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# EXISTENCE AND NONEXISTENCE OF LIMIT CYCLES FOR A CERTAIN CLASS OF PLANAR SYSTEMS OF LIÉNARD TYPE

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**ABSTRACT.** The phase-portrait of a Liénard system $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$ is investigated under a classical assumption on $F(x)$, namely that there are $\alpha < 0 < \beta$ such that $F(\alpha) = F(0) = F(\beta) = 0$ with $F(x)x < 0$ for $x \in (\alpha, \beta) \setminus \{0\}$ and $F$ is monotone increasing outside $(\alpha, \beta)$. It is well known that such a system has at least a limit cycle provided that $G(x) \to +\infty$ or $F(x)\text{sign}(x) \to +\infty$ for $x \to \pm\infty$. Clearly, such assumptions imply that $G(x) \pm F(x) \to +\infty$ as $x \to \pm\infty$. In [19] it has been proved that this is actually a necessary and sufficient condition for the intersection of the trajectories with the vertical isocline. In this paper we treat the case in which this assumption is not fulfilled and prove that there are cases where both existence and nonexistence may occur.

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**Key Words and Phrases.** Liénard equation, limit cycles, existence/non-existence.

## 1. INTRODUCTION

The Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

has been widely investigated starting from the seminal papers by Van der Pol [17] and Liénard [14], and there is an enormous quantity of results dealing with the problem of existence or uniqueness of limit cycles for both the equivalent systems

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x). \end{cases} \tag{1.2}$$

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Here and in what follows, the usual regularity assumptions on $g$ and $F$ are assumed. We just observe that the particular case in which $F$ has three zeros,

$$\alpha < 0 < \beta, \quad \text{with } F(x) > 0 \text{ on } (\alpha, 0) \text{ and } F(x) < 0 \text{ on } (0, \beta) \tag{1.3}$$

and

$$F \text{ is monotone increasing for } x < \alpha \text{ and } x > \beta, \tag{1.4}$$

has a relevant place in the literature. This because in a fundamental work, Duff and Levinson [6], using the averaging method, produced an important example with three limit cycles, proving in this way that the conjecture that with these assumptions of $F$ there was at most a limit cycle, was actually incorrect (see [1, 2, 11] for further examples inspired by Duff and Levinson work). Later on, the uniqueness of the limit cycles was proved under the additional assumption that all the possible limit cycles must intersect both the lines $x = \alpha$ and $x = \beta$, as observed by Conti in [4], starting from the well-known assumption $G(\alpha) = G(\beta)$ (see [13, 16, 1]). It is worth to note that these ideas were already in the pioneering work of Liénard [14].

More recently, assumptions guaranteeing the above mentioned intersection property were proposed, in order to have the uniqueness of the limit cycle [9, 10, 2, 21, 11]. It should be noticed that the particular shape of $F$ gives easily the existence of the limit cycle under the necessary and sufficient condition for the intersection of the positive semi-trajectories with the vertical isocline $y = F(x)$ in system (1.2), which is clearly equivalent to the intersection of the $x$-axis in system (1.1).

A classical existence result is the following, which is an equivalent version of the result of Draghilev [5].

**Theorem 1.1.** *Assume* (1.3) *and* $G(x) \to +\infty$ *for* $x \to \pm\infty$. *Then system* (1.2) *has at least a limit cycle if*

$$\limsup_{x \to -\infty} F(x) < 0 < \liminf_{x \to +\infty} F(x).$$

We observe that the conditions at infinity for $F(x)$ and $G(x)$ imply the uniform ultimate boundedness for the solutions and this is true even for the forced Liénard system

$$\begin{cases} \dot{x} = y - F(x) + E(t) \\ \dot{y} = -g(x) \end{cases}$$

with $\|E\|_\infty < \liminf_{x \to +\infty} F(x) - \limsup_{x \to -\infty} F(x)$ (see [12, Example p. 119] and [23, Remark 4]).

Further extensions of Theorem 1.1 may be found in in [3, 7, 18, 19].

Summarizing the above mentioned achievements, we have the following.

**Theorem 1.2.** *Assume* (1.3) *and*

$$\limsup_{x\to\pm\infty}(G(x) \pm F(x)) = +\infty. \tag{1.5}$$

*Then system* (1.2) *has at least a limit cycle.*

However, as proved in [19], condition (1.5) is actually necessary and sufficient for the intersection of the trajectories with the vertical isocline.

Therefore a natural question arises: If condition (1.5) is not fulfilled, does the limit cycle actually exist? The aim of this note is to discuss this problem.

The plan of the paper is the following: In Section 2 we briefly discuss the problem for system (1.2) and the assumptions which will be requested throughout the paper. Section 2 will also present some nonexistence results in which the maximum of $F(x)$ for $\alpha < x < 0$ (or in the dual case the minimum of $F(x)$ in $0 < x < \beta$) plays a crucial role. Constructive examples for the existence of the limit cycle will be also presented, which complete all the possible cases.

## 2. THE PHASE-PORTRAIT AND NONEXISTENCE RESULTS

Throughout the paper we assume the following conditions on system (1.2), namely

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x). \end{cases}$$

1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and therefore $F$ is of class $C^1$ and hence it is Lipschitz continuous. Moreover, $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.
2) $F$ has the following shape: there are $\alpha < 0 < \beta$ such that

   $$F(\alpha) = F(0) = F(\beta) = 0, \text{ with } F(x)x < 0 \text{ if and only if } x \in (\alpha, \beta), x \neq 0$$

3) $F$ is monotone increasing for $x < \alpha$ and $x > \beta$.

Clearly, assumption 1) guarantees the (local) existence and the uniqueness of the solutions for the initial value problems. Accordingly, for any initial point $P_0$ we will denote by $\gamma^+(P_0)$ the semi-positive trajectory departing from $P_0$.
As mentioned in the Introduction, if $G(\alpha) = G(\beta)$, there is at most a limit cycle. However, as we are interested in nonexistence, we will not assume explicitly this condition, as it will be discussed in the sequel.

Notice also that the monotonicity condition on $F(x)$ can be relaxed provided that $F(x)$ is eventually greater that the maximal $y$-amplitude of the cycle for $x > 0$ and large and $F(x)$ is eventually smaller that the minimal $y$-amplitude of the cycle for $x > 0$ and large in absolute value, [22]. Therefore, it is not necessarily required that a limit at $\pm\infty$ for $F(x)$ does exists.

It is also well known that if $G(x) \to +\infty$ as $x \to \pm\infty$ (see [5, 18]), or

$$\liminf_{x\to-\infty} F(x) = -\infty \quad \text{and} \quad \limsup_{x\to+\infty} F(x) = +\infty$$

(see [19]), then at least a limit cycle always exists.

We observe that the two assumptions on $G(x)$ and $F(x)$, respectively, imply

$$\limsup_{x\to\pm\infty}(G(x) \pm F(x)) = +\infty \tag{2.1}$$

and hence we enter in the setting of Theorem 1.2. However, as proved in [19], such an assumption is a necessary and sufficient condition for the intersection with the vertical isocline $y = F(x)$ of any positive trajectory $\gamma^+(0, y_0)$ with $y_0 \neq 0$. This because assumption 2) implies that $F(x)$ is bounded below for $x > 0$ and above for $x < 0$.

At this point, we investigate the case in which condition (2.1) is not satisfied and prove the main result of the paper, namely that there are situations in which no limit cycle exists. For sake of simplicity, we suppose that condition (2.1) fails only for $x > 0$, being the other case completely symmetric. As expected, the maximum value $\hat{F}$ of $F(x)$ in $\alpha < x < 0$ will play a crucial role.

**Theorem 2.1.** *Assume 1)-2)-3) and* $\limsup_{x\to-\infty}(G(x) - F(x)) = +\infty$. *Suppose* $G(+\infty) = H < +\infty$ *and* $F(+\infty) = K < +\infty$.
*Moreover, assume that* $G(\alpha) \leq G(\beta)$.

a) *If* $\hat{F} \geq K + \sqrt{2H}$, *system* (2.1) *has no limit cycles.*
b) *If* $K < \hat{F} < K + \sqrt{2H}$, *there is* $\hat{\lambda} > 0$ *such that system*

$$\dot{x} = y - \lambda F(x), \quad \dot{y} = -g(x)$$

*has no limit cycles for* $\lambda \geq \hat{\lambda}$.

*Proof.* The proof is elementary. At first we observe that for every $P_\beta := (\beta, y_\beta)$ with $y_\beta < 0$, the positive trajectory $\gamma^+(P_\beta)$ intersects the axis $x = \alpha$. This is well known (see, for instance [8]) and we give just a sketch of the proof.
In light of the assumption[1] $G(\alpha) \leq G(\beta)$ the level curve of the energy function

$$E(x, y) = \frac{1}{2}y^2 + G(x) = L$$

with $L = G(\beta)$ intersects also $x = \alpha$ at a point with $y \leq 0$. Taking $E$ as a Lyapunov function for system (1.2) and considering the derivative of $E$ along the trajectories,

$$\dot{E} = -g(x)F(x) \geq 0 \quad \text{for} \;\; \alpha < x < \beta,$$

we see that $\gamma^+(P_\beta)$ is bounded away from this level line and hence it intersects the line $x = \alpha$ at a point $(\alpha, y_\alpha)$ with $y_\alpha < 0$.

[1]This assumption is not restrictive, because, in the other case, we start from an initial point $P_\alpha := (\alpha, y_\alpha)$ with $y_\alpha > 0$ and consider similar assumptions on the minimum of $F$ in $(0, \beta)$.

Now the assumption $\limsup_{x\to-\infty}(G(x)-F(x))=+\infty$, guarantees the intersection with the vertical isocline. Elementary phase-plane analysis shows that the trajectory intersects the $x$-axis in a point $x<\alpha$ and then achieves the $y$-axis at a point $(0,y_0)$ with $y_0>\hat{F}>K+\sqrt{2H}$. Following again [19] and [15, 12], if we consider the level curves of the shifted energy

$$E_K(x,y):=\frac{1}{2}(y-K)^2+G(x),$$

we see that from

$$\dot{E}_K=g(x)(K-F(x)),$$

the trajectory stays bounded away from the level curve $E_K(x,y)=H$. Clearly, such level curve does not intersect the line $y=K$ and therefore it does not intersect the vertical isocline for $x>0$ (see Figure 1).

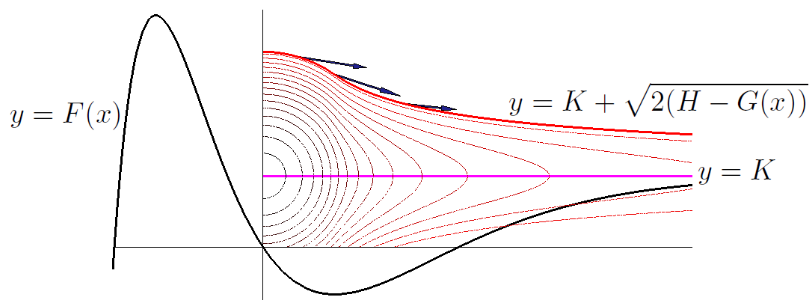


FIGURE 1. **A graphical illustration of the proof for the case** $a$). The trajectory $\gamma^+(P_\beta)$ after an half turn, gains a level higher that $\hat{F}$, then it crosses the line $x=0$ at a point $(0,y_0)$ with $y_0>K+\sqrt{2H}$, the latter constant being the intersection of the level line $E_K(x,y)=H$ with $x=0$. The graph is produced using [24].

This completes the proof of case $a$) as we actually have shown that no trajectory staring from $P_\beta$ is oscillatory. □

**Remark 2.2.** We just observe that the shape of $F$ plays a crucial role in the proof of Theorem 2.1. Indeed, if we change $F(x)$ allowing the existence of a limit cycle in a strip containing $[\alpha,\beta]\times\mathbb{R}$ but adding other two zeros in $(-\infty,\alpha)$, we may assume that $F(x)$ takes its maximum value $\hat{F}$ at a point between the two added new zeros with $\hat{F}>K+\sqrt{2H}$. In this manner, repeating the argument of the proof, we can show that all the trajectories large enough do not intersect the vertical isocline for $x$ positive and become eventually bounded away from the $x$-axis.

Notice that no other limit cycle may occur because there is at most a limit cycle intersecting both $x=\alpha$ and $x=\beta$. Therefore, the phase-portrait has a repulsive separatrix and all the trajectories above the separatrix are bounded away from the $x$-axis, while, below the separatrix, the trajectories tend to the stable limit cycle. Figure 2 provides an illustrative example of this situation.

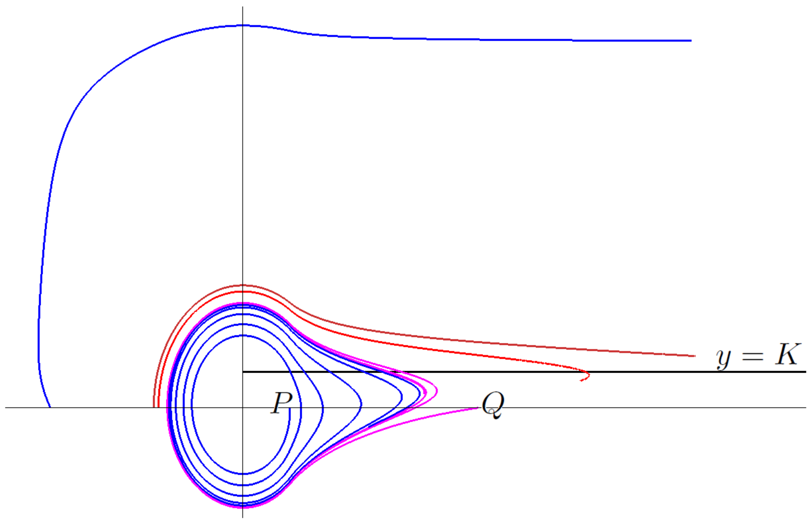


FIGURE 2. **Coexistence of limit cycles and unbounded solutions.** The numerical example illustrates the situation described in Remark 2.2. For the example, we have considered a Liénard system (1.2) with $g(x) = 10x$ for $x \leq 1$ and $g(x) = 10/x^2$ for $x \geq 1$. In this case, $G(x) = 5x^2$ for $x \leq 1$ and $G(x) = \frac{15x-10}{x}$ for $x \geq 1$ and therefore, $H = 15$. As $F$, we have: $F(x) = 1.8 - 1.11\exp(-x)x^2 - 1.8\exp(-x) - 0.03\exp(-x)x^4 - 0.3\exp(-x)x^3 - 2.04\exp(-x)x$. The (apparently complicated) choice of $F(x)$ comes after an integration of the simpler function $f(x) = r(x+4)(x+2)(x+1)(x-1)\exp(-x)$, where the constant $r$ is set to $r = 3/100$ just for the convenience of producing a better outcome. From the numerical simulation, it is also apparent that there exists an unstable separatrix such that the points below the separatrix are attracted by the limit cycles, while the semi-orbits of the points above the separatrix are unbounded. The graph is produced using [24].

For case $b)$ we observe that if we multiply the function $F$ with a real parameter $\lambda$, we have that both $\hat{F}$ and $K$ are now expanded by $\lambda$. In order to enter in the previous setting, namely $\hat{F} \geq K + \sqrt{2H}$ it is sufficient to take $\lambda$ such that $\lambda(\tilde{F} - K) \geq \sqrt{2H}$, that is $\lambda \geq \hat{\lambda} := \frac{\sqrt{2H}}{\tilde{F}-K}$. This completes the proof. $\square$

**Remark 2.3.** Clearly, the proof fails for $\lambda$ small and limit cycles may appear. In this light we produce the following numerical example (see Figure 3), where, as before, a repulsive separatrix appear even if not shown in the figure.

When the maximum of $F$ is such that $\hat{F} < K$, the previous argument is no more valid, and one expects the existence of limit cycles. This is clearly true. Indeed, one may use the Van der Pol equation where the unique limit cycle can be easily located, in any scientific computing environment, in a strip $C < x < D$, with $C, D$ suitable constants. Now it suffices to modify $F$ and $G$ for $x > D$ in such a way that they become bounded. Clearly, this modification does not affect the limit cycle, even if this approach is in some sense “tricky” because the limit cycle has been already located in the plane. Also in this case, we will have the phenomenon of the unstable separatrix.

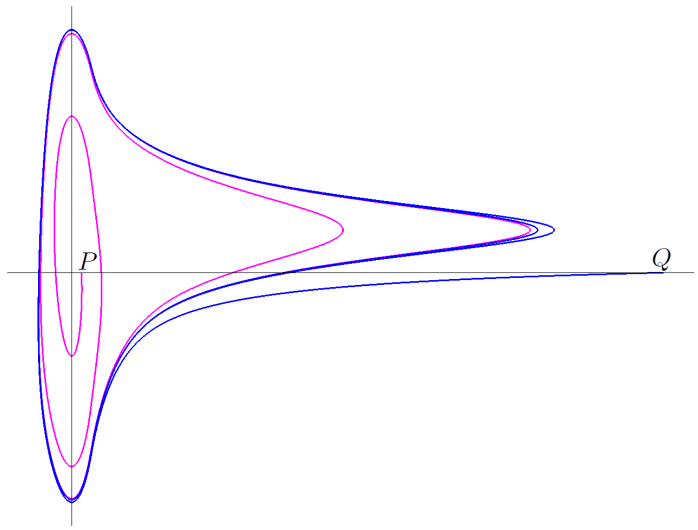


FIGURE 3. **Existence of a limit cycle for case $b$) when $\lambda$ is not large.** The numerical example shows the phase-portrait of the Liénard system (1.2) with $g(x) = 10x$ for $x \leq 1$ and $g(x) = 10/x^2$ for $x \geq 1$. In this case, $G(x) = 5x^2$ for $x \leq 1$ and $G(x) = \frac{15x-10}{x}$ for $x \geq 1$ and therefore, $H = 15$. Concerning the functions $F$ and $f$, we have made the following choice: $F(x) = 1 - (x^2 + 2.05x + 1)\exp(-x)$, with $f(x) = (x + 1.05)(x - 1)\exp(-x)$. Observe that $F$ has three zeros $\alpha < 0 < \beta$. Moreover, $K = 1$ and $\hat{F} = \max_{[\alpha,0]} F(x) = F(-1.05) \approxeq 1.14288$. Hence we are in case $b$) of Theorem 2.1, because $K = 1 < \hat{F} < K + \sqrt{2H} = 1 + \sqrt{30}$. We have proved that system $\dot{x} = y - \lambda F(x)$, $\dot{y} = -g(x)$ has no limit cycles for $\lambda$ larger than a constant which can be explicitly determined. In our case, a limit cycle appears, as is evident from the fact that the solution departing from $P = (1/2, 0)$ unwinds from the origin, while the solution with initial value in $Q = (30, 0)$ winds toward the origin. The graph is produced using [24].

Finally, we just observe that, using an approach similar to the one adopted for the construction of the not existence example presented in the first section, one can produce also in this case examples with no limit cycles. It suffices to take $G(\alpha) > G(\beta)$ and the absolute value of the minimum in the strip $0 < x < \beta$ large enough, together with conditions which guarantee that any trajectory starting from the positive y axis intersect back in time the curve $y = F(x)$. Such conditions may be founded in [20]. Being such a construction similar to the one already discussed, the details are omitted.

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