Dynamic Systems and Applications 31 (2022) No.4, 237-256

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLOCAL SCHRÖDINGER–KIRCHHOFF EQUATIONS WITH THE EXTERNAL MAGNETIC FIELD

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ABSTRACT. We are concerned with the existence of a nontrivial weak solution to Schrödinger–Kirchhoff type equations involving the fractional magnetic field without Ambrosetti and Rabinowitz condition using mountain pass theorem under suitable assumptions of the external force. Also, we present the existence of infinitely many large- or small- energy solutions to this problem. The strategy of the proof for these results is to approach the problem by applying the variational methods, namely, the fountain and the dual fountain theorem with Cerami condition.

AMS (MOS) Subject Classification. 58E05, 26A33, 35J60, 47G20.

Key Words and Phrases. Schrödinger-Kirchhoff equation; Fractional magnetic operators; Variational methods.

1. Introduction

The present paper is devoted to the existence of solutions for the following Schrödinger–Kirchhoff type equation with the fractional magnetic field

$$K\Big(|z|^2_{s,\mathcal{A}}\Big)(-\Delta)^s_{\mathcal{A}}z + P(x)z = \lambda h(x,|z|)z \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$|z|^2_{s,\mathcal{A}} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})}z(y)|^2}{|x-y|^{N+2s}}dxdy, \quad s\in(0,1),$$

and the fractional magnetic operator $(-\Delta)^s_{\mathcal{A}}$ is defined as

$$(-\Delta)^s_{\mathcal{A}}\varphi(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{\varphi(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})}\varphi(y)}{|x-y|^{N+2s}}dy, \quad x\in\mathbb{R}^N,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$. Here, $B_\varepsilon(x)$ denotes a ball in $\mathbb{R}^N$ centered at $x\in\mathbb{R}^N$ and radius $\varepsilon>0$. The functions $K:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a continuous function and $\mathcal{A}:\mathbb{R}^N\to\mathbb{R}^N$ is the magnetic potential. Also, the nonlinearity function $h:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ will be stated later (see Section 2). The operator $(-\Delta)^s_{\mathcal{A}}$ is a fractional

Received June 13, 2022 ISSN 1056-2176(Print); ISSN 2693-5295 (online)

https://doi.org/10.46719/dsa202231.04.02
www.dynamicpublishers.org.

Laplacian accompanied by the magnetic field, which is the nonlocal operator given in [21] as a fractional extension of the magnetic pseudorelativistic operator. We refer to [20] for more details for $s = 1/2$. In particular, in the viewpoint of quantum mechanics, many people have studied on quantum phenomena from various angles (see [1, 3, 9, 12, 18, 32, 33]). On the other hand, when $\mathcal{A} \equiv 0$, the standard fractional Laplacian $(-\Delta)^s$ has been become a classical topic for a long time and it is applied in various research areas: the social sciences, quantum or phase transition phenomena, continuum mechanics, game theory and Lévy processes [5, 6, 17, 22, 35] and the references therein.

Kirchhoff in [25] first provided a model given by the equation

$$\rho \frac{\partial^2 z}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial z}{\partial x} \right|^2 dx \Big) \frac{\partial^2 z}{\partial x^2} = 0,$$

which extends the classical D'Alembert's wave equation by considering the changes in the length of the strings during the vibrations. In this direction, the non-local problem of Kirchhoff type equations have been investigated in [8, 10, 14, 15].

In the recent paper [36], the authors consider the following limiting problem, which is a Bourgain-Brezis-Mironescu type result in the framework of magnetic Sobolev spaces

$$\lim_{s \to 1} (1-s) \int_\Omega \int_\Omega \frac{|z(x) - e^{i(x-y)\cdot \mathcal{A}(\frac{x+y}{2})} z(y)|^2}{|x-y|^{N+2s}} dxdy = C_N \int_\Omega |\nabla z - i\mathcal{A}(x) z|^2 dx,$$

where $C_N = \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\omega \cdot \mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega)$, and $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$ and $\mathbf{e}$ is any unit vector of $\mathbb{R}^N$. In fact, based on various methods, many researchers have established the existence of a solution to the following limiting equation (or Schrödinger equation with electromagnetic potential)

$$\big(-i\nabla + \mathcal{A}(x) \big)^2 z + P(x) z = h(x, |z|) z, \quad \text{in } \mathbb{R}^N$$

(see [2, 34, 41, 42]).

Now in order to ensure the existence of solutions to the nonlinear elliptic equations, we remind the Ambrosetti and Rabinowitz condition ((AR)-condition) [4], that is,

(AR) There exist $M > 0$ and $\zeta > 2$ such that

$$0 < \zeta H(x, \tau) \le h(x, \tau) \tau^2, \quad \text{for} \quad x \in \Omega, \quad \text{and} \quad \tau \ge M,$$

where $H(x, \tau) = \int_0^\tau h(x, s) s ds$ and Ω is a bounded domain in $\mathbb{R}^N$.

It is commonly well known that (AR)-condition plays an important role in applying the critical point theory. However, this condition is restrictive and eliminates many nonlinearities. Thus many researchers have attempted to drop the (AR)-condition in the elliptic problem of nonlocal type (see e.g. [16, 22, 23, 24, 28, 37, 39, 41]). In this regard, we are to show the existence of a nontrivial solution for problem (1.1) without (AR)-condition using the mountain pass theorem with Cerami condition. Furthermore, we present the existence of infinitely many large- or small- energy solutions weak solutions to our problem without (AR)-condition. Especially, following in [26, Remark 1.8], there are many examples which are not fulfilled the (AR)-condition of h in a elliptic problem. Thus, inspired by these examples, we investigate the existence and multiplicity of weak solutions to the fractional p-Laplacian equation (1.1) with the external magnetic potential. The strategy of the proof for these results is to approach the problem by applying the variational methods, namely, the fountain theorem and the dual fountain theorem with Cerami condition. The key point in the present paper is to provide the existence of multiple solutions to (1.1) under suitable conditions on nonlinear growth h that does not satisfy (AR). However the main difficulty for getting the multiplicity results under these assumptions on the nonlinear term h is to make sure the Cerami compactness condition of the energy functional corresponding to (1.1). It is worth pointing out that we overcome it from the coercivity of the potential function P. Hence our proof of these compactness condition of the Palais-Smale type slightly differs from those of previous related studies [16, 22, 23, 24, 28, 37, 39, 41].

This paper is organized as follows. In Section 2, we state some basic results to deal with this type equation with the fractional magnetic field and review well known facts for the fractional Sobolev space. And under certain assumptions on h, we establish the existence of a weak solution of problem (1.1) using the mountain pass theorem. And finally, we provide the existence of infinitely many large- or small- energy solutions weak solutions by employing the fountain theorem and the dual fountain theorem with Cerami condition.

2. **Preliminaries and main results**

We assume that $P : \mathbb{R}^N \to \mathbb{R}^+$ satisfies

(P) $P \in L^1_{loc}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} P(x) > 0$ and $\lim_{|x|\to\infty} P(x) = +\infty$.

Let $L^2_P(\mathbb{R}^N)$ denote the Lebesgue space of real valued functions with $P(x)|z|^2 \in L^1(\mathbb{R}^N)$, endowed with norm

$$\|z\|^2_{2,P} = \int_{\mathbb{R}^N} P(x)|z|^2\,dx.$$

The fractional Sobolev space $\mathcal{H}^s_P(\mathbb{R}^N)$ is then defined as for $s \in (0,1)$

$$\mathcal{H}^s_P(\mathbb{R}^N) = \left\{ z \in L^2_P(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x)-z(y)|^2}{|x-y|^{N+2s}}\,dxdy < +\infty \right\}.$$

The space $\mathcal{H}^s_P(\mathbb{R}^N)$ is endowed with the norm

$$\|z\|^2_{\mathcal{H}^s_P(\mathbb{R}^N)} := \left(\|z\|^2_{2,P} + [z]^2_s\right) \quad \text{with} \quad [z]^2_s := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x)-z(y)|^2}{|x-y|^{N+2s}}\,dxdy.$$

For further details on the fractional Sobolev spaces we refer the reader to [27] and the references therein. We recall the embedding theorem; see e.g. [22, 30].

Lemma 2.1. *Let* (P) *hold and let* 2^*_s *be the fractional critical Sobolev exponent, that is*

$$2^*_s := \begin{cases} \frac{2N}{N-2s}, & \text{if } 2s < N, \\ \infty, & \text{if } 2s \geq N. \end{cases}$$

Then, the embedding $\mathcal{H}^s_P(\mathbb{R}^N) \to L^\gamma(\mathbb{R}^N)$ *is continuous for any* $\gamma \in [2, 2^*_s]$ *and moreover, the embedding* $\mathcal{H}^s_P(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ *is compact for any* $\gamma \in [2, 2^*_s)$.

Let $L^2_P(\mathbb{R}^N, \mathbb{C})$ be the Lebesgue space of functions $z : \mathbb{R}^N \to \mathbb{C}$ with $P(x)|z|^2 \in L^1(\mathbb{R}^N)$, endowed with the (real) scalar product

$$\langle z, v\rangle_{L^2_P} = \Re\left(\int_{\mathbb{R}^N} P(x) z\bar{v}\,dx\right), \quad \forall\ z, v \in L^2(\mathbb{R}^N, \mathbb{C}),$$

where $\bar{v}$ denotes complex conjugation of $v \in \mathbb{C}$.

Also, due to [11], the magnetic Gagliardo seminorm is given by

$$|z|^2_{s,\mathcal{A}} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})} z(y)|^2}{|x-y|^{N+2s}} dxdy.$$

Define $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N)$ as the closure of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ with respect to the norm

$$\|z\|^2_{s,\mathcal{A}} = (\|z\|^2_{2,P} + |z|^2_{s,\mathcal{A}}).$$

A scalar product on $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N)$ is given by

$$\begin{aligned}\langle z, v\rangle_{s,\mathcal{A}} &= \langle z, v\rangle_{L^2_P} \\ &+ \Re\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[z(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})} z(y)] \cdot \overline{[v(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})} v(y)]}}{|x-y|^{N+2s}} dxdy\right).\end{aligned}$$

In fact, arguing as in [11, Proposition 2.1], we see that $(\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N), \langle\cdot,\cdot\rangle)$ is a real Hilbert space. Moreover, we can easily show that it is a reflexive and separable Banach space as the similar arguments in [29, 30, Appendix]. The following Lemmas 2.2 and 2.3 are given in [39, Lemmas 3.4 and 3.5].

Lemma 2.2. *If* (P) *holds and* $r \in [2, 2^*_s]$, *then the embedding*

$$\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$$

is continuous. Furthermore, for any compact subset $\Gamma \subset \mathbb{R}^N$ *and* $r \in [1, 2^*_s)$, *then the embedding*

$$\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow \mathcal{H}^s_P(\Gamma, \mathbb{C}) \hookrightarrow L^r(\Gamma, \mathbb{C})$$

is continuous and the latter is compact, where $\mathcal{H}^s_P(\Gamma,\mathbb{C})$ *is equipped with the following norm:*

$$\|z\|^2_{s,P} = \Big(\int_\Gamma P(x)|z|^2dx + \int_\Gamma\int_\Gamma \frac{|z(x)-z(y)|^2}{|x-y|^{N+2s}}\,dxdy\Big).$$

Lemma 2.3. *Under the assumption* (P)*, for all bounded sequence* $\{z_n\}$ *in* $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$ *the sequence* $\{|z_n|\}$ *has a subsequence converging strongly to some* z *in* $L^r(\mathbb{R}^N)$ *for all* $r\in[2,2^*_s)$.

For our problem, we suppose that $K:\mathbb{R}^+_0\to\mathbb{R}^+_0$ satisfies the following conditions:

(K1) $K\in C(\mathbb{R}^+_0)$ satisfies $\inf_{\tau\in\mathbb{R}^+}K(\tau)\geq a>0$, where $a>0$ is a constant.

(K2) There is a positive constant $\theta\in[1,\frac{N}{N-2s})$ such that $\theta\mathcal{K}(\tau)=\theta\int_0^\tau K(\eta)d\eta\geq K(\tau)\tau$ for any $\tau\geq 0$.

A typical example for K is given by $K(\tau)=b_0+b_1\tau^m$ with $m>0, b_0>0$ and $b_1\geq 0$.

Now we assume that for $1<2\theta<q<2^*_s$,

(H1) $h:\mathbb{R}^N\times\mathbb{R}^+\to\mathbb{R}$ satisfies the Carathéodory condition.

(H2) $h\in C(\mathbb{R}^N\times\mathbb{R}^+,\mathbb{R})$, and there exist constants $c_1,c_2>0$ such that

$$|h(x,\tau)|\leq c_1+c_2\tau^{q-2},\quad \text{for all } (x,\tau)\in\mathbb{R}^N\times\mathbb{R}^+,\quad q\in(2\theta,2^*_s).$$

(H3) $h(x,\tau)=o(\tau)$ as $\tau\to 0$ for $x\in\mathbb{R}^N$ uniformly.

(H4) $\lim_{\tau\to\infty}\frac{H(x,\tau)}{\tau^{2\theta}}=\infty$ uniformly for almost all $x\in\mathbb{R}^N$, where the number θ was given in (K2), and $H(x,\tau)=\int_0^\tau h(x,\eta)\eta\,d\eta$ for all $x\in\mathbb{R}^N$.

(H5) There are $\nu>2\theta$ and $\mathcal{C}>0$ such that

$$h(x,\tau)\tau^2-\nu H(x,\tau)\geq-\beta(x)\quad\text{for all}\quad x\in\mathbb{R}^N\quad\text{and}\quad\tau\geq\mathcal{C},$$

where $\beta\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $\beta(x)\geq 0$.

(H6) There exist $c_0\geq 0$, $r_0\geq 0$, and $\kappa>\frac{N}{2s}$ such that

$$|H(x,\tau)|^\kappa\leq c_0\tau^{2\kappa}\mathcal{H}(x,\tau)$$

for all $(x,\tau)\in\mathbb{R}^N\times\mathbb{R}^+$ and $\tau\geq r_0$, where $\mathcal{H}(x,\tau)=(\frac{1}{2\theta})h(x,\tau)\tau^2-H(x,\tau)\geq 0$.

The Euler functional corresponding to the problem (1.1) is $\mathcal{J}_\lambda:\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})\to\mathbb{R}$ defined as

$$\mathcal{J}_\lambda(z)=\frac{1}{2}(\mathcal{K}(|z|^2_{s,\mathcal{A}})+\|z\|^2_{2,P})-\lambda\int_{\mathbb{R}^N}H(x,|z|)\,dx.$$

The functional $\mathcal{J}_\lambda$ is Fréchet differentiable on $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$, and its derivative is

$$\begin{aligned}&\langle\mathcal{J}'_\lambda(z),v\rangle\\&=\Re\bigg(K(|z|^2_{s,\mathcal{A}})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[z(x)-e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})}z(y)]\cdot\overline{[v(x)-e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})}v(y)]}}{|x-y|^{N+2s}}dxdy\\&+\int_{\mathbb{R}^N}P(x)z\bar{v}\,dx-\lambda\int_{\mathbb{R}^N}h(x,|z|)z\bar{v}\,dx\bigg)\end{aligned}$$

for any $z, v \in \mathcal{H}^s_{A,P}(\mathbb{R}^N, \mathbb{C})$. Hereafter, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C}))'$ and $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$. Following in [39], we observe that the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions of (1.1) and the functional $\mathcal{J}_\lambda$ is weakly lower semi-continuous in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$.

The following result is to show that the energy functional $\mathcal{J}_\lambda$ satisfies the geometric conditions.

Lemma 2.4. *Let $s \in (0,1)$ and $N > 2s$. Assume that* (P), (K1), (K2) *and* (H1)–(H4) *hold. Then the geometric conditions in the mountain pass theorem hold, i.e.,*

(1) *$z = 0$ is a strict local minimum for $\mathcal{J}_\lambda$.*
(2) *$\mathcal{J}_\lambda$ is unbounded from below on $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$.*

Proof. Due to (H2) and (H3), for any $\varepsilon > 0$, we can choose a positive constant denoted $C(\varepsilon)$ such that

$$|h(x,\tau)\tau| \le \varepsilon\tau + C(\varepsilon)\tau^{q-1}, \quad \text{for all } (x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{2.1}$$

Assume that $\|z\|_{s,\mathcal{A}} < 1$. Owing to (K1), (K2) and (2.1), one has

$$\begin{aligned}
\mathcal{J}_\lambda(z) &= \frac{1}{2}(\mathcal{K}([z]^2_{s,\mathcal{A}}) + \|z\|^2_{2,P}) - \lambda \int_{\mathbb{R}^N} H(x,|z|)dx \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|^2_{s,\mathcal{A}} - \frac{\lambda\epsilon}{2}\|z\|^2_{L^2(\mathbb{R}^N)} - \frac{\lambda C(\varepsilon)}{q}\|z\|^q_{L^q(\mathbb{R}^N)} \\
&\ge \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|^2_{s,\mathcal{A}} - \frac{\lambda\epsilon C}{2}\|z\|^2_{s,\mathcal{A}} - \frac{\lambda C C(\varepsilon)}{q}\|z\|^q_{s,\mathcal{A}}
\end{aligned}$$

for some constant C. Choose $\epsilon > 0$ so small that $0 < \lambda\epsilon C < \frac{\min\{1, a\theta^{-1}\}}{4}$. Then

$$\mathcal{J}_\lambda(z) \ge \frac{\min\{1, a\theta^{-1}\}}{4}\|z\|^2_{s,\mathcal{A}} - C(\lambda,\epsilon)C\|z\|^q_{s,\mathcal{A}}.$$

Since $q > 2$, there are $R > 0$ small sufficiently and $\delta > 0$ such that $\mathcal{J}_\lambda(z) \ge \delta > 0$ when $\|z\|_{s,\mathcal{A}} = R$. Therefore $z = 0$ is a strict local minimum for $\mathcal{J}_\lambda$.

Next we prove the condition (2). By the condition (H4), for any $\tilde{C} > 0$, we can choose a constant $\delta > 0$ such that

$$H(x,\tau) \ge \tilde{C}\tau^{2\theta} \tag{2.2}$$

for $\tau > \delta$ and for almost all $x \in \mathbb{R}^N$. Under the assumption (K2), we note that for all $\xi \ge 0$,

$$\mathcal{K}(\xi) \le \mathcal{K}(1)(1 + \xi^\theta). \tag{2.3}$$

Relations (2.2) and (2.3) with Lemma 2.3 imply that for $v \in \mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$

$$\begin{aligned}\mathcal{J}_\lambda(tv) &= \frac{1}{2}(\mathcal{K}(|tv|^2_{s,\mathcal{A}}) + \|tv\|^2_{2,P}) - \int_{\mathbb{R}^N} H(x,|tv|)dx \\ &\le \frac{1}{2}\left(\mathcal{K}(1)(1+|tv|^{2\theta}_{s,\mathcal{A}}) + \|tv\|^2_{2,P}\right) - \lambda t^{2\theta}\tilde{C}\int_{\{|tv|>\delta\}} |v|^{2\theta}\, dx \\ &\le \frac{1}{2}\left(2\mathcal{K}(1)t^{2\theta}|v|^{2\theta}_{s,\mathcal{A}}) + t^{2\theta}\|v\|^2_{2,P}\right) - \lambda t^{2\theta}\tilde{C}\int_{\{|tv|>\delta\}} |v|^{2\theta}\, dx \\ &= t^{2\theta}\Big(\frac{1}{2}\left(2\mathcal{K}(1)\|v\|^{2\theta}_{s,\mathcal{A}} + \|v\|^2_{2,P}\right) - \lambda\tilde{C}\int_{\{|tv|>\delta\}} |v|^{2\theta}\, dx\Big)\end{aligned}$$

for $t > 0$. If $\tilde{C}$ is large sufficiently, then we deduce that $\mathcal{J}_\lambda(tv) \to -\infty$ as $t \to \infty$. Hence the functional $\mathcal{J}_\lambda$ is unbounded from below. The proof is completed. □

First of all, we introduce the Cerami condition, which was initially provided by Cerami [7].

Definition 2.5. Let the functional Φ be C^1 and $c \in \mathbb{R}$. If any sequence $\{z_n\}$ satisfying

$$\Phi(z_n) \to c \quad \text{and} \quad (1+\|z_n\|)\|\Phi'(z_n)\| \to 0$$

possesses a convergent subsequence, then we say that Φ fulfils Cerami condition ($(C)_c$-condition in short) at the level c.

Definition 2.6. A function $z \in \mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$ is called a weak solution of (1.1) if z satisfies

$$\begin{aligned}&\Re\Big(K(|z|^2_{s,\mathcal{A}})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[z(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})}z(y)]\cdot\overline{[\varphi(x) - e^{i(x-y)\cdot\mathcal{A}(\frac{x+y}{2})}\varphi(y)]}}{|x-y|^{N+2s}}dxdy \\ &+ \int_{\mathbb{R}^N} P(x)z\bar{\varphi}\,dx\Big) = \Re\Big(\lambda\int_\Omega h(x,|z|)z\bar{\varphi}\,dx\Big)\end{aligned}$$

for all $\varphi \in \mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$.

The following lemmas are essential in establishing the existence of a nontrivial weak solution for the given problem.

Lemma 2.7. *Let $s \in (0,1)$ and $N > 2s$. Assume that* (P), (K1), (K2), (H1)–(H3) *and* (H5) *hold. Then the functional $\mathcal{J}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda > 0$.*

Proof. For $c \in \mathbb{R}$, let $\{z_n\}$ be a $(C)_c$-sequence in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$, that is,

$$\mathcal{J}_\lambda(z_n) \to c \quad \text{and} \quad \|\mathcal{J}'_\lambda(z_n)\|_{s,\mathcal{A}'}(1+\|z_n\|_{s,\mathcal{A}}) \to 0 \qquad \text{as} \quad n\to\infty, \tag{2.4}$$

which means

$$c = \mathcal{J}_\lambda(z_n) + o(1) \quad \text{and} \quad \langle \mathcal{J}'_\lambda(z_n), z_n\rangle = o(1), \tag{2.5}$$

where $o(1) \to 0$ as $n \to \infty$. If $\{z_n\}$ is bounded in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$, it follows from the analogous argument as in the proof of Lemma 4.2 in [39] that sequence $\{z_n\}$ converges strongly to z in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$. Hence, it suffices to ensure that the sequence $\{z_n\}$ is bounded in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$. Notice that $P(x)\to+\infty$ as $|x|\to\infty$, then

$$\begin{aligned}&\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx-C_1\int_{|z_n|\le\mathcal{C}}(c_1\,|z_n|^2+c_2\,|z_n|^q)\,dx\\&\ge\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\|z_n\|^2_{2,P}-\mathcal{C}_0,\end{aligned}$$

where $C_1,\mathcal{C}_0$ are positive constants and c_1,c_2 are given in (H2). Indeed we know that

$$\begin{aligned}&\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx-C_1\int_{|z_n|\le\mathcal{C}}(c_1\,|z_n|^2+c_2\,|z_n|^q)\,dx\\&\ge\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx+\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{|z_n|\le1}P(x)\,|z_n|^2\,dx\\&-C_1\int_{|z_n|\le1}(c_1\,|z_n|^2+c_2\,|z_n|^q)\,dx-C_1\int_{1<|z_n|\le\mathcal{C}}(c_1\,|z_n|^2+c_2\,|z_n|^q)\,dx\\&\ge\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx+\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{|z_n|\le1}P(x)\,|z_n|^2\,dx\\&-C_1(c_1+c_2)\int_{|z_n|\le1}|z_n|^2\,dx-\widetilde{C}_1,\end{aligned}$$

where $\widetilde{C}_1>0$ is a constant. Since $|\{x\in\mathbb{R}^N:|z_n|>1\}|<\infty$, we know that there are a bounded set B and a set M of measure zero such that $\{x\in\mathbb{R}^N:|z_n|>1\}=B\cup M$ where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^N$. Without loss of generality, suppose that there exists $B_\tau\subseteq\mathbb{R}^N$ such that $\{x\in\mathbb{R}^N:|z_n|>1\}\subset B_\tau$. Since $P(x)\to+\infty$ as $|x|\to\infty$, there is $\tau_0>0$ such that $|x|\ge\tau_0>\tau$ implies $P(x)\ge 2C_2(c_1+c_2)\frac{2\theta\nu}{\nu-2\theta}$. Hence one has

$$\begin{aligned}&\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx-C_2\int_{|z_n|\le\mathcal{C}}(c_1\,|z_n|^2+c_2\,|z_n|^q)\,dx\\&\ge\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx+\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\{|z_n|\le1\}\cap B^c_{\tau_0}}P(x)\,|z_n|^2\,dx\\&+\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\{|z_n|\le1\}\cap B_{\tau_0}}P(x)\,|z_n|^2\,dx-C_2(c_1+c_2)\int_{\{|z_n|\le1\}\cap B^c_{\tau_0}}|z_n|^2\,dx\\&-C_2(c_1+c_2)\int_{\{|z_n|\le1\}\cap B_{\tau_0}}|z_n|^2\,dx-\widetilde{C}_2\\&\ge\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx+\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\{|z_n|\le1\}\cap B^c_{\tau_0}}P(x)\,|z_n|^2\,dx\\&-C_2(c_1+c_2)\int_{\{|z_n|\le1\}\cap B^c_{\tau_0}}|z_n|^2\,dx-\mathcal{C}_0\\&\ge\frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}P(x)\,|z_n|^2\,dx-\mathcal{C}_0,\end{aligned}$$

as claimed. This together with (K1)–(K2), (H2)–(H3) and (H5) yields

$$\begin{aligned}
c+1 &\geq \mathcal{J}_\lambda(z_n) - \frac{1}{\nu}\left\langle \mathcal{J}'_\lambda(z_n), z_n\right\rangle \\
&\geq \frac{1}{p}\mathcal{K}(|z_n|_{s,\mathcal{A}}^2) - \frac{1}{\nu}K(|z_n|_{s,\mathcal{A}}^2)|z_n|_{s,\mathcal{A}}^2 + \left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} P(x)\,|z_n|^2\,dx \\
&+ \lambda\int_{\mathbb{R}^N}\left(\frac{1}{\nu}h(x,|z_n|)|z_n|^2 - H(x,|z_n|)\right)dx \\
&\geq \left(\frac{1}{2\theta}-\frac{1}{\nu}\right)K(|z_n|_{s,\mathcal{A}}^2)|z_n|_{s,\mathcal{A}}^2 + \left(\frac{1}{2\theta}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} P(x)\,|z_n|^2\,dx \\
&+ \lambda\int_{|z_n|>\mathcal{C}}\left(\frac{1}{\nu}h(x,|z_n|)|z_n|^2 - H(x,|z_n|)\right)dx - C_2\int_{|z_n|\leq\mathcal{C}}(c_1\,|z_n|^2 + c_2\,|z_n|^q)\,dx \\
&\geq \frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\min\{1,a\}\|z_n\|_{s,\mathcal{A}}^2 - \frac{\lambda}{\nu}\int_{\mathbb{R}^N}\beta(x)\,dx - \mathcal{C}_0 \\
&\geq \frac{1}{2}\Big(\frac{1}{2\theta}-\frac{1}{\nu}\Big)\min\{1,a\}\|z_n\|_{s,\mathcal{A}}^2 - \frac{\lambda}{\nu}\|\beta\|_{L^1(\mathbb{R}^N)} - \mathcal{C}_0.
\end{aligned}$$

Therefore, the sequence $\{z_n\}$ is bounded in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$. This completes the proof. □

Lemma 2.8. *Let $s\in(0,1)$ and $N>2s$. Assume that* (P), (K1), (K2), (H1)–(H4) *and* (H6) *hold. Then the functional $\mathcal{J}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda>0$.*

Proof. For $c\in\mathbb{R}$, let $\{z_n\}$ be a $(C)_c$-sequence in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$ satisfying (2.4) and (2.5). As in Lemma 2.7, it suffices to ensure that the sequence $\{z_n\}$ is bounded in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$. We argue by contradiction. Assume that the sequence $\{z_n\}$ is unbounded in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$. So then we may assume that

$$\|z_n\|_{s,\mathcal{A}}\to\infty, \quad \text{as} \quad n\to\infty.$$

Due to the condition (2.5), we have that

$$c = \mathcal{J}_\lambda(z_n)+o(1) = \frac{1}{2}(\mathcal{K}(|z_n|_{s,\mathcal{A}}^2)+\|z_n\|_{2,P}^2) - \lambda\int_{\mathbb{R}^N}H(x,|z_n|)\,dx + o(1). \tag{2.6}$$

Since $\|z_n\|_{s,\mathcal{A}}\to\infty$ as $n\to\infty$, we assert by (2.6) that

$$\begin{aligned}
\int_{\mathbb{R}^N}H(x,|z_n|)\,dx &\geq \frac{1}{2\lambda}(\mathcal{K}(|z_n|_{s,\mathcal{A}}^2)+\|z_n\|_{2,P}^2) - \frac{c}{\lambda}+\frac{o(1)}{\lambda} \\
&\geq \frac{1}{2\lambda}\min\{1,a\theta^{-1}\}\|z_n\|_{s,\mathcal{A}}^2 - \frac{c}{\lambda}+\frac{o(1)}{\lambda}\to\infty \quad \text{as} \quad n\to\infty. \qquad (2.7)
\end{aligned}$$

Define a sequence $\{\omega_n\}$ by $\omega_n = z_n/\|z_n\|_{s,\mathcal{A}}$. Then it is immediate that $\{\omega_n\}\subset\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$ and $\|\omega_n\|_{s,\mathcal{A}}=1$. Hence, up to a subsequence, still denoted by $\{\omega_n\}$, we obtain $\omega_n\rightharpoonup\omega$ in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N,\mathbb{C})$ as $n\to\infty$, we have

$$\omega_n(x)\to\omega(x) \text{ for a.e. } x\in\mathbb{R}^N \text{ and } \quad |\omega_n|\to|\omega| \text{ in } L^r(\mathbb{R}^N) \tag{2.8}$$

as $n \to \infty$ for $2 \le r < 2_s^*$. Set $\Sigma = \left\{x \in \mathbb{R}^N : \omega(x) \neq 0\right\}$. By the convergence (2.8), we know that

$$|z_n(x)| = |w_n(x)|\, \|z_n\|_{s,\mathcal{A}} \to \infty \quad \text{as} \quad n \to \infty$$

for all $x \in \Sigma$. Then it follows from (K2) and (H4) that for all $x \in \Sigma$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{H(x,|z_n|)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}} &\ge \lim_{n\to\infty} \frac{H(x,|z_n|)}{\mathcal{K}(1)(1+|z_n|^{2\theta}_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}} \\
&\ge \lim_{n\to\infty} \frac{H(x,|z_n|)}{2\mathcal{K}(1)\|z_n\|^{2\theta}_{s,\mathcal{A}} + \|z_n\|^{2\theta}_{2,P}} \\
&\ge \lim_{n\to\infty} \frac{H(x,|z_n|)}{(2\mathcal{K}(1)+1)\,\|z_n\|^{2\theta}_{s,\mathcal{A}}} \\
&\ge \lim_{n\to\infty} \frac{H(x,|z_n|)}{(2\mathcal{K}(1)+1)\,|z_n|^{2\theta}}\,|w_n|^{2\theta} \\
&= \infty, \qquad (2.9)
\end{aligned}$$

where the inequality $\mathcal{K}(\eta) \le \mathcal{K}(1)(1+\eta^\theta)$ is used for all $\eta \in \mathbb{R}_0^+$ (see (K1)) because if $0 \le \eta < 1$, then $\mathcal{K}(\eta) = \int_0^\eta K(s)\,ds \le \mathcal{K}(1)$, and if $\eta > 1$, then $\mathcal{K}(\eta) \le K(1)\eta^\theta$. Thus we obtain that $|\Sigma| = 0$. Indeed, assume that $|\Sigma| \neq 0$. Taking account into (H4) we can choose $\tau_0 > 1$ such that $H(x,\tau) > \tau^{2\theta}$ for all $x \in \mathbb{R}^N$ and $\tau_0 < \tau$. By means of (H1) and (H2), we derive that there is $K > 0$ such that $|H(x,\tau)| \le K$ for all $(x,\tau) \in \mathbb{R}^N \times (0,\tau_0]$. Hence there is a $K_0 \in \mathbb{R}$ such that $H(x,\tau) \ge K_0$ for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+$, and thus

$$\frac{H(x,|z_n|) - K_0}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}} \ge 0, \qquad (2.10)$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. In accordance with relations (2.7), (2.9), (2.10) and the Fatou lemma, we infer that

$$\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} H(x,|z_n|)\,dx}{\lambda\int_{\mathbb{R}^N} H(x,|z_n|)\,dx + c - o(1)} \\
&\ge \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{2H(x,|z_n|)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&\ge \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{2H(x,|z_n|)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx - \limsup_{n\to\infty}\int_{\mathbb{R}^N} \frac{2K_0}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&\ge \liminf_{n\to\infty} \int_{\Sigma} \frac{2(H(x,|z_n|) - K_0)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&\ge \int_{\Sigma} \liminf_{n\to\infty} \frac{2(H(x,|z_n|) - K_0)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&= \int_{\Sigma} \liminf_{n\to\infty} \frac{2H(x,|z_n|)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx - \int_{\Sigma} \limsup_{n\to\infty} \frac{2K_0}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&= \infty,
\end{aligned}$$

which is a contradiction. This means $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$.

Note that for a sufficiently large n,

$$
\begin{aligned}
c+1 &\geq \mathcal{J}_\lambda(z_n) - \frac{1}{2\theta}\left\langle \mathcal{J}'_\lambda(z_n), z_n\right\rangle \\
&= \frac{1}{2}(\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}) - \lambda\int_{\mathbb{R}^N} H(x,|z_n|)\,dx \\
&\quad - \frac{1}{2\theta}(K(|z_n|^2_{s,\mathcal{A}})|z_n|^2_{s,\mathcal{A}} + \|z_n\|^2_{2,P}) + \frac{\lambda}{2\theta}\int_{\mathbb{R}^N} h(x,|z_n|)\,|z_n|^2\,dx \\
&\geq \lambda\int_{\mathbb{R}^N}\mathcal{H}(x,z_n)\,dx. \qquad (2.11)
\end{aligned}
$$

Let us define $\Sigma_n(\tilde{a}, b) := \{x \in \mathbb{R}^N : \tilde{a} \leq |z_n| < b\}$ for $0 \leq \tilde{a} < b$. By the convergence (2.8), we know that

$$
|\omega_n| \to 0 \quad \text{in} \quad L^r(\mathbb{R}^N) \quad \text{and} \quad \omega_n(x) \to 0 \quad \text{for a.e. } x \in \mathbb{R}^N \qquad (2.12)
$$

for $2 \leq r < 2^*_s$. Hence from the relation (2.6) we get

$$
0 < \frac{1}{2\lambda} \leq \limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{|H(x,|z_n|)|}{\mathcal{K}(|z|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx. \qquad (2.13)
$$

On the other hand, from (H2), we know that

$$
|H(x,|z|)| \leq \frac{c_1}{2}|z|^2 + \frac{c_2}{q}|z|^q. \qquad (2.14)
$$

Then, from the conditions (K1)–(K2), (2.12) and (2.14), we have

$$
\begin{aligned}
&\int_{\Sigma_n(0,r_0)}\frac{H(x,|z_n|)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&\leq \int_{\Sigma_n(0,r_0)}\frac{|z_n|^2 + \frac{1}{q}\,|z_n|^q}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\,dx \\
&\leq \frac{\|z_n\|^2_{L^2(\mathbb{R}^N)}}{\min\{1,a\theta^{-1}\}\|z_n\|^2_{s,\mathcal{A}}} + \frac{1}{\min\{1,a\theta^{-1}\}q}\int_{\Sigma_n(0,r_0)}|z_n|^{q-2}\,|\omega_n|^2\,dx \\
&\leq \frac{\|z_n\|^2_{L^2(\mathbb{R}^N)}}{\min\{1,a\theta^{-1}\}\|z_n\|^2_{s,\mathcal{A}}} + \frac{1}{\min\{1,a\theta^{-1}\}q}r_0^{q-2}\int_{\mathbb{R}^N}|\omega_n|^2\,dx \\
&\leq \frac{\|\omega_n\|^2_{L^2(\mathbb{R}^N)}}{\min\{1,a\theta^{-1}\}} + \frac{1}{\min\{1,a\theta^{-1}\}q}r_0^{q-2}\int_{\mathbb{R}^N}|\omega_n|^2\,dx \to 0, \qquad (2.15)
\end{aligned}
$$

as $n \to \infty$, where we use the inequality

$$
\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P} \geq \min\{1,a\theta^{-1}\}\|z_n\|^2_{s,\mathcal{A}}
$$

by the definition of the Kirchhoff function K and norm $\|\cdot\|_{s,\mathcal{A}}$. Set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > \frac{N}{2}$, we see that $2 < 2\kappa' < 2^*_s$. Hence, it follows from (H5), (2.11) and (2.12) that

$$
\begin{aligned}
&\int_{\Sigma_n(r_0,\infty)} \frac{|H(x,|z_n|)|}{\mathcal{K}(|z_n|_{s,\mathcal{A}}^2)+\|z_n\|_{2,P}^2}\,dx \le \int_{\Sigma_n(r_0,\infty)} \frac{|H(x,|z_n|)|}{\min\{1,a\theta^{-1}\}\,|z_n|^2}\,|\omega_n|^2\,dx \\
&\le \frac{1}{\min\{1,a\theta^{-1}\}}\left\{\int_{\Sigma_n(r_0,\infty)} \left(\frac{|H(x,|z_n|)|}{|z_n|^2}\right)^{\kappa} dx\right\}^{\frac{1}{\kappa}}\left\{\int_{\Sigma_n(r_0,\infty)} |\omega_n|^{2\kappa'}\,dx\right\}^{\frac{1}{\kappa'}} \\
&\le \frac{c_0^{\frac{1}{\kappa}}}{\min\{1,a\theta^{-1}\}}\left\{\int_{\Sigma_n(r_0,\infty)} \mathcal{H}(x,|z_n|)\,dx\right\}^{\frac{1}{\kappa}}\left\{\int_{\mathbb{R}^N} |\omega_n|^{2\kappa'}\,dx\right\}^{\frac{1}{\kappa'}} \\
&\le \frac{c_0^{\frac{1}{\kappa}}}{\min\{1,a\theta^{-1}\}}\left(\frac{c+1}{\lambda}\right)^{\frac{1}{\kappa}}\left\{\int_{\mathbb{R}^N} |w_n|^{2\kappa'}\,dx\right\}^{\frac{1}{\kappa'}} \to 0, \quad \text{as } n\to\infty. \qquad (2.16)
\end{aligned}
$$

In combination with (2.15) and (2.16), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|H(x,|z_n|)|}{\mathcal{K}(|z_n|_{s,\mathcal{A}}^2)+\|z_n\|_{2,P}^2}\,dx \\
&\quad = \int_{\Sigma_n(0,r_0)} \frac{|H(x,|z_n|)|}{\mathcal{K}(|z_n|_{s,\mathcal{A}}^2)+\|z_n\|_{2,P}^2}\,dx + \int_{\Sigma_n(r_0,\infty)} \frac{|H(x,|z_n|)|}{\mathcal{K}(|z_n|_{s,\mathcal{A}}^2)+\|z_n\|_{2,P}^2}\,dx \to 0,
\end{aligned}
$$

as $n\to\infty$, which contradicts (2.13). The proof is completed. □

Using Lemma 2.7, we prove the existence of a nontrivial weak solution to our problem.

Theorem 2.9. *Under the same assumptions of Lemma 2.7, then the problem* (1.1) *has a nontrivial weak solution for all* $\lambda > 0$.

Proof. Note that $\mathcal{J}_\lambda(0) = 0$. By Lemma 2.4, the mountain pass geometric conditions are satisfied. From Lemma 2.7, $\mathcal{J}_\lambda$ fulfils the $(C)_c$-condition for any $\lambda > 0$. Subsequently, problem (1.1) admits a nontrivial weak solution for all $\lambda > 0$. □

With the help of Lemma 2.8, we obtain the following assertion.

Theorem 2.10. *Under the same assumptions of Lemma 2.8, then the problem* (1.1) *has a nontrivial weak solution for all* $\lambda > 0$.

Proof. The proof is completely the same as that of Theorem 2.9. □

At last, we are ready to prove our multiplicity results. By using the fountain theorem in [38, Theorem 3.6], we demonstrate infinitely many weak solutions for problem (1.1). Let E be a reflexive and separable Banach space, then it is known (see [13]) that there exist $\{e_n\}\subseteq E$ and $\{f_n^*\}\subseteq E^*$ such that

$$
E = \overline{\operatorname{span}\{e_n : n=1,2,\cdots\}}, \quad E^* = \overline{\operatorname{span}\{f_n^* : n=1,2,\cdots\}},
$$

and

$$\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j. \end{cases}$$

Let us denote $\mathcal{E}_n = \operatorname{span}\{e_n\}$, $\mathcal{Y}_k = \bigoplus_{n=1}^{k} \mathcal{E}_n$, and $\mathcal{Z}_k = \overline{\bigoplus_{n=k}^{\infty} \mathcal{E}_n}$. In order to obtain our first multiplicity result, we use the following Fountain theorem.

Lemma 2.11. ([31, 38]) *Let E be a real Banach space, $\mathcal{I} \in C^1(E, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$ and $\mathcal{I}$ is even. If for each sufficiently large $k \in \mathbb{N}$, there exist $\varrho_k > \sigma_k > 0$ such that the following conditions hold:*

(1) $\beta_k := \inf\{\mathcal{I}(z) : z \in \mathcal{Z}_k, \|z\|_E = \sigma_k\} \to \infty \quad as \quad k \to \infty;$
(2) $\alpha_k := \max\{\mathcal{I}(z) : z \in \mathcal{Y}_k, \|z\|_E = \varrho_k\} \leq 0.$

Then the functional $\mathcal{I}$ has an unbounded sequence of critical values, i.e., there exists a sequence $\{z_n\} \subset E$ such that $\mathcal{I}'(z_n) = 0$ and $\mathcal{I}(z_n) \to +\infty$ as $n \to +\infty$.

Theorem 2.12. *Let $s \in (0, 1)$ and $N > 2s$. Assume that* (P), (K1), (K2) *and* (H1)–(H5) *hold. Then for any $\lambda > 0$, problem* (1.1) *possesses an unbounded sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to \infty$ as $n \to \infty$.*

Proof. The proof follows the lines of that of Lemma 3.2 in [40]. To apply Lemma 2.11, let us denote $E := \mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$ and $\mathcal{I} := \mathcal{J}_\lambda$. Plainly, $\mathcal{J}_\lambda$ is an even functional and ensures the $(C)_c$-condition by Lemma 2.7. It suffices to show that there exist $\varrho_k > \sigma_k > 0$ with the conditions (1) and (2) in Lemma 2.11. Let us denote

$$\varsigma_k = \sup_{\|z\|_{s,\mathcal{A}}=1, z\in\mathcal{Z}_k} \|z\|_{L^q(\mathbb{R}^N)}.$$

Then, it is easy to verify that $\varsigma_k \to 0$ as $k \to \infty$. For any $z \in \mathcal{Z}_k$, assume that $\|z\|_{s,\mathcal{A}} > 1$. Choose $\epsilon > 0$ so small that $0 < \lambda\epsilon C_{imb} < \frac{\min\{1, a\theta^{-1}\}}{4}$ where C_{imb} is an imbedding constant of $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^2(\mathbb{R}^N)$. Then it follows from (2.1) that

$$\begin{aligned} \mathcal{J}_\lambda(z) &= \frac{1}{2}(\mathcal{K}(|z|^2_{s,\mathcal{A}}) + \|z\|^2_{2,P}) - \lambda \int_{\mathbb{R}^N} H(x, |z|)dx \\ &\geq \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|^2_{s,\mathcal{A}} - \lambda \int_{\mathbb{R}^N} H(x, |z|)dx \\ &\geq \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|^2_{s,\mathcal{A}} - \frac{\lambda\epsilon}{2}\|z\|^2_{L^2(\mathbb{R}^N)} - \frac{\lambda C(\epsilon)}{q}\|z\|^q_{L^q(\mathbb{R}^N)} \\ &\geq \frac{\min\{1, a\theta^{-1}\}}{4}\|z\|^2_{s,\mathcal{A}} - \lambda C(\epsilon)\varsigma_k^q\|z\|^q_{s,\mathcal{A}} \\ &= \Big(\frac{\min\{1, a\theta^{-1}\}}{4} - \lambda C(\epsilon)\varsigma_k^q\|z\|^{q-2}_{s,\mathcal{A}}\Big)\|z\|^2_{s,A}. \end{aligned} \tag{2.17}$$

Choose $\sigma_k = \left[\frac{4\lambda C(\epsilon)}{\min\{1,a\theta^{-1}\}}\varsigma_k^q\right]^{\frac{1}{2-q}}$. Since $2 < q$ and $\varsigma_k \to 0$ as $k \to \infty$, we infer $\sigma_k \to \infty$ as $k \to \infty$. Hence, if $z \in Z_k$ and $\|z\|_{s,\mathcal{A}} = \sigma_k$, then we deduce that

$$\mathcal{J}_\lambda(z) \geq \frac{\min\{1, a\theta^{-1}\}}{4}\sigma_k^2 \to \infty \quad \text{as} \quad k \to \infty,$$

which implies (1).

Now we prove condition (2). To do this, we claim that $\mathcal{J}_\lambda(z) \to -\infty$ as $\|z\|_{s,\mathcal{A}} \to \infty$ for all $z \in \mathcal{Y}_k$. Let us assume that this is false for some k. Then we can choose a sequence $\{z_n\}$ in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$ such that

$$\|z_n\|_{s,\mathcal{A}} \to \infty \text{ as } n \to \infty \quad \text{and} \quad \mathcal{J}_\lambda(z_n) \geq -K.$$

Let $\omega_n = z_n/\|z_n\|_{s,\mathcal{A}}$. Then it is obvious that $\|\omega_n\|_{s,\mathcal{A}} = 1$. Since $\dim \mathcal{Y}_k < \infty$, there is $\omega \in \mathcal{Y}_k \setminus \{0\}$ such that up to a subsequence,

$$\|\omega_n - \omega\|_{s,\mathcal{A}} \to 0 \quad \text{and} \quad \omega_n(x) \to \omega(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. Thus we have by (2.17) that

$$\begin{aligned}
\frac{1}{2} + \frac{K}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}} &\geq \frac{1}{2} - \frac{\mathcal{J}_\lambda(z_n)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}} \\
&= \lambda \int_{\mathbb{R}^N} \frac{H(x, |z_n|)}{\mathcal{K}(|z_n|^2_{s,\mathcal{A}}) + \|z_n\|^2_{2,P}}\, dx \\
&\geq \lambda \int_{\{\omega_n(x) \neq 0\}} \frac{H(x, |z_n|)}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx.
\end{aligned} \tag{2.18}$$

If we follow the analogous argument as in the proof of Lemma 2.8, we derive by (2.10), (2.18), (H4) and Fatou's lemma that

$$\begin{aligned}
\frac{1}{2\lambda} &\geq \liminf_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{H(x, |z_n|)}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx \\
&\quad - \limsup_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{K_0}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx \\
&= \liminf_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{H(x, |z_n|) - K_0}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx \\
&\geq \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \frac{H(x, |z_n|) - K_0}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx \\
&= \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \frac{H(x, |z_n|)}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx \\
&\quad - \int_{\{\omega_n(x)\neq 0\}} \limsup_{n\to\infty} \frac{K_0}{(2\mathcal{K}(1) + 1)\, \|z_n\|^{2\theta}_{s,\mathcal{A}}}\, dx \\
&\geq \frac{1}{2\mathcal{K}(1) + 1} \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \left(\frac{H(x, |z_n|)}{|z_n|^{2\theta}} |\omega_n|^{2\theta} \right) dx = \infty,
\end{aligned}$$

where K_0 was given in the proof of Lemma 2.8. This is impossible. Thus, $\mathcal{J}_\lambda(z) \to -\infty$ as $\|z\|_{s,\mathcal{A}} \to \infty$ for all $z \in \mathcal{Y}_k$. Choose $\varrho_k > \sigma_k > 0$ large sufficiently and let $\|z\|_{s,\mathcal{A}} = \varrho_k$, we finally obtain

$$a_k = \max\{\mathcal{J}_\lambda(z) : z \in \mathcal{Y}_k, \|z\|_{s,\mathcal{A}} = \varrho_k\} \leq 0.$$

This completes the proof. □

Theorem 2.13. *Let $s \in (0,1)$ and $N > 2s$. Assume that* (P), (K1), (K2), (H1)–(H4) *and* (H6) *hold. Then for any $\lambda > 0$, problem* (1.1) *possesses an unbounded sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to \infty$ as $n \to \infty$.*

Proof. By a similar fashion as in Theorem 2.12, instead of Lemma 2.7, by Lemma 2.8, the conclusion holds. □

Definition 2.14. Let E be a real separable and reflexive Banach space. We say that $\mathcal{I}$ satisfies the $(C)^*_c$-condition (with respect to $\mathcal{Y}_n$) if any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ for which $z_n \in \mathcal{Y}_n$, for any $n \in \mathbb{N}$,

$$\mathcal{I}(z_n) \to c \quad \text{and} \quad \|(\mathcal{I}|_{\mathcal{Y}_n})'(z_n)\|_{E^*}(1 + \|z_n\|_E) \to 0 \text{ as } n \to \infty,$$

contains a subsequence converging to a critical point of $\mathcal{I}$.

Lemma 2.15. (Dual Fountain Theorem [19, Theorem 3.11]) *Assume that E is a real Banach space, $\mathcal{I} \in C^1(E, \mathbb{R})$ is an even functional. If there is $k_0 > 0$ so that, for each $k \geq k_0$, there are $\varrho_k > \sigma_k > 0$ such that*

(A1) $\inf\{\mathcal{I}(z) : z \in \mathcal{Z}_k, \|z\|_E = \varrho_k\} \geq 0.$
(A2) $\beta_k := \max\{\mathcal{I}(z) : z \in \mathcal{Y}_k, \|z\|_E = \sigma_k\} < 0.$
(A3) $\gamma_k := \inf\{\mathcal{I}(z) : z \in \mathcal{Z}_k, \|z\|_E \leq \varrho_k\} \to 0$ *as* $k \to \infty$.
(A4) *$\mathcal{I}$ satisfies the $(C)^*_c$-condition for every $c \in [d_{k_0}, 0)$.*

Then $\mathcal{I}$ has a sequence of negative critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

Lemma 2.16. *Under the same assumptions of Lemma 2.7 (resp. Lemma 2.8), the functional $\mathcal{J}_\lambda$ satisfies the $(C)^*_c$-condition.*

Proof. The proof is carried out by the analogous argument as in [40]. □

With the help of Lemmas 2.15 and 2.16 we are ready to demonstrate the following assertion.

Theorem 2.17. *Under the same assumptions of Theorem 2.12, the problem* (1.1) *has a sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}^s_{\mathcal{A},P}(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$.*

Proof. Invoking Lemma 2.16, we get that $\mathcal{J}_\lambda$ is even and satisfies the $(C)_c^*$-condition for all $c \in \mathbb{R}$. Now it remains to show that conditions (A1), (A2) and (A3) of Lemma 2.15 are satisfied.

(A1): Let us denote

$$\theta_{1,k} = \sup_{\|z\|_{s,\mathcal{A}}=1, z\in\mathcal{Z}_k} \|z\|_{L^2(\mathbb{R}^N)}, \quad \theta_{2,k} = \sup_{\|z\|_{s,\mathcal{A}}=1, z\in\mathcal{Z}_k} \|z\|_{L^q(\mathbb{R}^N)}.$$

Then, it is immediate to verify that $\theta_{1,k} \to 0$ and $\theta_{2,k} \to 0$ as $k \to \infty$. Set $\vartheta_k = \max\{\theta_{1,k},\ \theta_{2,k}\}$. Then it follows that

$$\begin{aligned}
\mathcal{J}_\lambda(z) &= \frac{1}{2}(\mathcal{K}(|z|_{s,\mathcal{A}}^2) + \|z\|_{2,P}^2) - \lambda \int_{\mathbb{R}^N} H(x,|z|)dx \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|_{s,\mathcal{A}}^2 - \frac{\lambda c_1}{2}\|z\|_{L^2(\mathbb{R}^N)}^2 - \frac{\lambda c_2}{q}\|z\|_{L^q(\mathbb{R}^N)}^q \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|_{s,\mathcal{A}}^2 - \frac{\lambda c_1}{2}\vartheta_{1,k}^2\|z\|_{s,\mathcal{A}}^2 - \frac{\lambda c_2}{q}\vartheta_{2,k}^q\|z\|_{s,\mathcal{A}}^q \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|_{s,\mathcal{A}}^2 - \lambda\left(\frac{c_1}{2} + \frac{c_2}{q}\right)\vartheta_k^2\|z\|_{s,\mathcal{A}}^q
\end{aligned}$$

for sufficiently large k and $\|z\|_{s,\mathcal{A}} \geq 1$. Choose

$$\rho_k = \left[\frac{4\lambda}{\min\{1, a\theta^{-1}\}}\left(\frac{c_1}{2} + \frac{c_2}{q}\right)\vartheta_k^2\right]^{\frac{1}{2-2q}}.$$

Let $z \in \mathcal{Z}_k$ with $\|z\|_{s,\mathcal{A}} = \varrho_k > 1$ for k large enough. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathcal{J}_\lambda(z) &\geq \frac{\min\{1, a\theta^{-1}\}}{2}\|z\|_{s,\mathcal{A}}^2 - \lambda\left(\frac{c_1}{2} + \frac{c_2}{q}\right)\vartheta_k^2\|z\|_{s,\mathcal{A}}^{2q} \\
&= \frac{\min\{1, a\theta^{-1}\}}{4}\varrho_k^2 \geq 0
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, because

$$\lim_{k\to\infty} \frac{\min\{1, a\theta^{-1}\}}{4}\varrho_k^2 = \infty.$$

Therefore,

$$\inf\{\mathcal{J}_\lambda(z) : z \in \mathcal{Z}_k, \|z\|_{s,\mathcal{A}} = \varrho_k\} \geq 0.$$

(A2): Observe that $\|\cdot\|_{L^2(\mathbb{R}^N)}$, $\|\cdot\|_{L^{2\theta}(\mathbb{R}^N)}$ and $\|\cdot\|_{s,\mathcal{A}}$ are equivalent on $\mathcal{Y}_k$. Then there exist positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$\|z\|_{L^2(\mathbb{R}^N)} \leq \varsigma_{1,k}\|z\|_{s,\mathcal{A}} \text{ and } \|z\|_{s,\mathcal{A}} \leq \varsigma_{2,k}\|z\|_{L^{2\theta}(\mathbb{R}^N)}$$

for any $z \in \mathcal{Y}_k$. From (H2)–(H4), for any $\mathcal{M} > 0$ there is positive constant $C_7(\mathcal{M})$ such that

$$H(x,\tau) \geq \mathcal{M}\varsigma_{2,k}^{2\theta}\tau^{2\theta} - C_7(\mathcal{M})\tau^2$$

for almost all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+$. Since $\mathcal{K}(\eta) \le \mathcal{K}(1)(1+\eta^\theta)$ for all $\eta \in \mathbb{R}_0^+$, it follows that

$$\begin{aligned}
\mathcal{J}_\lambda(z) &= \frac{1}{2}(\mathcal{K}(|z|_{s,\mathcal{A}}^2) + \|z\|_{2,P}^2) - \int_{\mathbb{R}^N} H(x,|z|)dx \\
&\le \frac{1}{2}\left(\mathcal{K}(1)(1+|z|_{s,\mathcal{A}}^{2\theta}) + \|z\|_{2,P}^2\right) - \mathcal{M}\varsigma_{2,k}^{2\theta}\int_{\mathbb{R}^N} |z|^{2\theta}dx + C_7(\mathcal{M})\int_{\mathbb{R}^N}|z|^2dx \\
&\le \frac{1}{2}\left(2\mathcal{K}(1)\|z\|_{s,\mathcal{A}}^{2\theta} + \|z\|_{s,\mathcal{A}}^{2\theta}\right) - \mathcal{M}\varsigma_{2,k}^{2\theta}\int_{\mathbb{R}^N} |z|^{2\theta}dx + C_7(\mathcal{M})\int_{\mathbb{R}^N}|z|^2dx \\
&\le \frac{1}{2}\left(2\mathcal{K}(1)+1\right)\|z\|_{s,A}^{2\theta} - \mathcal{M}\|z\|_{s,\mathcal{A}}^{2\theta} + C_7(\mathcal{M})\varsigma_{1,k}^2\|z\|_{s,\mathcal{A}}^2
\end{aligned}$$

for any $z \in \mathcal{Y}_k$ with $\|z\|_{s,\mathcal{A}} \ge 1$. Let $h(\tau) = \frac{1}{2}\left(2\mathcal{K}(1)+1\right)\tau^{2\theta} - \mathcal{M}\tau^{2\theta} + C_7(\mathcal{M})\varsigma_{1,k}^2\tau^2$. If $\mathcal{M}$ is large enough, then $\lim_{\tau\to\infty} h(\tau) = -\infty$, and thus there is $\tau_0 \in (1,\infty)$ such that $h(\tau) < 0$ for all $\tau \in [\tau_0,\infty)$. Hence $\mathcal{J}_\lambda(z) < 0$ for all $z \in \mathcal{Y}_k$ with $\|z\|_{s,\mathcal{A}} = t_0$. Choosing $\sigma_k = t_0$ for all $k \in \mathbb{N}$, one has

$$\beta_k := \max\{\mathcal{J}_\lambda(z) : z \in \mathcal{Y}_k, \|z\|_{s,\mathcal{A}} = \sigma_k\} < 0.$$

If necessary, we can change k_0 to a large value, so that $\varrho_k > \sigma_k > 0$ for all $k \ge k_0$.

(A3): Because $\mathcal{Y}_k \cap \mathcal{Z}_k \ne \phi$ and $0 < \sigma_k < \varrho_k$, we have $\gamma_k \le \beta_k < 0$ for all $k \ge k_0$. For any $z \in \mathcal{Z}_k$ with $\|z\|_{s,\mathcal{A}} = 1$ and $0 < \tau < \varrho_k$, one has

$$\begin{aligned}
\mathcal{J}_\lambda(\tau z) &\ge \frac{\min\{1, a\theta^{-1}\}}{2}\|\tau z\|_{s,\mathcal{A}}^2 - \frac{\lambda c_1}{2}\|\tau z\|_{L^2(\mathbb{R}^N)}^2 - \frac{\lambda c_2}{q}\|\tau z\|_{L^q(\mathbb{R}^N)}^q \\
&\ge -\frac{\lambda c_1}{2}\tau^2\|z\|_{L^2(\mathbb{R}^N)}^2 - \frac{\lambda c_2}{q}\tau^q\|z\|_{L^q(\mathbb{R}^N)}^q \\
&\ge -\frac{\lambda c_1}{2}\varrho_k^2\vartheta_k^2 - \frac{\lambda c_2}{q}\varrho_k^q\vartheta_k^q
\end{aligned}$$

for large enough k. Hence, it follows from the definition of ϱ_k that

$$\begin{aligned}
\gamma_k &\ge -\frac{\lambda c_1}{2}\varrho_k^2\vartheta_k^2 - \frac{\lambda c_2}{q}\varrho_k^q\vartheta_k^q \\
&= -\frac{\lambda c_1}{2}\left[\frac{4\lambda}{\min\{1, a\theta^{-1}\}}\left(\frac{c_1}{2} + \frac{c_2}{q}\right)\right]^{\frac{1}{1-q}}\vartheta_k^{\frac{4-2q}{1-q}} \\
&\quad - \frac{\lambda c_2}{q}\left[\frac{4\lambda}{\min\{1, a\theta^{-1}\}}\left(\frac{c_1}{2} + \frac{c_2}{q}\right)\right]^{\frac{q}{2-2q}}\vartheta_k^{\frac{(2-q)q}{1-q}}.
\end{aligned}$$

Because $2 < q$ and $\vartheta_k \to 0$ as $k \to \infty$, we derive that $\lim_{k\to\infty}\gamma_k = 0$.

Hence all conditions of Lemma 2.15 are required. Consequently, we assert that problem (1.1) has a sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}_{\mathcal{A},P}^s(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$. □

Theorem 2.18. *Under the same assumptions of Theorem 2.13, the problem* (1.1) *has a sequence of nontrivial weak solutions* $\{z_n\}$ *in* $\mathcal{H}_{\mathcal{A},P}^s(\mathbb{R}^N,\mathbb{C})$ *such that* $\mathcal{J}_\lambda(z_n) \to 0$ *as* $n \to \infty$ *for any* $\lambda > 0$.

Proof. The proof is carried out by a similar fashion as in Theorem 2.17. □

3. Conclusion

In this paper, we take into account the variational methods to get the existence of nontrivial solutions to nonlocal Schrödinger–Kirchhoff equations with the external magnetic field. In particular we obtain these results under the various conditions on h when the nonlinear growth h does not satisfy the condition of Ambrosetti-Rabinowitz type. We point out that with an analogous analysis our main assertions still hold when $(-\Delta)^s_A z$ in (1.1) is changed into any non-local integro-differential operator $\mathcal{L}_\phi$ defined as follows:

$$\mathcal{L}_\phi z(x) = 2\int_{\mathbb{R}^N} (z(x) - \mathbb{E}(x,y)z(y))\phi(x-y)dy \quad \text{for all } x \in \mathbb{R}^N.$$

where $\mathbb{E}(x,y) := e^{i(x-y)\cdot A(\frac{x+y}{2})}$ and $\phi : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a kernel function satisfying properties that

(K1) $m\phi \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;
(K2) there exists $\mu > 0$ such that $\phi(x) \geq \mu|x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
(K3) $\phi(x) = \phi(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

Funding

This research was funded by a 2020 research Grant from Sangmyung University.

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