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# PERIODIC SOLUTIONS OF RANDOM NONLINEAR EVOLUTION INCLUSIONS IN BANACH SPACES

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**ABSTRACT.** In this work, we investigate the existence of random integral solutions for convex and non-convex evolution differential inclusions with periodic conditions. Also, we give the random version of Bader's fixed point theorem. The existence results are established by means of random fixed point theory. Finally, an example is given to illustrate the result..

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**Key Words and Phrases.** Evolution inclusion, random integral solution, periodic problem, $m-$accretive operator, multivalued map, continuous selection.

## 1. INTRODUCTION

The study of periodic problems has received great attention from many authors. For evolution equations, we refer to the works by Pruss[26], Vrabie[32], Becker[7] and the references therein. In addition, for evolution inclusions, we cite the works by Hu and Papageorgiou [16], Kandilakis and Papageorgiou [20], Lakshmikantham and Papageorgiou [23], and we also mention recent articles by Bader and Papageorgiou [4], and Hu and Papageorgiou [19].

The topic of random differential equations and inclusions is a great field. This theory is used in many different applications such as statistics, control theory, biological sciences, etc. For more information on such applications, see the books of Bharucha-Reid [8] and Skorohod [30]. Due to different applications, various studies of differential equations with random coefficients have been considered recently; see for instance [5, 14, 15, 24, 28, 29] and their references.

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In view of this we study the existence of random integral solutions for the following problem

$$\begin{cases} -x'(\omega,t) \in A(\omega)x(\omega,t) + F(\omega,t,x), t \in [0,b], \\ x(\omega,0) = x(\omega,b), \quad \omega \in \Omega, \end{cases} \tag{1.1}$$

where for every $\omega \in \Omega$, $A(\omega)$ is an $m-$accretive operator in a reflexive Banach space $X$, $F : \Omega \times [0,b] \times X \to 2^X$ is a multivalued map (perturbation), and $x : \Omega \to D(A(\omega)) \subset X$ is random variable.

We first collect some background material and basic results from multivalued analysis and random variable calculus in Section 2. The existence of random integral solutions of (1.1) is investigated in Section 3. The extremal solution is considered in Section 4. Finally, in Section 5, an example is described to illustrate the applicability of our results.

## 2. PRELIMINARIES

Let $(E, |\cdot|)$ be a Banach space. Denote by $\mathcal{P}(E) = \{Y \subset E\colon Y \neq \emptyset\}$, $\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ closed}\}$, $\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ bounded}\}$, $\mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ convex}\}$, $\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ compact}\}$, and $\mathcal{P}_{wkcp}(E) = \{Y \in \mathcal{P}(E)\colon Y \text{ weakly compact}\}$.

Let $X$ be a real reflexive, separable Banach space with norm $\|\cdot\|$, $X^*$ be the dual space of $X$, with norm $\|\cdot\|_*$, $\sigma(X,X^*)$ be the weak topology on $X$, and denote by $X_w$ the space $X$ endowed with the topology $\sigma(X,X^*)$ : The duality pairing between $X$ and $X^*$ will be denoted by $\langle\cdot,\cdot\rangle$. The duality mapping $J(x) : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|_*^2\right\}, \forall x \in X,$$

and the upper semi-inner product on $X$ is defined by

$$\langle y, x\rangle_+ = \sup\left\{x^*(y) : x^* \in J(x)\right\}.$$

The duality mapping $J$ is single-valued and uniformly continuous on bounded subsets of $X$, if $X^*$ is uniformly convex.

Let $A : X \to 2^X$ be a multivalued operator on $X$. The domain and, respectively, the range of $A$ are given by

$$D(A) := \{x \in X : Ax \neq \emptyset\}, \quad R(A) := \bigcup_{x \in D(A)} Ax.$$

The operator $A$ is called $m$-accretive if the following conditions are satisfied:

- $A$ is a monotone operator,

$$\langle y' - y, x' - x\rangle_+ \geq 0, \forall x, x' \in D(A), \forall y \in Ax, \forall y' \in Ax',$$

- $$R(I + \lambda A) = X, \forall \lambda > 0,$$

where $I$ is the identity map on $X$.

If $A$ is $m$-accretive, then $-A$ generates a semigroup of contractions $\{S(t) : t \geq 0\}$ on $\overline{D(A)}$, according to a celebrated conclusion by Crandall and Liggett [11]. The semigroup $\{S(t) : t \geq 0\}$ is said to be a compact semigroup, if $S(t)$ maps bounded subsets of $\overline{D(A)}$ into precompact subsets of $\overline{D(A)}$, for each $t > 0$.

Throughout this paper we will be using the following notations: $C([0, b], X)$ is the Banach space of all continuous functions $u : [0, b] \times X$ with norm

$$\|u\|_\infty = \sup_{t \in J} \|u(t)\|,$$

and for $1 \leq p < \infty$, $L^p([0, b], X)$ is the Banach space of measurable functions $u : [0, b] \times X$ such that $\|u\|_p$ is Lebesgue integrable, endowed with the norm

$$\|u\|_p = \left( \int_0^b \|u(t)\|^p dt \right)^{1/p}.$$

We consider the following weak norm in the space $L^1([0, b], X)$, given by

$$\|u\|_w = \sup \left\{ \left\| \int_s^t u(\tau) d\tau \right\| : 0 \leq s \leq t \leq b \right\}, \forall u \in L^1([0, b], X).$$

The norm $\| \cdot \|_w$ is weaker than the usual norm $\| \cdot \|_1$ and for a broad class of subsets of $L^1([0, b], X)$, the topology defined by the weak norm coincides with the usual weak topology (see Proposition 4.14 in [17]). The space $L^1([0, b], X)$, equipped with the weak norm, will be denoted by $L^1_w([0, b], X)$ . This notation is to be distinguished from $L^1([0, b], X)_w$, which designates the space $L^1([0, b], X)$ with the $\sigma(L^1([0, b], X), L^\infty([0, b], X^*))$ topology.

Let $A$ is $m$-accretive on $X$. For $f \in L^1([0, b], X)$ we consider the evolution equation,

$$-u'(t) = Au(t) + f(t), \quad t \in [0, b]. \tag{2.1}$$

**Definition 2.1.** A continuous function $u : [0, b] \to \overline{D(A)}$ is called an integral solution of (2.1) if for all $x \in D(A)$, $y \in Ax$ and all $0 \leq s \leq t \leq b$,

$$\|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2 \int_s^t \langle -f(\tau) - y, u(\tau) - x \rangle_+ d\tau. \tag{2.2}$$

It is well-known that for each $u_0 \in D(A)$ and $f \in L^1([0, b], X)$ the equation (2.2) admits a unique integral solution satisfying the initial condition $u(0) = u_0$.

Let $(\Omega, \Sigma, \mu)$ is a complete, $\sigma-$finite measure space, $\mathcal{L}([0, b])$ is the Lebesgue $\sigma-$field of $[0, b]$, and $\mathcal{B}(X)$ the Borel $\sigma-$algebra on $X$. Then $(\Omega \times [0, b] \times X, \Sigma \times$

$\mathcal{L}([0,b])\times\mathcal{B}(X))$ and $(\Omega\times[0,b],\Sigma\times\mathcal{L}([0,b]))$ are, respectively, the product $\sigma-$algebra on $\Omega\times[0,b]\times X$ and $\Omega\times[0,b]$.

The multifunction $\varphi:\Omega\to\mathcal{P}_{cl}(X)$ is called measurable if it satisfies any of the following equivalent conditions :

- $\mathcal{G}r\varphi=\{(\omega,x)\in\Omega\times X : x\in\varphi(\omega)\}\in\Sigma\times\mathcal{B}(X)$.
- The function $\omega\to d(x,\varphi(\omega))=\inf\{\|x-z\| : z\in\varphi(\omega)\}$ is measurable.

The set of all measurable selections of $\varphi$ that belong to the Bochner-Lebesgue space $L^p(\Omega,X)$, we denote by $S^p_\varphi$,

$$\mathcal{S}^p_\varphi=\{\psi\in L^p(\Omega,X):\psi(\omega)\in\varphi(\omega),\ \text{a.e. on }\Omega\}.$$

According to the Kuratowski-Ryll Nardzewski Theorem (see, e.g. [17], p.154) one has that for a measurable multifunction $\varphi:\Omega\to\mathcal{P}_f(X)$, the function $\omega\to\inf\{\|z\|:z\in\varphi(\omega)\}$ belongs to $L^p_+(\Omega)=L^p(\Omega,\mathbb{R}^+)$, if and only if the set $S^p_\varphi$ is nonempty.

A subset $\mathcal{A}$ of $L^p([0,b],X)$ is decomposable if for all $u,v\in\mathcal{A}$ and $N\subset\Sigma$ measurable, the function $u_{\chi_N}+v_{\chi_{[0,b]\setminus N}}\in\mathcal{A}$, where $\chi$ stands for the characteristic function. Clearly $\mathcal{S}^p_\varphi$ is decomposable.

Let now $Y$ be a Hausdorff topological space and let $\phi:Y\to 2^X$. For $A\in 2^X$, we set

$$\phi^{-1}(A):=\{y\in Y:\phi(y)\cap A\neq\emptyset\},$$
$$\phi^{+1}(A):=\{y\in Y:\phi(y)\subset A\}.$$

**Definition 2.2.** The multifunction $\phi$ is said to be

(A) Upper semi-continuous on $X$ (*u.s.c.*, for short) if the set $\phi^+(A)$ is open in $Y$ for any open subset of $A$ of $X$.(Equivalently, is u.s.c. if $\phi^{-1}(C)$ is closed in $Y$ for each closed subset $C$ of $Z$).

(B) Lower semicontinuous on $X$ (*l.s.c.*, for short) if $\phi^{+1}(C)$ is closed in $Y$ for each closed subset $C$ of $X$.

**Proposition 2.1.** If $\phi:Y\to\mathcal{P}_f(X)$ is upper semicontinuous, then $\phi$ is closed (its graph $\mathcal{G}r\phi$ is closed in $Y\times X$ )

**Proposition 2.2.** if $\phi:Y\to\mathcal{P}_f(X)$ is closed and locally compact (is, for every $y\in Y$, there exists a $U\in N(y)$ such that $\overline{\phi(U)}\in\mathcal{P}_k(X)$, then $\phi$ is upper semicontinuous

Consider the Hausdorff pseudo-metric distance

$$H_d(A,B):\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}^+\cup\infty$$

defined by

$$H_d(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\}$$

where $d(A,b)=\inf_{a\in A}d(a,b)$ and $d(a,B)=\inf_{b\in B}d(a,b)$.

A multifunction $\phi : Y \to \mathcal{P}_f(X)$ is called Hausdorff continuous if $\phi : Y \to (\mathcal{P}_f(X), H_d)$ is a continuous map, that is, for every $y_0 \in Y$ and any $\varepsilon > 0$ there exists a neighborhood $U_0$ of $y_0$ such that for any $y \in U_0$, we get $H_d(\phi(y), \phi(y_0)) < \varepsilon$.

By a space we always mean a separable Banach space.

**Definition 2.3.** Let $X, Y$ be two real separable Banach spaces, a multivalued map $F : \Omega \times X \to Y$ is called a random operator if $\omega \to F(\omega, x)$ is measurable for every $x \in X$.

**Definition 2.4.** A random fixed point of $F$ is a measurable function $z : \Omega \to X$ such that

$$z(\omega) \in F(\omega, z(\omega)) \quad \text{for all} \quad \omega \in \Omega.$$

**Definition 2.5.** An random multivalued operator $F : \Omega \times X \to \mathcal{P}(X)$ is called has a decomposition if there exists, closed convex subset $Y$ of separable Banach space, an random operator $\Phi : \Omega \times X \to \mathcal{P}_{cv,cp}(Y)$, for all $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is upper semicontinuous and continuous map $f : Y \to X$ such that

$$F(\omega, x) = (f \circ \Phi)(\omega, x) \quad (\omega, x) \in \Omega \times X.$$

The multifunction $F$ is called compact if for every $\omega \in \Omega$, the multivlued operator $F(\omega, \cdot)$ is compact.

**Theorem 2.6.** *Let $X$ be a separable Banach space and $F : \Omega \times \bar{B}(0, r) \to X$ be a compact random operator and that has decomposition. Suppose*

a) *The random multivalued operator $F$ has a measurable graph.*

b) *There exists $r > 0$ such there is not $x \in X$ such that*

$$x \notin \lambda(\omega) F(\omega, x), \|x(\omega)\| = r, \quad \lambda(\omega) \in (0, 1).$$

*Then $F$ has at least one random fixed point.*

*Proof.* From [3, Theorem 7], there exists $x(\omega) \in B(0, r)$ such that $x(\omega) \in F(\omega, x)$. Then for each $\omega \in \Omega$ we have

$$\{x \in \bar{B}(0, r) : x \in F(\omega, x)\} \neq \emptyset.$$

We define $G : \Omega \to \mathcal{P}(\bar{B}(0, r))$ by

$$G(\omega) = \{(x, x) \in \bar{B}(0, r) \times \bar{B}(0, r) : x \in F(\omega, x)\}.$$

Then

$$\mathcal{G}rG \quad = \quad (\Omega \times \Delta) \cap \mathcal{G}rF$$

where

$$\Delta = \{(x, x) \in \bar{B}(0, r) \times \bar{B}(0, r) : x \in \bar{B}(0, r)\}.$$

Since $\Delta$ is closed and by $a)$, we deduce that $\mathcal{G}rG \in \Sigma \bigotimes \mathcal{B}(\bar{B}(0,r))$. Applying the selection Von-Neumann-Aumann's theorem type [27], we get $x : \Omega \to \bar{B}(0,r)$ is a measurable function such that

$$x(\omega) \in G(\omega), \quad \omega \in \Omega.$$

Then clearly

$$x(\omega) \in F(\omega, x(\omega)), \quad \omega \in \Omega.$$

□

**Theorem 2.7.** *Let $(\Omega, \Sigma, \mu)$ be a complete measurable space, $X$ be a separable Banach space and $F : \Omega \times X \to \mathcal{P}(X)$ be a random multivalued map with decomposion $\mathcal{D}(F)$ and $F$ has a measurable graph. If $F$ is compact and the set for each $\omega \in \Omega$,*

$$\mathcal{A}(\omega) = \{x \in X : x(\omega) \in \lambda F(\omega, x(\omega)) \text{ for some } \lambda(\omega) \in (0,1)\}$$

*is bounded. Then $F$ has a random fixed point.*

*Proof.* Let $r > 0$ such that for all $x \in \mathcal{A}(\omega)$ we have $\|x\| \leq r$. We consider

$$\hat{F}(\omega, x) = \begin{cases} F(\omega, x), & \text{if } \|x\| \leq r, \\ F(\omega, M_r(x)) & \text{if } \|x\| > r, \end{cases}$$

where

$$M_r(x) = \begin{cases} x, & \text{if } \|x\| \leq r, \\ r\frac{x}{\|x\|} & \text{if } \|x\| > r. \end{cases}$$

It is clear that

$$(\hat{F} \circ \hat{M}_r)(\omega, x), \quad (\omega, x) \in \Omega \times X,$$

where $\hat{M}_r : \Omega \times X \to \Omega \times X$ is a measurable function defined by

$$\hat{M}_r(\omega, x) = (\omega, M_r(x)), \quad (\omega, x) \in \Omega \times X.$$

Let $I_X$ be the identity map, hence we deduce

$$\mathcal{G}r\hat{F} = (\hat{M}_r \times I_X)^{-1}(\mathcal{G}rF).$$

Since $F$ has a measurable graph, it remains to check that $\hat{F}$ has a measurable graph; then from Theorem 2.6 there exists a random variable $x : \Omega \to \bar{B}(0,r)$ such that

$$x(\omega) \in \hat{F}(\omega, x(\omega)), \quad \omega \in \Omega,$$

and hence from the definition of $\hat{F}$, we get

$$x(\omega) \in F(\omega, x(\omega)), \quad \omega \in \Omega.$$

□

## 3. CONVEX AND NON-CONVEX CASES

In this section, we denote by $X$ a real separable Banach space with a uniformly convex dual $X^*$.

**Definition 3.1.** By an random integral solution of (1.1) we mean a continuous function $x : \Omega \times [0, b] \to D(A(\omega))$ for every $\omega \in \Omega$ with the property that $x(0, \omega) = x(\omega, b)$ and there exists $f(\omega, \cdot) \in L^1([0, b], X)$ such that $f(\omega, t) \in F(\omega, t, x(\omega, t))$, a.e. on $T$, and for every $\omega \in \Omega$, $x(\omega, \cdot)$ is an integral solution (in the sense of Definition 2.1) of equation (2.1) .

**Theorem 3.2.** *Assume the following conditions hold:*

($\mathcal{H}_1$) *The operator $A(\omega)$ is an $m$-accretive operator in $X$ for every $\omega \in \Omega$, in addition $0 \in A(\omega)0$, such that $-A(\omega)$ generates a semigroup $\{S(\omega, t), t \geq 0\}$ which is compact on $D(A(\omega))$.*

($\mathcal{H}_2$) *$F : \Omega \times [0, b] \times X \to \mathcal{P}_{wkc}(X)$ satisfies:*

(a) *$(\omega, t, x) \to F(\omega, t, x)$ is measurable,*

(b) *for every $(\omega, t) \in \Omega \times [0, b]$, the graph of $x \to F(\omega, t, x)$ is sequentially closed in $X \times X_w$,*

(c) *for each $\rho > 0$ there exists a function $a_\rho(\omega, \cdot) \in L^1_+([0, b])$ for any $\omega \in \Omega$ and $a(\cdot, \cdot)$ is jointly measurable such that*

$$|F(\omega, t, x)| := \sup\{\|w\| : w \in F(\omega, t, x)\} \leq a_\rho(\omega, t),$$

*for every $(\omega, t, x) \in \Omega \times [0, b] \times X$ with $\|x\| \leq \rho$,*

(d) *there exists $r > 0$ such that $\langle v, Jx\rangle > 0$ for all $v \in F(\omega, t, x)$, for any $(\omega, t, x) \in \Omega \times J \times X \times$ $\|x\| = r$.*

*Then the problem (1.1) admits at least one random integral solution.*

*Proof.* For all $\omega \in \Omega$, and $g \in L^1([0, b], X)$. The following problem has a unique integral solution

$$\begin{cases} -x'(\omega, t) \in A(\omega)x(\omega, t) + g(t), & t \in [0, b], \\ x(\omega, 0) = x_0(\omega) \end{cases} \tag{3.1}$$

defined as follows

$$x(\omega, t) = S(\omega, t)x_0(\omega) + \int_0^t S(\omega, t - s)g(s)ds.$$

According to the Crandall-Liggett formula we have

$$S(\omega, t)x = \lim_{n\to\infty} \left(I + \frac{t}{n}A(\omega)\right)^{-n} x.$$

From [21, Proposition 4.1], we know that $\omega \to (I+tA(\omega)/n)^{-n}x$ is measurable which implies that $\omega \to S(\omega,t)x$ is measurable. Using the continuity properties of the semigroup $S(\omega,t)$ and our hypotheses we get that the mapping

$$\omega \to S(\omega,t)x_0(\omega)$$

is measurable. By Fubini's theorem

$$\omega \to \int_0^t S(\omega,t-s)g(s)ds$$

is measurable. It then follows that $\omega \to x(\omega,t)$ is measurable.

For $g \in L^1([0,b],X)$, $x_0(\omega) = x(\omega,0) \in D(A(\omega))$, we denote by $x(g,x_0(\omega))$ the unique random integral solution of (3.1). Let $x := x(g,x_0(\omega))$ and $\tilde{x} := x(\tilde{g},\tilde{x}_0(\omega))$ be two integral solutions of (3.1) corresponding, respectively, to $(g,x_0(\omega))$, $(\tilde{g},\tilde{x}_0(\omega)) \in L^1([0,b],X) \times D(A(\omega))$. Applying [10, Proposition 2], if $A(\omega)-\alpha I$ is $m$-accretive for some $\alpha > 0$, then for all $t \in [0,b]$ we have

$$\|x(\omega,t)-\tilde{x}(\omega,t)\| \le e^{-\alpha t}\|x_0(\omega)-\tilde{x}_0(\omega)\| + \int_0^t e^{-\alpha(t-s)}\|g(s)-\widehat{g}(s)\|ds. \tag{3.2}$$

In particular (when $g = \tilde{g}$, and $t = b$ in (3.2)) the Poincaré map $x_0(\omega) = x(\omega,b)$ is a strict contraction on $D(A(\omega))$. As a result, the periodic problem

$$\begin{cases} -x'(\omega,t) \in A(\omega)x(\omega,t)+g(t), t \in [0,b], \\ x(\omega,0) = x(\omega,b) \end{cases} \tag{3.3}$$

admits a unique random integral solution $x^g \in C([0,b],\overline{D(A(\omega))})$ for any $g \in L^1([0,b],X)$. The map $g_\omega \to x^g$ will be denoted by $\psi$.

It clear that $A+\alpha I$ is $m$-accretive and by $A(\omega)$ satisfies $(\mathcal{H}_1)$. Then for any $\varepsilon > 0$ and $g \in L^1([0,b],X)$ there exists a unique integral solution $x_\varepsilon(\cdot,\omega) = x_\varepsilon^g \in C([0,b],\overline{D(A(\omega))})$ of the problem

$$\begin{cases} -x_\varepsilon'(\omega,t) \in (A(\omega)+\varepsilon I)x_\varepsilon(\omega,t)+g(t), \quad t \in [0,b], \\ x_\varepsilon(\omega,0) = x_\varepsilon(\omega,b). \end{cases} \tag{3.4}$$

Thus, we can define the solution map $\psi_\varepsilon : L^1([0,b],X) \to C([0,b],\overline{D(A(\omega))})$ for every $\varepsilon > 0$ by

$$\psi_\varepsilon(g) = x_\varepsilon^g(\cdot), \tag{3.5}$$

where $x_\varepsilon^g$ is the integral solution of (3.4). We show that $\psi_\varepsilon$ is weakly-strongly sequentially continuous. Indeed, let $g_n(\cdot) \to g(\cdot)$ weakly in $L^1([0,b],X)$, as $n \to \infty$, and set $x_n(\omega,\cdot) := \psi_\varepsilon(g_n(\cdot))$. By Bénilan's inequality in [31, Theorem 17.5] and the fact that $0 \in A(\omega)0$, we get

$$\|x_n(\omega,t)\| \le \|x_n(\omega,s)\| + \int_s^t \|g_n(\tau)\|\,d\tau, \quad \forall 0 \le s \le t < b. \tag{3.6}$$

Let

$$m_n := \min_{t\in[0,b]} \|x_n(\omega,t)\|, \quad M_n := \max_{t\in[0,b]} \|x_n(\omega,t)\|. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$M_n \le m_n + C, \quad \text{where} \quad C = \sup_n \int_0^b \|g_n(\tau)\| d\tau. \tag{3.8}$$

Since $X^*$ is uniformly convex space then the duality map $J$ is single-valued, according to Definition 2.1 we find for each $0 \le s \le t \le b$,

$$\|x_n(\omega,t)\|^2 + 2\varepsilon \int_s^t \|x_n(\omega,\tau)\|^2 \, d\tau \le \|x_n(\omega,s)\|^2 - 2\int_s^t \langle g_n(\tau), Jx_n(\tau,\omega)\rangle \, d\tau.$$

Supposing $s = 0$ and $t = b$ in the last inequality, we conclude that

$$\varepsilon \int_0^b \|x_n(\omega,\tau)\|^2 \, d\tau \le \int_0^b \|x_n(\omega,\tau)\| \, \|g_n(\tau)\| \, d\tau,$$

hence by (3.7), we have

$$\varepsilon b m_n^2 \le M_n C.$$

The sequence $\{x_n(\omega,\cdot)\}_{n\in\mathbb{N}}$ for all $\omega \in \Omega$ is bounded in $C([0,b],X)$, according to $m_n$ and $M_n$ are bounded from (3.8) and the last inequality. In particular, $\{x_n(\omega,0)\}_{n\in\mathbb{N}}$ is bounded in $X$. Also observe that the $g_n(\cdot)$ vary in a uniformly integrable subset of $L^1([0,b],X)$. According to [32, Theorem 2] that $\{x_n(\omega,b)\}_{n\in\mathbb{N}}$ is relatively compact in $X$. Since $x_n(\omega,0) = x_n(\omega,b)$, thus $x_n(\omega,0)$ varies in a relatively compact subset of $X$. Using [32, Theorem 2], we deduce also that $\{x_n(\omega,\cdot)\}_{n\in\mathbb{N}}$ is relatively compact in $C([0,b],X)$. For any $\omega \in \Omega$ thus without loss of generality, we can assume that $x_n(\omega,\cdot) \to x(\omega,\cdot)$ in $C([0,b],X)$, as $n \to \infty$. Obviously, $x(\omega,\cdot) \in C([0,b],\overline{D(A(\omega))})$ and $x(\omega,0) = x(\omega,b)$. Moreover, since $x_n(\omega,\cdot) = \psi_\varepsilon(g_n(\cdot))$, with $\psi_\varepsilon$ defined by (3.5). By inequality (2.2) we obtain that

$$\|x_n(\omega,t) - y\|^2 \le \|x_n(\omega,s) - y\|^2 - 2\int_s^t \langle g_n(\tau) + z, J\left(x_n(\omega,\tau) - y\right)\rangle_+ \, d\tau,$$

for each $0 \le s \le t \le b$ and for all $y, z$ with $z \in (A(\omega) + \varepsilon I)y$. As $x_n(\omega,\cdot) \to x(\omega,\cdot)$ strongly in $C([0,b],X)$, $g_n(\cdot) \to g(\cdot)$ weakly in $L^1([0,b],X)$ for all $\omega \in \Omega$ and $J$ is uniformly continuous from compact subsets of $X$ to $X^*$ we may pass to the limit in last inequality as $n \to \infty$ to deduce that

$$x(\omega,\cdot) = \psi_\varepsilon(g(\cdot)).$$

Let $\hat{F} : \Omega \times [0,b] \times X \to \mathcal{P}_{wkc}(X)$ be given by

$$\hat{F}(\omega,t,x) = \begin{cases} F(\omega,t,x), & \text{if } \|x\| \le r, \\ F\left(\omega,t,M_r(x)\right) & \text{if } \|x\| > r. \end{cases} \tag{3.9}$$

where

$$M_r(x) = \begin{cases} x, & \text{if } \|x\| \le r, \\ r\frac{x}{\|x\|} & \text{if } \|x\| > r. \end{cases}$$

with $r$ as in $(\mathcal{H}_2)$. Clear that, $\hat{F}$ satisfies $(\mathcal{H}_2)$, to be specific

$$\left|\hat{F}(\omega, t, x)\right| \le a_r(t, \omega), \text{ a. e. on } [0, b], \forall x \in X, \omega \in \Omega. \tag{3.10}$$

Consider the operator $N : \Omega \times C([0, b], X) \to 2^{L^1([0,b],X)}$ defined by

$$N(\omega, x) = \{g(\cdot) \in L^1([0, b], X) : g(t) \in \hat{F}(\omega, t, x(\omega, t)) \quad \text{a.e } t \in [0, b]\}, \tag{3.11}$$

where $x : \Omega \to C([0, b], X)$ is a measurable function. It is easy to prove that $N(\omega, \cdot)$ has nonempty, convex, and weakly compact values. In addition, by [17, Proposition 2.23, p.43] and the convergence theorem [2, Theorem p.60], $N$ is an upper semicontinuous multifunction from $C([0, b], X)$ into $L^1([0, b], X)_w$. We conclude that

$$N(\omega, \cdot) \in \mathcal{P}_{clcv}(L^1([0, b], X), \text{ is u.s.c}, \quad \omega \in \Omega. \tag{3.12}$$

Now, we show that for each measurable function $x : \Omega \to C([0, b], X)$, the multifunction $\omega \to N(\omega, x)$ is measurable. Indeed, since $(\omega, t, x) \to \hat{F}(\omega, t, x)$ is measurable, then for any $y \in X$,

$$(\omega, t, x(\omega, \cdot)) \to d(y, \hat{F}(\omega, t, x(\omega, \cdot))$$

is measurable. Since the distance function is continuous in $y$, for each $h \in L^1([0, b], X)$,

$$(\omega, t) \to d(h(t), \hat{F}(\omega, t, x(\omega, t))),$$

is measurable. From Fubini's theorem we get that

$$\omega \to \int_0^b d(h(t), \hat{F}(\omega, t, x(\omega, t)))dt = d\left(h, S^1_{\hat{F}(\omega,\cdot,x(\omega,\cdot))}\right),$$

is measurable, which implies that $\omega \to N(\omega, x)$ is measurable and $\omega \to \psi_\varepsilon \circ N(\omega, \cdot)$ is measurable and the function

$$(\omega, h, x) \to d\left(h, S^1_{\hat{F}(\omega,\cdot,x(\omega,\cdot))}\right) \tag{3.13}$$

is measurable. Consider the approximating problem

$$\begin{cases} -x'_\varepsilon(\omega, t) \in (A(\omega) + \varepsilon I)x_\varepsilon(\omega, t) + \hat{F}\left(\omega, t, x_\varepsilon(\omega, t)\right), t \in [0, b], \\ x_\varepsilon(\omega, 0) = x_\varepsilon(\omega, b). \end{cases} \tag{3.14}$$

From (3.5) and (3.11), we compare with Definition 3.1. The presence of a random integral solution to (3.14) is clearly equivalent to the random fixed point for the map $\psi_\varepsilon \circ N(\omega, \cdot)$ in $C([0, b], X)$. We shall use Theorem 2.7 to prove that $\psi_\varepsilon \circ N(\omega, \cdot)$ has a random fixed point. The proof will be given in several steps.

**Step 1** $\psi_\varepsilon \circ N$ has a measurable graph.

From (3.13), we conclude

$$\begin{aligned}\mathcal{G}r\psi_\varepsilon \circ N &= \{(\omega, x, y) \in \Omega \times C([0,b], X) \times C([0,b], X) : \ d(y, \psi_\varepsilon \circ N) = 0\} \\ &\in \sum \times \mathcal{B}(C([0,b], X) \times C([0,b], X)) = \sum \otimes \mathcal{B}(C([0,b], X)) \otimes \mathcal{B}(C([0,b], X)).\end{aligned}$$

**Step 2** From (3.12), the multivalued. $N(\omega, \cdot)$ is upper semicontinuous from $C([0,b], X)$ to $L^1([0,b], X)_w$, and $\psi_\varepsilon$ is sequentially continuous from $L^1([0,b], X)_w$ into $C([0,b], X)$.

**Step 3** We will prove that $\psi_\varepsilon \circ N(\omega, x(\omega, \cdot))$ is compact. Indeed, let $\{x_n(\omega, \cdot)\}_{n\geq 1}$ be a sequence bounded in $C([0,b], X)$ such that $x_n(\omega, \cdot) \longrightarrow x(\omega, \cdot), n \to +\infty$, As $N(\omega, \cdot)$ is upper semicontinuous and has weakly compact values, then $\{N(\omega, x_n(\omega, \cdot))\}_{n\geq 1}$ is relatively compact in $L^1_w([0,b], X)$, and then $N(\omega, x_{n_k}(\omega, \cdot)) \to N(\omega, x(\omega, \cdot))$ in $L^1([0,b], X)_w$, and as $\psi_\varepsilon$ is sequentially continuous, thus $\psi_\varepsilon(N(\omega, x_{n_k}(\omega, \cdot)))$ converges to $\psi_\varepsilon(N(\omega, x(\omega, \cdot)))$. Hence for each $\omega \in \Omega$,

$$\psi_\varepsilon \circ N(\omega, \cdot)$$

is compact.

**Step 4** It remains to show

$$\mathcal{A}(\omega) := \{x(\omega, \cdot) C([0,b], X) : \quad x(\omega, \cdot) \in \lambda \psi_\varepsilon(N(\omega, x(\omega, \cdot)), \quad \lambda \in (0,1]\}, \tag{3.15}$$

is bounded. We will prove by contradiction that

$$\|x(\omega, \cdot)\| \leq r, \forall x(\omega, \cdot) \in \mathcal{A}(\omega). \tag{3.16}$$

Assume that (3.16) is false. Then either $\|x(\omega, t)\| > r$, $\forall t \in [0,b], \omega \in \Omega$, or there exist $\eta, \theta \in [0,b], \eta < \theta$, such that

$$\|x(\omega, \eta)\| = r$$

and

$$\|x(\omega, t)\| > r, \quad \forall t \in (\eta, \theta].$$

In the first case, from (3.5), (3.11) (3.14), (3.15) and Definition 2.1, we get

$$\|x(\omega, b)\|^2 + 2\varepsilon \int_0^b \|x(\omega, t)\|^2 dt \leq \|x(\omega, 0)\|^2 - 2\lambda^2 \int_0^b \langle \widehat{f}(\omega, t), J(\lambda^{-1} x(\omega, t))\rangle dt, \tag{3.17}$$

where $\widehat{f}(\omega, \cdot) \in L^1([0,b], X), \widehat{f}(\omega, t) \in \hat{F}(\omega, t, x(\omega, t))$, a.e. $t \in [0,b]$. From the homogeneity of $J$, (3.9) and $(H_2)$, we obtain that for each $t \in [0,b]$,

$$\left\langle \widehat{f}(\omega, t), J\left(\lambda^{-1} x(\omega, t)\right)\right\rangle = \lambda^{-1} r^{-1} \|x(\omega, t)\| \left\langle \widehat{f}(\omega, t), J\left(M_r(x(\omega, t))\right)\right\rangle \geq 0. \tag{3.18}$$

We combine (3.17) and (3.18) to obtain

$$\|x(\omega, b)\| < \|x(\omega, 0)\|, \forall \omega \in \Omega, \tag{3.19}$$

which is absurd. In the second case, the inequality (3.17) holds. When, we change 0 and $b$ to $\eta$ and $\theta$ respectively, and (3.18) is satisfied on $[\eta,\theta]$, then (3.19) changes to

$$\|x(\omega,\theta)\| < \|x(\omega,\eta)\|, \forall\omega\in\Omega,$$

which contradicts the choice of $\eta$ and $\theta$. Then (3.16) has been proved. From Theorem 2.7, we deduce that $\psi_\varepsilon \circ N$ has a random fixed point $x_\varepsilon(\omega)$, which is the solution to (3.14).

**Step 5** Now we are in the position to prove that $x_\varepsilon(\omega,\cdot)$ converges to some random integral solution of the problem (1.1). Since $x_\varepsilon(\omega,\cdot)$ must satisfy (3.16), it follows that $\hat{F}(\omega,t,x_\varepsilon(\omega,t)) = F(\omega,t,x_\varepsilon(\omega,t))$ (see (3.9)), so that $x_\varepsilon(\omega,\cdot)$ is an integral solution of

$$\begin{cases} -x'_\varepsilon(\omega,t)\in (A(\omega)+\varepsilon I)x_\varepsilon(\omega,t)+f_\varepsilon(\omega,t), t\in[0,b], \\ x_\varepsilon(\omega,0)=x_\varepsilon(\omega,b), \end{cases} \tag{3.20}$$

where $f(\omega,\cdot)\in L^1([0,b],X)$, $f(\omega,t)\in F\left(\omega,t,x_\varepsilon(\omega,t)\right)$ a.e. $t\in[0,b]$.

The sequence $\{x_\varepsilon(\omega,\cdot)\}_{\varepsilon>0}$ is bounded in $C([0,b],X)$ from (3.16). According to $(\mathcal{H}_2)$ we deduce that $\{f_\varepsilon(\omega,\cdot)\}_{\varepsilon>0}$ is uniformly integrable in $L^1([0,b],X)$. On the basis of $(\mathcal{H}_1)$, we can think in the same way as in the second section of the proof of the weak-strong sequential continuity of $\psi_\varepsilon$ to infer that (on a subsequence, as $\varepsilon\to 0$),

$$x_\varepsilon(\omega,\cdot)\to x(\omega,\cdot) \text{ in } C([0,b],X), \quad f_\varepsilon(\omega,\cdot)\to f(\omega,\cdot) \text{ weakly in } L^1([0,b],X). \tag{3.21}$$

According to $(\mathcal{H}_2)$,and ([32], p.120), it follows that $f(\omega,t)\in F(\omega,t,x(\omega,t))$, a. e. $t\in T$. Then, using the continuity of $J$ and (3.21) we have

$$\|x_\varepsilon(\omega,t)-y\|^2 \le \|x_\varepsilon(\omega,s)-y\|^2 - 2\int_s^t \langle f_\varepsilon(\omega,\tau)+z, J\left(x_\varepsilon(\omega,\tau)-y\right)\rangle_+ d\tau, \tag{3.22}$$

for all $0\le s\le t\le b$ and all $y,z$ with $z\in(A(\omega)+\varepsilon I)y$. So, we can pass to the limit in (3.22) as $\varepsilon\to 0$, and then

$$\|x(\omega,t)-y\|^2 \le \|x(\omega,s)-y\|^2 - 2\int_s^t \langle f(\omega,\tau)+z, J\left(x(\omega,\tau)-y\right)\rangle_+ d\tau \tag{3.23}$$

for all $0\le s\le t\le b$ and all $y,z$ with $z\in A(\omega)y$. So, we may pass to the limit in (3.20) as $\varepsilon\to 0$ and conclude that $x(\omega,\cdot)$ is an integral solution to the problem (1.1) in the sense of Definition 3.1.

□

Now, our result for the non-convex problem is the the following.

**Theorem 3.3.** *Assume that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)((c)-(d))$ *and*

$(\mathcal{H}_3)$ $F:\Omega\times[0,b]\times X\to\mathcal{P}_f(X)$ *satisfies:*

$(\bar{a})$ $(\omega,t,x)\to F(\omega,t,x)$ *is* $\Omega\otimes\mathcal{L}([0,b])\otimes\mathcal{B}(X)$ *measurable,*

($\bar{b}$) *$x \to F(\omega, t, x)$ is lower semicontinuous for all $(\omega, t) \in \Omega \times [0, b]$,*

*hold. Then there is a random integral solution to problem (1.1).*

*Proof.* Let again $\hat{F} : \Omega \times [0, b] \times X \to \mathcal{P}_f(X)$ and $N : \Omega \times C([0, b], X) \to 2^{L^1([0,b],X)}$ be given by (3.9) and (3.11), respectively. $N$ is well-defined, having closed decomposable values, which can be easily confirmed. Furthermore, by simple modification of the proof of [12, Proposition 1] or [17, Theorem 7.28, p.238], we deduce that the multivalued $N(\omega, \cdot)$ is l.s.c. for all $\omega \in \Omega$. Hence, we can utilize the Bressan-Colombo selection theorem [9] for existence of a continuous function $u(\omega, \cdot) : C([0, b], X) \to L^1([0, b], X)$ such that

$$u(\omega, x(\omega, \cdot)) \in N(\omega, x(\omega, \cdot)), \forall x(\omega, \cdot) \in C([0, b], X). \tag{3.24}$$

We consider the approximating problem :

$$\begin{cases} -x_\varepsilon'(\omega, t) \in (A(\omega) + \varepsilon I)x_\varepsilon(\omega, t) + u(\omega, x_\varepsilon(\omega, \cdot))(t), t \in [0, b], \\ x_\varepsilon(\omega, 0) = x_\varepsilon(\omega, b). \end{cases} \tag{3.25}$$

The existence of an integral solution of (3.25) is identical to existence a fixed point for the map $\psi_\varepsilon \circ u(\omega, \cdot)$ in $C([0, b], X)$, where $\psi_\varepsilon$ is given by (3.5). Since $\psi_\varepsilon \circ u(\omega, \cdot)$ is continuous, compact and measurable, we can apply the Leray-Schauder type random fixed point theorem (see [14, Theorem 9.26]) to demonstrate the existence of a random fixed point.

We consider the set :

$$\hat{\mathcal{S}} := \{x(\omega, \cdot) \in C([0, b], X) : x(\omega, \cdot) = \lambda(\psi_\varepsilon \circ u(\omega, x(\omega, \cdot))), \lambda \in (0, 1)\}.$$

Similar to the proof of Theorem 3.2, we conclude that (3.16) holds. Hence by [14, Theorem 9.26], the operator $\psi_\varepsilon \circ u(\omega, \cdot)$ has at least one random integral fixed point which is solution of (3.25). We denote this solution by $x_\varepsilon(\omega, \cdot)$. Moreover

$$\|x_\varepsilon(\omega, \cdot)\| \leq r, \quad \omega \in \Omega.$$

Combining (3.9) and (3.24) we get

$$u(\omega, x_\varepsilon(\omega, \cdot))(t) \in F(\omega, t, x_\varepsilon(\omega, t)), \text{ a. e. on} [0, b].$$

Applying the similar argument of the proof of Theorem 3.2, we can prove that $x_\varepsilon(\omega, \cdot) \to x(\omega, \cdot)$ in $C([0, b], X)$ as $\varepsilon \to 0$. In view of the continuity of $u(\omega, \cdot)$ and $J$, passage to the limit in (3.25), as $\varepsilon \to 0$ yields that $x(\omega, \cdot)$ is a solution of (1.1). Then the proof is finished. □

## 4. EXTREMAL SOLUTIONS

In this section, we study the existence of so-called extremal solutions to problem (1.1). Now we consider the evolution inclusion

$$\begin{cases} -x'(\omega,t) \in A(\omega)x(\omega,t) + \operatorname{ext} F(\omega,t,x(\omega,t)), & t \in [0,b], \\ x(\omega,0) = x(\omega,b) \quad \omega \in \Omega, \end{cases} \tag{4.1}$$

where $\operatorname{ext} F(\omega,t,x(t,\omega))$ denotes the set of extreme points of $F(\omega,t,x(\omega,t))$.

We suppose that $F$ has nonempty, weakly compact values, thus $\operatorname{ext} F(\omega,t,x) \neq \emptyset$ for all $(\omega,t,x) \in \Omega \times [0,b] \times X$. In general, Theorems 3.2 and 3.3 do not apply to (4.1) because the multivalued map $(\omega,t,x) \to \operatorname{ext} F(\omega,t,x)$ is neither convex nor closed valued. The following conditions are imposed on $A(\omega)$ and $F$.

**Theorem 4.1.** *Assume the following conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)((c)-(d))$ and*

$(\mathcal{H}_4)$ *There exists $\alpha > 0$ such that $A(\omega) - \alpha I$ is accretive,*
$(\mathcal{H}_5)$ *$F : \Omega \times [0,b] \times X \to \mathcal{P}_f(X)$ satisfies:*
  $(\bar{c})$ *$(\omega,t) \to F(\omega,t,x)$ is measurable for all $x \in X$,*
  $(\bar{d})$ *$x \to F(\omega,t,x)$ is continuous for the Hausdorff pseudometric for a.a. $(\omega,t) \in \Omega \times [0,b]$,*

*hold. Then the problem (4.1) has at least on integral random solution.*

*Proof.* Let

$$V := \left\{ g(\omega,\cdot) \in L^1([0,b],X) : \|g(\omega,t)\| \leq a_r(t,\omega) \text{a. e. on } [0,b] \right\}. \tag{4.2}$$

Because $X$ is reflexive, from The Dunford-Pettis theorem, then $V$ is weakly compact in $L^1([0,b],X)$. In view of the strong accretivity of $A(\omega)$ (cf. $(\mathcal{H}_4)$ ), for each $g(\omega,\cdot) \in V$ there exists a unique random integral solution $x^{g_\omega}(\omega,t) = \psi(g(\omega,t))$ of (3.2).

$\psi$ is weakly-strongly continuous as a map from $V$ to $C([0,b],X)$, as proved in the proof of Theorem 3.2 (see the properties of $\psi_\varepsilon$), then $\psi(V) \subseteq C([0,b],X)$ is compact.

Let $K := \overline{conv}\psi(V)$ and note that $K$ is a convex, compact subset of $C(T,X)$.

For only a measurable function $x : \Omega \to C([0,b],X)$, consider the multifunction $N : \Omega \times K \to \mathcal{P}(L^1([0,b],X))$ defined by

$$N(\omega,x) = \operatorname{ext} S^1_{\hat{F}(\omega,\cdot,x(\cdot))} \tag{4.3}$$

where $\hat{F}$ be defined by (3.9) and recall that (3.10) is satisfied.
Moreover, by [17, Theorem 4.6, p.192], we know that

$$\operatorname{ext} S^1_{\hat{F}(\omega,\cdot,x(\omega,\cdot))} = S^1_{\operatorname{ext} \hat{F}(\omega,\cdot,x(\omega,\cdot))}, \forall x(\omega,\cdot) \in K,$$

and then

$$N(\omega,x) = S^1_{\operatorname{ext} \hat{F}(\omega,\cdot,x(\cdot))}, \forall x \in K.$$

For each $x(\omega,\cdot)\in K$, the multifunction $\omega\to N(\omega,x)$ is measurable. Indeed, since $(\omega,t)\to\hat{F}(\omega,t,x)$ is measurable, $x\to\hat{F}(\omega,t,x)$ is $h$-continuous and $\operatorname{ext}\hat{F}(\omega,t,x)\subseteq\hat{F}(\omega,t,x)$. Then, from [25, Theorem 3.3], the multifunction, $\operatorname{ext}\hat{F}(\cdot,\cdot,\cdot)$ is jointly measurable. Hence for every $y\in X$,

$$(\omega,t,x)\to d(y,\operatorname{ext}\hat{F}(\omega,t,x(\omega,\cdot))$$

is measurable. By the continuity of the distance function, for each $h\in L^1([0,b],X)$,

$$(\omega,t)\to d(h(t),\operatorname{ext}\hat{F}(\omega,t,x(\omega,t)))$$

is measurable. From Fubini's theorem we get that

$$\omega\to\int_0^b d(h(t),\operatorname{ext}\hat{F}(\omega,t,x(\omega,t)))dt=d\left(h,S^1_{\operatorname{ext}\hat{F}(\omega,\cdot,x(\omega,\cdot))}\right)$$

is measurable, which implies that $\omega\to N(\omega,x)$ is measurable.

Next, we consider the multifunction $R:\Omega\to\mathcal{P}(C(K,L^1_w([0,b],X)))$ given by

$$R(\omega)=\left\{u\in C\left(K,L^1_w([0,b],X)\right):u(x)\in N(\omega,x)\text{ for all }x(\omega,\cdot)\in K\right\}.$$

From Fryszkowski's selection theorem [13], we know that, for all $\omega\in\Omega, R(\omega)\neq\emptyset$. We have

$$R(\omega)=\left\{u\in C\left(K,L^1_w([0,b],X)\right):d(u(x),N(\omega,x))=0\text{ for all }x\in K\right\}.$$

Since $x\to\operatorname{ext}\hat{F}(\omega,t,x)$ is $h$-continuous, the multifunction $x\to R(\omega,x)$ is $h$-continuous. It follows that

$$x\to d(u(x),N(\omega,x))$$

is continuous. Now, for fixed $x\in K$, $\omega\to d(u(x),N(\omega,x))$ is a measurable function. Therefore

$$(\omega,x)\to d(u(x),N(\omega,x)),$$

is a Carathéodory map. So, the multifunction $R$ is measurable. According to the selection theorem of Kuratowski and Ryll-Nardzewski [22], there exists $u:\Omega\to C(K,L^1_w([0,b],X))$ which is measurable such that $u(\omega)\in R(\omega)$ for all $\omega\in\Omega$, and

$$u(\omega)(x(\cdot))\in S^1_{\operatorname{ext}\hat{F}(\omega,\cdot,x(\cdot))}. \tag{4.4}$$

Now, we consider the operator $\psi\circ u(\omega)$. Combining with (3.10), (4.2) and (4.4), we obtain that $\psi\circ u(\omega)(K)\subset K$. In addition, $\psi\circ u(\omega)$ is continuous. Indeed, let $x_n(\omega,\cdot)\to x(\omega,\cdot)$ in $C([0,b],X)$ as $n\to\infty$ with $x_n(\omega,\cdot),x(\omega,\cdot)\in K$. The continuity of $u(\omega)(\cdot)$, implies that $u(\omega)(x_n(\omega,\cdot))\to u(\omega)(x(\omega,\cdot))$ in $L^1_w([0,b],X)$, as $n\to\infty$. Since

$$\operatorname{ext}\hat{F}(\omega,t,x_n(\omega,t))\subseteq\hat{F}(\omega,t,x_n(\omega,t))\text{ a. e. on}[0,b],\forall n\in\mathbb{N},$$

it follows by (3.10) and (4.4) that

$$\|u(\omega)\left(x_n(\omega,.)\right)(t)\|\leq a_r(\omega,t)\quad\text{a. e. on }\ [0,b],\forall n\in\mathbb{N} \tag{4.5}$$

where (see $(\mathcal{H}_2)((c)-(d))$ and $(\overline{\mathcal{H}}_5)$), $a_r(\omega,\cdot)\in L^p_+([0,b])$ with $1<p<\infty$. As a result, we may use [18, Lemma 2.8, p.24] to deduce that $u(\omega)(x_n(\omega,\cdot))\to u(\omega)(x(\omega,\cdot))$, weakly in $L^1([0,b],X)$, as $n\to\infty$. Then by the weak-strong continuity of $\psi$ we have

$$(\psi\circ u(\omega))(x_n(\omega,t))\to(\psi\circ u(\omega))(x(\omega,t)),\ \text{as } n\to\infty.$$

Therefore $\psi\circ u(\omega)$ is continuous. Consequently by Schauder's theorem of type random fixed point, we conclude that there exists $x:\Omega\to K$ such that $x(\omega,\cdot)=(\psi\circ u(\omega))(x(\omega,\cdot))$. As a result, $x(\omega,\cdot)$ is an integral solution of

$$\begin{cases} -x'_\varepsilon(\omega,t)\in(A(\omega)+\varepsilon I)x_\varepsilon(\omega,t)+u(\omega)(x_\varepsilon(\omega,\cdot))(t), t\in[0,b],\\ x_\varepsilon(\omega,0)=x_\varepsilon(\omega,b),\end{cases}\tag{4.6}$$

where $u(\omega)(\cdot)$ satisfies (4.4). Using $(\mathcal{H}_2)((c)-(d))$ and $(\overline{\mathcal{H}}_5)$ one shows, exactly as in the proof of Theorem 3.2, that $\|x(\omega,\cdot)\|\le r$, so that $\hat F(\omega,t,x(\omega,t))=F(\omega,t,x(\omega,t))$ a.e. on $[0,b]$. According to (4.4) and (4.6), $x(\omega,\cdot)$ is an integral solution of (4.2). □

## 5. **EXAMPLE**

In this section we introduce an example to which Theorem 3.2 can be applied. Let $(\Omega,\Sigma,\mu)$ be a complete probability space and $W$ an open domain in $\mathbb{R}^n(n\ge1)$, with smooth boundary $\partial W=\Gamma$, be a measurable space, and let $\rho:\mathbb{R}\to\mathbb{R}$ be a nondecreasing and continues function with $\rho(0)=0$. We consider the boundary value problem

$$\begin{cases} -\frac{\partial x}{\partial t}(\omega,t,z)=\Delta\rho(u(\omega,t,z))+F(\omega,t,x), \quad \text{on } \Omega\times[0,b]\times W,\\ x(\omega,0,z)=x(\omega,b,z), \quad \text{a.e. on } \Omega\times W.\end{cases}$$

Set $X=L^2(W)$ which is a separable Banach space, and let us define $A:D(A)\subset L^2(W)\to L^2(W)$ by

$$Au=-\Delta\rho(u),$$

for all $u\in D(A)$, where

$$D(A)=\left\{u\in L^2(W);\rho(u)\in H^1_0(W),\Delta\rho(u)\in L^2(W)\right\}.$$

According to [31, p.27], $A$ is $m$-accretive on $X$, $0\in A0$ and $-A$ generates a compact semigroup on $D(A)=X$, then $(\mathcal{H}_1)$ is satisfied.

Let $F:\Omega\times[0,b]\times X\to 2^X$ be a multivalued map defined by

$$F(\omega,t,x(\omega,t,z))=\{v\in X: v(z)\in\widehat{f}(\omega,t,x(z)),\ \text{a.e. } (\omega,x)\in\Omega\times W\},$$

where $\widehat{f}:\Omega\times[0,b]\times\mathbb{R}\to2^{\mathbb{R}}$ is defined by

$$\widehat{f}(\omega,t,x)=[f_1(\omega,t,x),f_2(\omega,t,x)],\tag{5.1}$$

and $f_1,f_2:\Omega\times[0,b]\times\mathbb{R}\to\mathbb{R}$ can be defined as:

$$f_1(\omega,t,x)=\liminf_{x'\to x}f(\omega,t,x'),\ f_2(\omega,t,x)=\limsup_{x'\to x}f(\omega,t,x'),$$

where $f : \Omega \times [0, b] \times \mathbb{R} \to \mathbb{R}$ satisfies

($\overline{\mathcal{H}}_1$) $(\omega, t, x) \to f(\omega, t, x)$ is measurable,

($\overline{\mathcal{H}}_2$) there exist $\alpha, \beta : \Omega \to L^1_+([0, b])$ such that

$$|f(\omega, t, x)| \leq \alpha(\omega, t)|x| + \beta(\omega, t), \text{ a.e. } t \in [0, b], \forall(\omega, x) \in \Omega \times \mathbb{R} \tag{5.2}$$

($\overline{\mathcal{H}}_3$) $xf(\omega, t, x) \geq 0$ a.e. $t \in [0, b], \forall x \in \mathbb{R}$ and $\omega \in \Omega$.

By [18, p. 97], the function $f_1$ is $l.s.c.$, and $f_2$ is $u.s.c.$ with respect to $x$. It is easy to prove (see [18, p.96]) that $F$ satisfies ($\mathcal{H}_2$).

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