplete if the metric space (X, d^s) is complete.

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A basic and important example of a bicomplete quasi-metric space is the pair $(\mathbb{R}, d)$ where $d(x,y) = \max\{x-y, 0\}$ for all $x, y \in \mathbb{R}$ (note that d^s is the Euclidean metric on $\mathbb{R}$).

Given $M > 0$, we denote by $[0,M]^\omega$ the set of all functions (sequences) from ω to $[0,M]$.

In our context the so-called supremum quasi-metric on $[0,M]^\omega$, defined below, will play a central role because it allows calculations and estimates to be carried out in a simple and relatively fast way.

Denote by $d_{\sup}$ the function from $[0,M]^\omega \times [0,M]^\omega$ to $[0,M]$ given by

$$d_{\sup}(f,g) = \sup_{n\in\omega} \max\{g(n) - f(n), 0\},$$

for all $f, g \in [0,M]^\omega$.

Then $d_{\sup}$ is a quasi-metric on $[0,M]^\omega$ which will be called the supremum quasi-metric on $[0,M]^\omega$.

Since $(d_{\sup})^s$ is the well-known supremum metric on $[0,M]^\omega$, $([0,M]^\omega, (d_{\sup})^s)$ is a complete metric space, so we have the following.

Proposition 1. *The quasi-metric space* $([0,M]^\omega, d_{\sup})$ *is bicomplete.*

The use of the supremum quasi-metric instead of the supremum metric has the advantage that we have $d_{\sup}(f,g) = 0$ whenever $g(n) \leq f(n)$ for all $n \in \omega$. If, in addition, $f \neq g$, the equality $d_{\sup}(f,g) = 0$ can be interpreted, from a computational view, as g is more 'efficient' than f. Of course, that relevant information remains hidden if one only computes $(d_{\sup})^s(f,g)$.

Besides, our start point will be the famous Browder fixed point theorem [3], which is established as follows.

Theorem 1 (Browder). *Let* (X,d) *be a complete metric space. If* T *is a self map of* X *for which there exists a non-decreasing right continuous function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying* $\varphi(t) < t$ *for all* $t > 0$, *and* $d(Tx,Ty) \leq \varphi(d(x,y))$ *for all* $x, y \in X$, *then* T *has a unique fixed point* $z \in X$. *Furthermore* $\lim_{n\to\infty} d(z, T^n x) = 0$ *for all* $x \in X$.

Although there exist several well-known improvements of Browder's theorem due, among others, to Boyd and Wong [2], Matkowski[7], and Meir and Keeler [8], it will be sufficient to our purposes here to apply the following quasi-metric extension of Theorem 1 (compare [1, Theorem 2.4]).

Theorem 2. *Let* (X,d) *be a bicomplete quasi-metric space. If* T *is a self map of* X *for which there exists a non-decreasing right continuous function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying* $\varphi(t) < t$ *for all* $t > 0$, *and* $d(Tx,Ty) \leq \varphi(d(x,y))$ *for all* $x, y \in X$, *then* T *has a unique fixed point* $z \in X$. *Furthermore* $\lim_{n\to\infty} d^s(z, T^n x) = 0$ *for all* $x \in X$.

Proof. Let T be a self map of X and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing right continuous function such that $\varphi(t) < t$ for all $t > 0$, and $d(Tx,Ty) \leq \varphi(d(x,y))$ for all $x, y \in X$. Given $x, y \in X$, suppose, without loss of generality, that $d^s(Tx,Ty) = d(Tx,Ty)$. Then $d^s(Tx,Ty) \leq \varphi(d(x,y)) \leq \varphi(d^s(x,y))$. By Theorem 1, T has a unique fixed point $z \in X$ and $\lim_{n\to\infty} d^s(z, T^n x) = 0$ for all $x \in X$.

A self map T on a quasi-metric space (X,d) for which there is a function φ satisfying the conditions of Theorem 2 will be called a Browder contraction on (X,d).

As an immediate consequence of Theorem 2 we obtain the following well-known quasi-metric version of the Banach contraction principle (see e.g [1, 9]).

Corollary 1. *Let (X, d) be a bicomplete quasi-metric space. If T is a self map of X for which there exists a constant $\alpha \in (0, 1)$ such that $d(Tx, Ty) \le \alpha d(x, y)$ for all $x, y \in X$, then T has a unique fixed point $z \in X$. Furthermore $\lim_{n\to\infty} d^s(z, T^n x) = 0$ for all $x \in X$.*

A self map T on a quasi-metric space (X, d) for which there exists a constant $\alpha \in (0, 1)$ satisfying the conditions of Corollary 1 is said to be a Banach contraction on (X, d).

Remark 1. Recall that condition 'non-decreasing' can be omitted in the statement of Theorem 1 and, hence, in the statement of Theorem 2 (see e.g. [2, Theorem 1] or [6, Remark 1]).

2. EXISTENCE AND UNIQUENESS OF SOLUTION FOR TWO TYPES OF RATIONAL DIFFERENCE EQUATIONS

2.1. Difference equations of type I

We say that a difference equation with initial values $x_0, ..., x_j$, ($x_i > 0$ for all $i \in \{0, ..., j\}$), is of type I if for every $n > j$,

$$x_n = \sum_{k=1}^{n} a_{n,k} \frac{q x_{n-k}}{r + s x_{n-k}},$$

where:

(i) $a_{n,k} \ge 0$, and $0 < \sum_{k=1}^{n} a_{n,k} \le 1$ for all $n > j$;

(ii) $0 < q \le r$ and $s > 0$.

If we replace condition (ii) with

(ii') $0 < q < r$ and $s \ge 0$,

we will say that the difference equation is of type I'.

Remark 2. Notice that the difference equation considered in [9, 10] is of type I with $a_{n,1} = 1$, $a_{n,k} = 0$ for all $n \in \mathbb{N}$ and $k > 1$, and $q = r = s = 1$.

The following is an easy but representative example of a difference equation of type I:

$$x_n = a_{n,1} \frac{q x_{n-1}}{r + s x_{n-1}} + a_{n,2} \frac{q x_{n-2}}{r + s x_{n-2}},$$

for $n \ge 2$, with initial (positive) values x_0, x_1, and $0 < a_{n,1} + a_{n,2} \le 1$, $0 < q \le r$, $s > 0$.

In particular, for $a_{n,1} = a_{n,2} = 1/2$, and $q = r = s = 1$, we get the difference equation

$$x_n = \frac{x_{n-1} + x_{n-2} + 2x_{n-1}x_{n-2}}{2(1 + x_{n-1} + x_{n-2} + x_{n-1}x_{n-2})},$$

for $n \ge 2$.

Next we establish and prove the two main results of this subsection:

(R1) *Every difference equation of type I has a unique solution, which will be obtained by applying Theorem 2 to the bicomplete quasi-metric space $([0, M]^\omega, d_{\sup})$, where M is any positive constant such that $M \ge \max\{x_0, ..., x_j\}$.*

(R2) *Every difference equation of type I' has a unique solution which will be obtained by applying Corollary 1 to the bicomplete quasi-metric space $([0, M]^\omega, d_{\sup})$.*

(We shall also give an example of a difference equation of type I for which Corollary 1 does not work.)

Proof of (R1) and (R2). Given a difference equation of type I (resp. of type I') we define, in a natural way, the map Φ on $[0, M]^\omega$ such that, for any $f \in [0, M]^\omega$, $\Phi f(n) = x_n$ for all $n \in \{0, ...j\}$, and

$$\Phi f(n) = \sum_{k=1}^{n} a_{n,k} \frac{qf(n-k)}{r + sf(n-k)},$$

for all $n > j$.

Note that actually Φ is a self map of $[0, M]^\omega$ because for any $f \in [0, M]^\omega$ we have $\Phi f(n) = x_n \leq M$ if $n \in \{0, ...j\}$, and also $qf(n-k) \leq qM \leq rM \leq (r + sf(n-k))M$, by condition (ii) (resp. by condition (ii')). Hence

$$\Phi f(n) \leq \sum_{k=1}^{n} a_{n,k} M \leq M,$$

for all $n > j$, by condition (i).

Let $f, g \in [0, M]^\omega$.

If $\Phi g(n) \leq \Phi f(n)$ for all $n \in \omega$, we get $d_{\sup}(\Phi f, \Phi g) = 0$.

Otherwise, for each $n \in \mathbb{N}$ such that $0 < \Phi g(n) - \Phi f(n)$, we get

$$0 < \Phi g(n) - \Phi f(n) = \sum_{k=1}^{n} a_{n,k} \frac{qr(g(n-k) - f(n-k))}{(r + sg(n-k))(r + sf(n-k))}.$$

Put $A_n := \{k \in \{1, \ldots, n\} : g(n-k) > f(n-k)\}$.

Since $(r + sg(n-k))(r + sf(n-k)) \geq r^2 + rsg(n-k)$, we deduce that

$$\begin{aligned} 0 \quad < \quad & \Phi g(n) - \Phi f(n) \leq \sum_{k \in A_n} a_{n,k} \frac{qr(g(n-k) - f(n-k))}{(r + sg(n-k))(r + sf(n-k))} \\ \leq \quad & \sum_{k \in A_n} a_{n,k} \frac{q(g(n-k) - f(n-k))}{r + sg(n-k)}. \end{aligned}$$

For each $k \in A_n$ we have

$$q(g(n-k) - f(n-k))(r + sd_{\sup}(f, g)) \leq qrd_{\sup}(f, g) + qsg(n-k)d_{\sup}(f, g)$$

$$= q(r + sg(n-k))d_{\sup}(f, g),$$

so

$$0 < \Phi g(n) - \Phi f(n) \leq \sum_{k \in A_n} a_{n,k} \frac{qd_{\sup}(f, g)}{r + sd_{\sup}(f, g)}.$$

Since $\sum_{k \in A_n} a_{n,k} \leq 1$, we get

$$0 < \Phi g(n) - \Phi f(n) \leq \frac{qd_{\sup}(f, g)}{r + sd_{\sup}(f, g)}.$$

- If the difference equation is of type I, we define a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = qt/(r+st)$ for all $t \in \mathbb{R}^+$. Obviously φ is non-decreasing and continuous on $\mathbb{R}^+$. Furthermore $\varphi(t) < t$ for all $t > 0$ because $q \leq r$ and $s > 0$.
  Consequently
  $$d_{\sup}(\Phi f, \Phi g) = \sup_{n \varepsilon \omega} \max\{\Phi g(n) - \Phi f(n), 0\} \leq \frac{q d_{\sup}(f,g)}{r + s d_{\sup}(f,g)} = \varphi(d_{\sup}(f,g)).$$
  Therefore Φ is a Browder contraction on $([0,M]^\omega, d_{\sup})$. By Proposition 1 and Theorem 2 we conclude that Φ has a unique fixed point in $[0,M]^\omega$ which is obviously the unique solution of the given difference equation. This completes the proof of the result **(R1)**.
- If the difference equation is of type I', we get
  $$d_{\sup}(\Phi f, \Phi g) \leq \frac{q d_{\sup}(f,g)}{r + s d_{\sup}(f,g)} \leq \alpha d_{\sup}(f,g),$$
  where $\alpha = q/r < 1$.

  Therefore Φ is a Banach contraction on $([0,M]^\omega, d_{\sup})$. By Proposition 1 and Corollary 1 we conclude that Φ has a unique fixed point in $[0,M]^\omega$ which is obviously the unique solution of the given difference equation. This completes the proof of the result **(R2)**.

Finally, we show that Φ is not a Banach contraction on $([0,M]^\omega, d_{\sup})$, in general; i.e., there are difference equations of type I for which Corollary 1 does not work.

To this end, suppose the case where we have a unique initial value $x_0 \leq 1$, with $\sum_{k=1}^{n_0} a_{n_0,k} = 1$ for some $n_0 \in \mathbb{N}$ and $q = r$. Fix $\alpha \in (0,1)$. Put $\varepsilon = \min\{q(1-\alpha)/2s\alpha, 1\}$ and let $f, g \in [0,1]^\omega$ given by $f(n) = 0$ for all $n \in \omega$, and $g(n) = \varepsilon$ whenever $0 \leq n < n_0$ and $g(n) = 0$ whenever $n \geq n_0$.

Then $\Phi f(0) = x_0$ and $\Phi f(n) = 0$ for all $n \in \mathbb{N}$, so
$$d_{\sup}(\Phi f, \Phi g) \quad \geq \quad \Phi g(n_0) = \sum_{k=1}^{n_0} a_{n_0,k} \frac{q\varepsilon}{q + s\varepsilon} = \frac{q\varepsilon}{q+s\varepsilon} > \alpha\varepsilon = \alpha d_{\sup}(f,g).$$

We finish this part by illustrating Theorem 2 with a simple but methodological instance.

Indeed, consider a difference equation of type I with initial values $x_0, ..., x_j$, and such that $a_{n,1} = 1$ for all $n > j$. Thus $a_{n,k} = 0$ for $1 < k \leq n$. By the result **(R1)**, we know that the self map Φ of $[0,M]^\omega$ given, for any $f \in [0,M]^\omega$, by $\Phi f(n) = x_n$ if $n \in \{0, ..., j\}$ and
$$\Phi f(n) = \frac{q f(n-1)}{r + s f(n-1)},$$
if $n > j$, has a unique solution $h \in [0,M]^\omega$ which is also the unique solution of the recurrence equation.

Denote by f_0 the function defined on ω by $f_0(n) = 0$ for all $n \in \omega$. From Theorem 2 it follows that $\lim_{k\to\infty} (d_{\sup})^s(h, \Phi^k f_0) = 0$, so, in particular, $\lim_{k\to\infty} d_{\sup}(\Phi^k f_0, h) = 0$.

An easy computation shows that $\Phi^k f_0(n) = 0$ whenever $n \geq j + k$.

Since for each $\varepsilon > 0$ there is as $k_\varepsilon \in \mathbb{N}$ such that $d_{\sup}(\Phi^k f_0, h) < \varepsilon$ for all $k \geq k_\varepsilon$, we deduce that, in particular,
$$h(n) - \Phi^k f(n) < \varepsilon,$$
for all $n \in \omega$ and $k \geq k_\varepsilon$. Thus $h(n) < \varepsilon$ for all $n \geq j + k_\varepsilon$. Therefore $\lim_{n\to\infty} h(n) = 0$.

2.2. **Difference equations of type II**

We say that a difference equation with initial values $x_0, ..., x_j$, ($x_i > 0$ for all $i \in \{0, ..., j\}$), is of type II if for every $n > j$,

$$x_n = \frac{\sum_{k=1}^{n} a_{n,k} x_{n-k}}{c + \sum_{k=1}^{n} a_{n,k} x_{n-k}},$$

where:

(iii) $a_{n,k} \geq 0$, and $0 < \sum_{k=1}^{n} a_{n,k} \leq 1$ for all $n > j$.

(iv) $c = 1$.

If we replace condition (iv) with

(iv') $c > 1$,

we will say that the difference equation is of type II'.

Note that condition (iii) coincides with condition (i) for the difference equations of type I.

Notice also (compare Remark 2) that the difference equation considered in [9, 10] is of type II with $a_{n,1} = 1$ and $a_{n,k} = 0$ for all $n \in \mathbb{N}$ and $k > 1$.

Next we establish and prove the two main results of this subsection:

(R3) *Every difference equation of type II has a unique solution, which will be obtained by applying Theorem 2 to the bicomplete quasi-metric space* $([0, M]^\omega, d_{\sup})$, *where* M *is any positive constant such that* $M \geq \max\{x_0, ..., x_j\}$.

(R4) *Every difference equation of type II' has a unique solution which will be obtained by applying Corollary 1 to the bicomplete quasi-metric space* $([0, M]^\omega, d_{\sup})$.

(We shall also give an example of a difference equation of type II for which Corollary 1 does not work.)

Proof of (R3) and (R4). As in the case of difference equations of type I or I', given a difference equation of type II (resp. of type II') we define, in a natural way, the map Ψ on $[0, M]^\omega$ such that, for any $f \in [0, M]^\omega$, $\Psi f(n) = x_n$ for all $n \in \{0, ...j\}$, and

$$\Psi f(n) = \frac{\sum_{k=1}^{n} a_{n,k} f(n-k)}{c + \sum_{k=1}^{n} a_{n,k} f(n-k)},$$

for all $n > j$.

Note that actually Ψ is a self map of $[0, M]^\omega$ because for any $f \in [0, M]^\omega$ we have $\Psi f(n) = x_n \leq M$ if $n \in \{0, ...j\}$, and, for $n > j$, we get

$$\Psi f(n) \leq M/c \leq M,$$

by conditions (iii) and (iv) (resp. by conditions (iii) and (iv')).

Let $f, g \in [0, 1]^\omega$.

If $\Psi g(n) \leq \Psi f(n)$ for all $n \in \omega$, we get $d_{\sup}(\Psi f, \Psi g) = 0$.

Otherwise, for each $n \in \mathbb{N}$ such that $0 < \Psi g(n) - \Psi f(n)$, we get

$$0 < \Psi g(n) - \Psi f(n) \ = \ \frac{\displaystyle\sum_{k=1}^{n} a_{n,k} g(n-k)}{c + \displaystyle\sum_{k=1}^{n} a_{n,k} g(n-k)} - \frac{\displaystyle\sum_{k=1}^{n} a_{n,k} f(n-k)}{c + \displaystyle\sum_{k=1}^{n} a_{n,k} f(n-k)}.$$

Put $A_n := \{k \in \{1, \ldots, n\} : g(n-k) > f(n-k)\}$.

Then

$$0 < \Psi g(n) - \Psi f(n) \leq \frac{c \displaystyle\sum_{k \in A_n} (g(n-k) - f(n-k))}{c^2 + c \displaystyle\sum_{k \in A_n} a_{n,k} g(n-k)} \leq \frac{d_{\sup}(f,g)}{c + d_{\sup}(f,g)}.$$

- If the difference equation is of type II, we define a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = t/(1+t)$ for all $t \in \mathbb{R}^+$ (recall that, in this case, we have $c = 1$). Obviously φ is non-decreasing and continuous on $\mathbb{R}^+$. Furthermore $\varphi(t) < t$ for all $t > 0$.
  Consequently
  $$d_{\sup}(\Psi f, \Psi g) = \sup_{n \varepsilon \omega} \max(\Psi g(n) - \Psi f(n), 0) \leq \varphi(d_{\sup}(f,g)).$$
  Therefore Ψ is a Browder contraction on $([0, M]^\omega, d_{\sup})$. By Proposition 1 and Theorem 2 we conclude that Ψ has a unique fixed point in $[0, M]^\omega$ which is obviously the unique solution of the given difference equation. This completes the proof of the result **(R3)**.
- If the difference equation is of type II', we get
  $$d_{\sup}(\Psi f, \Psi g) \leq \frac{d_{\sup}(f,g)}{c + d_{\sup}(f,g)} \leq \alpha d_{\sup}(f,g),$$
  where $\alpha = 1/c < 1$.

  Therefore Ψ is a Banach contraction on $([0, M]^\omega, d_{\sup})$. By Proposition 1 and Corollary 1 we conclude that Ψ has a unique fixed point in $[0, M]^\omega$ which is obviously the unique solution of the given difference equation. This completes the proof of the result **(R4)**.

Finally, we show that Ψ is not a Banach contraction on $([0, M]^\omega, d_{\sup})$, in general; i.e., there are difference equations of type II for which Corollary 1 does not work.

To this end, suppose, similarly to the equations of type I, the case where we have a unique initial value $x_0 \leq 1$, with $\displaystyle\sum_{k=1}^{n_0} a_{n_0,k} = 1$ for some $n_0 \in \mathbb{N}$ and $c = 1$. Fix $\alpha \in (0,1)$. Choose $\varepsilon \in (0,1)$ such that $1/(1+\varepsilon) > \alpha$. Let $f, g \in [0,1]^\omega$ given by $f(n) = 0$ for all $n \in \omega$, and $g(n) = \varepsilon$ whenever $0 \leq n < n_0$ and $g(n) = 0$ whenever $n \geq n_0$.

Then $\Psi f(0) = x_0$ and $\Psi f(n) = 0$ for all $n \in \mathbb{N}$, so
$$d_{\sup}(\Psi f, \Psi g) \geq \Psi g(n_0) = \frac{\varepsilon}{1+\varepsilon} > \alpha\varepsilon = \alpha d_{\sup}(f,g).$$

2.3. Particular cases

The following particular cases of difference equations of type I and I' are also special cases of equations of type #23, #30, #37, #53, #61 and #129, respectively, in [4].

a)

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n},$$

equation of type I for $0 < \beta \leq A$ and $B > 0$.

b)

$$x_{n+1} = \frac{\gamma x_{n-1}}{A + Cx_{n-1}},$$

equation of type I for $0 < \gamma \leq A$ and $C > 0$.

c)

$$x_{n+1} = \frac{\delta x_{n-2}}{A + Dx_{n-2}},$$

equation of type I for $0 < \delta \leq A$ and $D > 0$.

d)

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A},$$

equation of type I' for $0 < \beta + \gamma \leq A$.

e)

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{A},$$

equation of type I' for $0 < \gamma + \delta \leq A$.

f)

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A},$$

equation of type I' for $0 < \beta + \gamma + \delta \leq A$.

Finally we present a handful of particular cases of difference equations of type II and II' that are also special cases of equations of type #153, #158, #163 and #220, respectively, in [4].

g)

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}},$$

equation of type II for $0 < \beta + \gamma \leq 1$, $\beta = B$, $\gamma = C$ and $A = 1$, and of type II' if we replace $A = 1$ with $A > 1$.

h)

$$x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{A + Bx_n + Dx_{n-2}},$$

equation of type II for $0 < \beta + \delta \leq 1$, $\beta = B$, $\delta = D$ and $A = 1$, and of type II' if we replace $A = 1$ with $A > 1$.

i)

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{A + Cx_{n-1} + Dx_{n-2}},$$

equation of type II for $0 < \gamma + \delta \leq 1$, $\gamma = C$, $\delta = D$ and $A = 1$, and of type II' if we replace $A = 1$ with $A > 1$.

j)

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}},$$

equation of type II for $0 < \beta + \gamma + \delta \leq 1$, $\beta = B$, $\gamma = C$, $\delta = D$ and $A = 1$, and of type II' if we replace $A = 1$ with $A > 1$.

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