Neural, Parallel, and Scientific Computations 30 (2022), No.1, 23-41

# ON RANDOM POLYNOMIALS-III: A SURVEY

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**ABSTRACT.** In this paper, we continue to summarise a final set of fundamental results of various random polynomials We collect selected results in random polynomials on the real/complex zeros and distribution of zeros of random polynomials in higher dimensions, in algebraic structures, random fields, manifolds, and Lie groups.

**Mathematics Subject Classification(2020):** 60H25, 47B80, 47H40

**Key Words and Phrases:** Random Polynomials in algebraic structures, Random Polynomials in higher dimensions, Random Polynomials in Random Fields, manifolds, and Lie groups.

## 1 INTRODUCTION

It is well-known fact that the study of the roots of random polynomials has become an active area of research among the most popular topics in Mathematics. This field has motivation and application in many branches of mathematical sciences. This field has a notably a rich history. We need to invoke and remember the results on random polynomials in Bharucha-Reid and Sambandham [14] and Farahmand [30]. This contribution is our earnest effort as a continuation of our earlier papers Thangaraj and Sambandhan [68], [69].

## 2 RANDOM SYSTEMS OF POLYNOMIAL EQUATIONS

Let $f = (f_1, f_2, \dots, f_m)$ where

$$f_i(t) = \sum_{\|j\| \le d_i} a_j^{(i)} t^j \quad (i = 1, \dots, m) \tag{2.1}$$

be a system of $m$ polynomials in $m$ real variables. The notation in (2.1) is the following: $t := (t_1, \dots, t_m)$ denotes a point in $\mathbb{R}^m$, $j = (j_1, \dots, j_m)$ a multi-index of non-negative

Received March 10, 2022 1061-5369 $15.00 ©Dynamic Publishers, Inc.
www.dynamicpublishers.org; www.dynamicpublishers.com;
https://doi.org/10.46719/npsc202230.01.03.

integers, $\|j\| = \sum_{h=1}^{m} j_h, t^j = t^{j_1} \cdots t^{j_m}, a_j^{(i)} = a_{j_1,\ldots,j_m}^{(i)}$. Further $d_i$ is the degree of the polynomial $f_i$.

The next question arises to find the number of solutions of the system of equations

$$f_i(x) = 0 \quad (i = 1, \ldots, m) \tag{2.2}$$

lying in some subset $V$ of $\mathbb{K}^m$. Let $N^f(V)$ the number of roots of the system of equations (2.2) lying in the subset $V$ of $\mathbb{R}^m$. Throughout this section we discuss the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Also $N^f = N^f(\mathbb{R}^m)$. If we treat the coefficients $\left\{a_j^{(i)}\right\}$ as random, $N^f(V)$ becomes a random variable.

Fundamental results in the case of one polynomial in one variable were started appearing with the work of Marc Kac [36] (see the book by Bharucha-Reid and Sambandham [14]). Now we want to expand the horizon by considering the systems with $m > 1$, and in particular, for large values of $m$. We see a striking difference other than the case $m = 1$ and in general, little is known on the distribution of the random variable $N^f(V)$ (or $N^f$) even for simple choices of the probability law on the coefficients.

## 2.1 AVERAGE NUMBER OF ROOTS OF RANDOM POLYNOMIAL SYSTEMS

At first we note down the real case. Here we record results for (i) Zero Mean Case, (ii) Non-zero Mean Case, and (iii) Other Polynomial Basis Case.

### *2.1.1 Real Roots: Zero Mean Case*

**Theorem 2.1** ([65]). *Let the number $N^f(V)$ of solutions of* (2.2) *lie in the Borel subset $V$ of $\mathbb{R}^m$. Assume that the coefficients are centered Gaussian independent random variables having variances*

$$\mathbb{E}\left[\left(a_j^{(i)}\right)^2\right] = \frac{d_i!}{j_1! \cdots j_m! \, (d_i - \|j\|)!}. \tag{2.3}$$

*Then the expectation of $N^f$ is*

$$\mathbb{E}\left(N^f\right) = (d_1 \cdots d_m)^{1/2}. \tag{2.4}$$

*ie. that is, the square root of the Bézout number*[1] *associated to the system* (2.2).

Some extensions of this theorem, including new results for one polynomial in one variable, can be found in [29]. There are also other extensions to multi-homogeneous systems in [52], and, partially, to sparse systems in [48] and [51]. A similar question for the number of critical points of real-valued polynomial random functions was considered by Dedieu and Malajovich in [24].

[1]Bézout's theorem is a statement in algebraic geometry concerning the number of common zeros of $n$ polynomials in $n$ indeterminates. In its original form the theorem states that in general the number of common zeros(Bézout's number) equals the product of the degrees of the polynomials

A general formula for $\mathbb{E}\left(N^f(V)\right)$ when the random functions $f_i (i = 1, \ldots, m)$ are stochastically independent and their law is centered and invariant under the orthogonal group on $\mathbb{R}^m$ can be found in [11]. The Shub-Smale formula (2.4) can be seen as a special case. Not many results are available on higher moments. The only known results are asymptotic variances as $m \to +\infty$ (see [11] for non-polynomial systems and [70] for the Kostlan-Shub-Smale model).

Let us consider the case where the coefficients have zero expectation (in some cases, this has been considered for one polynomial in one variable in [29]).

### *2.1.2 Real Roots: Non-Zero Mean Case*

Let us start with a non-random system

$$P_i(t) = 0 \quad (i = 1, \ldots, m) \tag{2.5}$$

with a polynomial noise $\{X_i(t) : i = 1, \ldots, m\}$. This leads to consider the system

$$P_i(t) + X_i(t) = 0 \quad (i = 1, \ldots, m)$$

and we discuss the number of roots of the new system.

In order to obtain results on $E\left(N^{P+X}\right)$, some assumptions both on the "noise" $X$ and the class of polynomial "signals" $P$, especially the relation between the size of $P$ and the probability distribution of $X$ are required.

Assume that the polynomial noise $X$ is Gaussian and centered, the real-valued random processes

$$X_1(\cdot), \ldots, X_m(\cdot)$$

defined on $\mathbb{R}^m$ are independent, with covariance functions

$$R^{X_i}(s,t) := \mathbb{E}\left(X_i(s)X_i(t)\right) \quad (i = 1, \ldots, m)$$

depending only on the scalar product $\langle s,t\rangle$, that is: $R^{X_i}(s,t) = Q^{(i)}(\langle s,t\rangle)$, where

$$Q^{(i)}(u) = \sum_{k=0}^{d_i} c_k^{(i)} u^k, \quad u \in \mathbb{R} (i = 1, \ldots, m)$$

In this case, it is known that a necessary and sufficient condition for $Q^{(i)}(\langle s,t\rangle)$ to be a covariance is that $c_k^{(i)} \geq 0$ for all $k = 0, \ldots, d_i$ and the process $X_i$ can be written as

$$X_i(t) = \sum_{\|j\| \leq d_i} a_j^{(i)} t^j$$

where the random variables $a_j^{(i)}$ are centered Gaussian, independent and

$$\operatorname{Var}\left(a_j^{(i)}\right) = c_{\|j\|}^{(i)} \frac{\|j\|!}{j!} \quad (i = 1, \ldots, m; \|j\| \leq d_i)$$

(for a proof, see [11]).

**Particular case:** The Shub-Smale model (2.3) corresponds to the particular choice

$$c_k^{(i)} = \begin{pmatrix} d_i \\ k \end{pmatrix} \quad (k = 0, 1, \dots, d_i)$$

which implies

$$Q^{(i)}(u) = (1+u)^{d_i} \quad (i = 1, \dots, m)$$

We need the following notations to discuss further.

$Q_u^{(i)}, Q_{uu}^{(i)}$ denote the successive derivatives of $Q^{(i)}$. We assume that $Q^{(i)}(u), Q_u^{(i)}(u)$ do not vanish for $u \geq 0$. Put, for $x \geq 0$ :

$$q_i(x) = \frac{Q_u^{(i)}}{Q^{(i)}}, \tag{2.6}$$

$$r_i(x) = \frac{Q^{(i)} Q_{uu}^{(i)} - \left(Q_u^{(i)}\right)^2}{\left(Q^{(i)}\right)^2}, \text{ and} \tag{2.7}$$

$$h_i(x) = 1 + x \frac{r_i(x)}{q_i(x)} \tag{2.8}$$

In (2.6) and (2.7), the functions in the right-hand side are computed at the point $x$. In [11] the following statement has been proved:

**Theorem 2.2** ([11])**.** *Let $V \subset \mathbb{R}^m$ be any Borel subset. Then*

$$\mathbb{E}\left(N^X(V)\right) = \frac{1}{\sqrt{2}\pi^{(m+1)/2}} \Gamma\left(\frac{m}{2}\right) \int_V \left[\prod_{i=1}^m q_i\left(\|t\|^2\right)\right]^{1/2} \cdot \mathbb{E}_h\left(\|t\|^2\right) \mathrm{d}t$$

*where*

$$\mathbb{E}_h(x) = \mathbb{E}\left(\left[\sum_{i=1}^m h_i(x) \xi_i^2\right]^{1/2}\right)$$

*and $\xi_1, \dots, \xi_m$ denote independent standard normal random variables.*

**Remark:** We notice that Theorem 2.2 is a special case of a general theorem (see [11]), in which the covariance function of the random field is invariant under the action of the orthogonal group, and not only a function of the scalar product.

Let us assume that each polynomial $Q^{(i)}$ does not vanish for $u \geq 0$, which amounts to saying that for each $t$ the (one-dimensional) distribution of $X_i(t)$ does not degenerate. Also, $Q^{(i)}$ has effective degree $d_i$, that is,

$$c_{d_i}^{(i)} > 0 \quad (i = 1, \dots, m)$$

After simplification, each polynomial $Q^{(i)}$, as $u \to +\infty$, gives

$$q_i(u) \sim \frac{d_i}{1+u} \text{ and} \tag{2.9}$$

$$h_i(u) \sim \frac{c_{d_i-1}^{(i)}}{d_i c_{d_i}^{(i)}} \cdot \frac{1}{1+u}. \tag{2.10}$$

Further we assume the following:

(H$_1$) $h_i$ is independent of $i(i = 1, \dots, m)$ (but may vary with $m$). We put $h = h_i$. Of course, if the polynomials $Q^{(i)}$ do not depend on $i$, this assumption is satisfied. But there are more general cases, such as covariances having the form $Q(u)^{l_i}$ $(i = 1, \dots, m)$.

(H$_2$) There exist positive constants $D_i, E_i(i = 1, \dots, m)$ and $\underline{q}$ such that

$$0 \le D_i - (1+u)q_i(u) \le \frac{E_i}{1+u} \quad \text{and} \quad (1+u)q_i(u) \ge \underline{q} \tag{2.11}$$

for all $u \ge 0$, and moreover

$$\max_{1\le i\le m} D_i, \quad \max_{1\le i\le m} E_i \tag{2.12}$$

are bounded by constants $\bar{D}, \bar{E}$, respectively, which are independent of $m$; $\underline{q}$ is also independent of $m$. Also, there exist positive constants $\underline{h}, \bar{h}$ such that

$$\underline{h} \le (1+u)h(u) \le \bar{h}$$

for $u \ge 0$.

Notice that the auxiliary functions $q_i, r_i, h(i = 1, \dots, m)$ will also vary with $m$. To simplify somewhat the notation, just drop the parameter $m$ in $P, Q, q_i, r_i, h$. However, in (H$_2$) the constants $\underline{h}, \bar{h}$ do not depend on $m$.

With respect to (H$_2$), it is clear that for each $i, q_i$ will satisfy (2.11) with the possible exception of the first inequality, and $(1+u)h(u) \le \bar{h}$ for some positive constant $\bar{h}$, from the definitions (2.6), (2.8), (2.9), (2.10) and the conditions on the coefficients of $Q^{(i)}$. However, it is not self-evident from the definition (2.8) that $h(u) \ge 0$ for $u \ge 0$.

It is assumed the system with has the relation between the "signal" $P$ and the "noise" $X$. Let $P$ be a polynomial in $m$ real variables with real coefficients having degree $d$ and $Q$ a polynomial in one variable with non-negative coefficients, also having degree $d$, $Q(u) = \sum_{k=0}^{d} c_k u^k$. Further it is assumed that $Q$ does not vanish on $u \ge 0$ and $c_d > 0$. Define

$$\begin{aligned} H(P,Q) &= \sup_{t\in\mathbb{R}^m} \left\{ (1+\|t\|) \cdot \left\| \nabla \left( \frac{P}{\sqrt{Q(\|t\|^2)}} \right)(t) \right\| \right\} \\ K(P,Q) &= \sup_{t\in\mathbb{R}^m\setminus\{0\}} \left\{ \left(1+\|t\|^2\right) \cdot \left| \frac{\partial}{\partial\rho} \left( \frac{P}{\sqrt{Q(\|t\|^2)}} \right)(t) \right| \right\} \end{aligned}$$

where $\frac{\partial}{\partial\rho}$ denotes the derivative in the direction defined by $\frac{t}{\|t\|}$, at each point $t \neq 0$. For $r > 0$, put:

$$L(P,Q,r) := \inf_{\|t\|\ge r} \frac{P(t)^2}{Q(\|t\|^2)}.$$

One can check by means of elementary computations that for each pair $P, Q$ as above, one has

$$H(P,Q) < \infty, \quad K(P,Q) < \infty.$$

With these notations, the following assumptions on the systems $P, Q$, as $m$ grows, have been defined.

($H_3$)

$$A_m = \frac{1}{m} \cdot \sum_{i=1}^{m} \frac{H^2\left(P_i, Q^{(i)}\right)}{i} = o(1) \quad \text{as } m \to +\infty, \tag{2.13}$$

$$B_m = \frac{1}{m} \cdot \sum_{i=1}^{m} \frac{K^2\left(P_i, Q^{(i)}\right)}{i} = o(1) \quad \text{as } m \to +\infty. \tag{2.14}$$

($H_4$) There exist positive constants $r_0, \ell$ such that if $r \geq r_0$ :

$$L\left(P_i, Q^{(i)}, r\right) \geq \ell \quad \text{for all } i = 1, \ldots, m$$

The following result has been proved in [12]

**Theorem 2.3** ([12])**.** *Under the assumptions* $(H_1), \ldots, (H_4)$*, one has*

$$\mathbb{E}\left(N^{P+X}\right) \leq C\theta^m \mathbb{E}\left(N^X\right)$$

*where* $C, \theta$ *are positive constants,* $0 < \theta < 1$.

*2.1.3 Real Roots: Other Polynomial Basis*

We have so far considered that all probability measures have been introduced in a particular basis, namely, the monomial basis $\left\{x^j\right\}_{\|j\| \leq d}$. But many situations demand that the polynomial systems are expressed in different basis, for example, orthogonal polynomials, harmonic polynomials, Bernstein polynomials, etc. This triggers to discuss a question of studying $N^f(V)$ when the randomization is performed in a different basis. For the case of random orthogonal polynomials one may consult Bharucha-Reid and Sambandham [14], and Edelman and Kostlan [29] for random harmonic polynomials.

Here, a nice answer to the average number of real roots of a random system of equations expressed in the Bernstein basis case by Armentano and Dedieu in [7]. The Bernstein basis is given by

$$b_{d,k}(x) = \binom{d}{k} x^k (1-x)^{d-k}, 0 \leq k \leq d,$$

in the case of univariate polynomials, and

$$b_{d,j}(x_1, \ldots, x_m) = \binom{d}{j} x_1^{j_1} \ldots x_m^{j_m} (1 - x_1 - \ldots - x_m)^{d-\|j\|}, \|j\| \leq d,$$

for polynomials in $m$ variables, where $j = (j_1, \ldots, j_m)$ is a multi-integer, and $\binom{d}{j}$ is the multinomial coefficient.

Let us consider the set of real polynomial systems in $m$ variables,

$$f_i(x_1,\ldots,x_m) = \sum_{\|j\|\le d_i} a_j^{(i)} b_{d,j}(x_1,\ldots,x_m) \quad (i = 1,\ldots,m)$$

Take the coefficients $a_j^{(i)}$ to be independent standard Gaussian random variables. Define

$$\tau : \mathbb{R}^m \to \mathbb{P}\left(\mathbb{R}^{m+1}\right)$$

by

$$\tau(x_1,\ldots,x_m) = [x_1,\ldots,x_m, 1 - x_1 - \ldots - x_m]$$

Notice that $\mathbb{P}\left(\mathbb{R}^{m+1}\right)$ is the projective space associated with $\mathbb{R}^{m+1}$, with $[y]$ as the class of the vector $y \in \mathbb{R}^{m+1}$, $y \neq 0$, for the equivalence relation defining this projective space. The (unique) orthogonally invariant probability measure in $\mathbb{P}\left(\mathbb{R}^{m+1}\right)$ is denoted by $\lambda_m$. These lead to state the following result.

**Theorem 2.4** ([7])**.**

(1) *For any Borel set $V$ in $\mathbb{R}^m$ we have*

$$\mathbb{E}\left(N^f(V)\right) = \lambda_m(\tau(V))\sqrt{d_1\ldots d_m}$$

(2) *In particular,*
$\mathbb{E}\left(N^f\right) = \sqrt{d_1\ldots d_m}$ *and also*

(3) $\mathbb{E}\left(N^f(\Delta^m)\right) = \sqrt{d_1\ldots d_m}/2^m$, *where*

$$\Delta^m = \{x \in \mathbb{R}^m : x_i \ge 0 \text{ and } x_1 + \ldots + x_m \le 1\},$$

(4) *When $m = 1$, for any interval $I = [\alpha,\beta] \subset \mathbb{R}$, one has*

$$\mathbb{E}\left(N^f(I)\right) = \frac{\sqrt{d}}{\pi}(\arctan(2\beta - 1) - \arctan(2\alpha - 1))$$

The fourth assertion in Theorem 2.4 is deduced from the first assertion but it also can be derived from Crofton's formula (see Edelman and Kostlan [29]).

For the proof of Theorem 2.4 see Armentano and Dedieu [7].

## 2.2 COMPLEX ROOTS CASE

In this section we see that points in the sphere associated with roots of Shub-Smale complex analogue random polynomials via the stereographic projection, are surprisingly well-suited with respect to the minimal logarithmic energy on the sphere. This observations provide a good approximation to a classical minimization problem over the sphere, namely, the Elliptic Fekete points problem.

We record here the results of Armentano et al. [6], where one can find proofs and more detailed references.

**ELLIPTIC FEKETE POINTS PROBLEM:** Given $x_1, \ldots, x_N \in \mathbb{S}^2 = \left\{x \in \mathbb{R}^3 : \|x\| = 1\right\}$, let

$$V(x_1, \ldots, x_N) = \ln \prod_{1 \le i < j \le N} \frac{1}{\left\|x_i - x_j\right\|} = - \sum_{1 \le i < j \le N} \ln \left\|x_i - x_j\right\| \tag{2.15}$$

be the logarithmic energy of the $N$ -tuple $x_1, \ldots, x_N$. Let

$$V_N = \min_{x_1, \ldots, x_N \in \mathbb{S}^2} V(x_1, \ldots, x_N)$$

denote the minimum of this function. $N$ =tuples minimizing the quantity (2.14) are usually called Elliptic Fekete Points.

One may consult Whyte [71] for origin and Saff anf Kuijlaars [60] for an informative survey on this problem. In the list of Smale's problems for the XXI Century [67], Problem Number 7 is stated as follows.

**SMALE'S SEVENTH PROBLEM:** *Can one find $x_1, \ldots, x_N \in \mathbb{S}^2$ such that*

$$V(x_1, \ldots, x_N) - V_N \le c \ln N \tag{2.16}$$

*$c$ a universal constant?*

It is to be noted here that the value of $V_N$ is not even known up to logarithmic precision. Rakhmanov et al. have established the following result in [57].

**Theorem 2.5** ([57])**.** *If $C_N$ is defined by*

$$V_N = -\frac{N^2}{4} \ln\left(\frac{4}{e}\right) - \frac{N \ln N}{4} + C_N N \tag{2.17}$$

*then,*

$$-0.112768770\ldots \le \liminf_{N \to \infty} C_N \le \limsup_{N \to \infty} C_N \le -0.0234973\ldots.$$

Now the random polynomial version gives some insight to the problem. Let $X_1, \ldots, X_N$ be independent random variables with common uniform distribution over the sphere. One obtains the expected value of the function $V(X_1, \ldots, X_N)$ in this case as,

$$\mathbb{E}\left(V(X_1, \ldots, X_N)\right) = -\frac{N^2}{4} \ln\left(\frac{4}{e}\right) + \frac{N}{4} \ln\left(\frac{4}{e}\right). \tag{2.18}$$

We notice here that this random choice of points in the sphere with independent uniform distribution already provides a reasonable approach to the minimal value $V_N$, accurate to the order of $O(N \ln N)$.

In this context a beautiful argument provided by Armentano [4] is reproduced here for the sake of understanding the real spirit of the problem. "On one side, this probability distribution has an important property, namely, invariance under the action of the orthogonal group on the sphere. However, on the other hand this probability distribution lacks on correlation between points. More precisely, in order to obtain well-suited configurations one needs some kind of repelling property between points, and in this direction independence is not favorable. Hence, it is a natural question whether other handy orthogonally invariant probability distributions may yield better expected values. Here is where complex random polynomials comes into account."

Now we are ready to state random polynomials version in [4].

Given $z \in \mathbb{C}$, let

$$\hat{z} = \frac{(z, 1)}{1 + |z|^2} \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$$

be the associated points in the Riemann Sphere, i.e. the sphere of radius $1/2$ centered at $(0, 0, 1/2)$. Finally, let

$$X = 2\hat{z} - (0, 0, 1) \in \mathbb{S}^2$$

be the associated points in the unit sphere. Given a polynomial $f$ in one complex variable of degree $N$, we consider the mapping

$$f \mapsto V(X_1, \ldots, X_N)$$

where $X_i (i = 1, \ldots, N)$ are the associated roots of $f$ in the unit sphere. Notice that this map is well defined in the sense that it does not depend on the way we choose to order the roots.

**Theorem 2.6** ([4]). *Let $f(z) = \sum_{k=0}^{N} a_k z^k$ be a complex random polynomial, such that the coefficients $a_k$ are independent complex random variables, such that the real and imaginary parts of $a_k$ are independent (real) Gaussian random variables centered at 0 with variance $\begin{pmatrix} N \\ k \end{pmatrix}$. Then, with the notations above,*

$$\mathbb{E}(V(X_1, \ldots, X_N)) = -\frac{N^2}{4} \ln\left(\frac{4}{e}\right) - \frac{N \ln N}{4} + \frac{N}{4} \ln \frac{4}{e}$$

We summarise here some interesting facts.

- We see that the value of $V$ is small at points coming from the solution set of this random polynomials when we compare Theorem 2.6 with equations (2.17) and (2.18).
- Notice that, taking the homogeneous counterpart of $f$, Theorem can be restated for random homogeneous polynomials and considering its complex projective solutions, under the identification of $\mathbb{P}\left(\mathbb{C}^2\right)$ with the Riemann sphere.
- In this fashion, the induced probability distribution over the space of homogeneous polynomials in two complex variables corresponds to the classical unitarily invariant Hermitian structure of the respective space (see Blum et al.[16]). Therefore, the probability distribution of the roots in $\mathbb{P}\left(\mathbb{C}^2\right)$ is invariant under the action of the unitary group.
- We note that the distribution of the associated random roots on the sphere is orthogonally invariant.

For a proof of Theorem 2.6 and a more detailed discussion on this account, one may consult Armentano et al.[6]. See also Shub and Smale [66].

We now notice from the results of Hammersley [32] and Šparo and Šur [61] that the most of complex zeros a random polynomial are near the unit circle. Also Ibragimov and Zaporochets (11) have shown that the convergence to the unit circle happens if and only if $\mathbb{E}\log(1 + |X_i|) < \infty$. The following question raises in this context: What is scale at which most of the zeros are near the unit circle?

Shepp and Vanderbei [62] have shown that most of zeros are on the scale $\frac{1}{n}$ away from the unit circle when the IID random coefficients in the algebraic random polynomial follow standard normal distribution. They have found an exact expression for the expected number of zeros in a set $A \subset \mathbb{C}$. A new observation in the study by Peres and Vińah [55] that zeros uniformly bounded away from unit circle tend to a determinantal point process.

**Conjecture:** Shepp and Vanderbei [62] have formulated the following conjectures when the coefficients follow standard normal distribution.

($a$) There is a complex zero within $O(1/n^2)$ of the unit circle with high probability. Further, if $\{\zeta_i\}$ is the set of complex zeros then $\{(|\zeta| - 1)n^2\}_\zeta$ tends to Poisson Process.
($b$) There is a real zero within $O(1/n)$ of the unit circle with high probability. Further, if $\{r\}$ is the set of zeros then $\{(|r| - 1)n\}$ tends to Poisson Process.

Ibragimov and Zeitouni [34], and, Konyagin and Schlag [38] have shown that $O(1/n^2)$ is the best possible for the conjecture ($a$) : For $\epsilon > 0$ small enough, there is a positive probability that there are no zeros within $\epsilon/n^2$ of unit circle. It is pertinent to note results of Michelen and Sahasrabudhe [54].

**Theorem 2.7** ([54])**.** *Conjecture* ($a$) *holds. For IID standard normal coefficients, if* $\{\zeta\}$ *are the zeros of the random algebraic polynomial in the upper half plane then* $\{(|\zeta| - 1)n^2\}_\zeta$ *tends to a homogeneous Poisson process with intensity* $1/12$ *in the vague topology.*

We have a deduction from the above result.

**Corollary 2.8** ([54])**.** *If* $d_n$ *is the distance from the unit circle to the closest zero of random algebraic polynomial* $F_n$*, then* $d_n n^2$ *converges to an exponential random variable of rate* $1/6$ *in distribution.*

Further Michelen[53] has established the following result.

**Theorem 2.9** ([53])**.** *First half of Conjecture* ($b$) *holds whereas the second half does not. For* $\mathbb{E}(X_i) = 0$ *and* $\mathbb{E}(X_i^2) = 1$, *there is a real zero within* $O(1/n)$ *of the unit circle with high probability but* $\{(|r| - 1)n\}$ *tends in distribution to non-Poisson limit.*

**Question:** This investigation leads to several avenues for further study when the coefficients follow other distributions.

## 2.3 VARIANCE NUMBER OF ROOTS OF RANDOM POLYNOMIAL SYSTEMS

We may look for analogous results of Maslova ([49],[50]) for the random polynomial system. In this connection, we present here the results in Armentano et al.[5].

The case of algebraic polynomials $P_d(t) = \sum_{j=1}^{d} a_j t^j$ with independent identically distributed coeffcients was the first one to be extensively studied and was completely understood during the 70s. If $a_1$ is centered, $P(a_1 = 0) = 0$ and $\mathbb{E}(|a_1|^{2+\delta}) < \infty$ for some $\delta > 0$, then, the asymptotic expectation and the asymptotic variance of the number of real roots of $P_d$, as the degree $d$ tends to infinity, are of order $\log d$ and, once normalized, the number of real roots converges in distribution towards a centered Gaussian random variable.

The case of systems of polynomial equations seems to be considerably harder and has received in consequence much less attention. The results in this direction are confined to the Shub-Smale model and some other invariant distributions. The ensemble of Shub-Smale random polynomials was introduced in the early 90's by Kostlan [39]. Kostlan argues that this is the most natural distribution for a polynomial system. The exact expectation was obtained in the early 90's by geometric means, see Edelman and Kostlan [29] for the one-dimensional case and Shub and Smale [65] for the multi-dimensional one. In 2004, 2005 Azaïs and Wschebor [11] and Wschebor [70] obtained by probabilistic methods the asymptotic variance as the number of equations and variables tends to infinity. Recently, Dalmao [23] obtained the asymptotic variance and a CLT for the number of zeros as the degree $d$ goes to infinity in the case of one equation in one variable. Letendre in [44] studied the asymptotic behavior of the volume of random real algebraic submanifolds. His results include the finiteness of the limit variance, when the degree tends to infinity, of the volume of the zero sets of Kostlan-Shub-Smale systems with strictly less equations than variables. Some results for the expectation and variance of related models are included in ([11], [42], [43]).

Now we state the interesting results obtained by Armentano et al.[5]. Consider a square system $\mathbf{P}$ of $m$ polynomial equations in $m$ variables with common degree $d > 1$. More precisely, let $\mathbf{P} = (P_1, \ldots, P_m)$ with

$$P_\ell(t) = \sum_{|\boldsymbol{j}| \le d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}}$$

where

(1) $j = (j_1, \ldots, j_m) \in \mathbb{N}^m$ and $|j| = \sum_{k=1}^{m} j_k$;
(2) $a_j^{(\ell)} = a_{j_1 \cdots j_m}^{(\ell)} \in \mathbb{R}, \ell = 1, \ldots, m, |j| \le d$
(3) $t = (t_1, \ldots, t_m)$ and $t^j = \prod_{k=1}^{m} t_k^{j_k}$.

We say that P has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_j^{(\ell)}$ are independent centered normally distributed random variables with variances

$$\operatorname{Var}\left(a_j^{(\ell)}\right) = \begin{pmatrix} d \\ j \end{pmatrix} = \frac{d!}{j_1! \dots j_m!(d-|j|)!}$$

We are interested in the number of real roots of **P** that we denote by $N_d^{\mathrm{P}}$. Shub and Smale [65] have proved that $\mathbb{E}\left(N_d^{\mathrm{P}}\right) = d^{m/2}$. The main interesting result is the following.

**Theorem 2.10** ([5]). *Let* P *be an KSS random polynomial system with m equations, m variables and degree d. Then, as $d \to \infty$ we have*

$$\lim_{d\to\infty} \frac{\operatorname{Var}\left(N_d^{\mathrm{P}}\right)}{d^{m/2}} = V_\infty^2$$

*where* $0 < V_\infty^2 < \infty$.

Also we present here an explicit expression of the variance.

**Theorem 2.11** ([5]). *For $k = 1, \dots, m$ let $\xi_k, \eta_k$ be independent standard normal random vectors on $\mathbb{R}^k$. Let us define*

- $\bar{\sigma}^2(t) = 1 - \frac{t^2 \exp(-t^2)}{1-\exp(-t^2)}$;
- $\bar{\rho}(t) = \frac{\left(1-t^2-\exp(-t^2)\right)\exp(-t^2/2)}{1-\left(1+t^2\right)\exp(-t^2)}$;
- $m_{k,j} = \mathbb{E}\left(\|\xi_k\|^j\right) = 2^{j/2}\frac{\Gamma((j+k)/2)}{\Gamma(k/2)}$, *where* $\|\cdot\|$ *is the Euclidean norm on* $\mathbb{R}^k$;
- *for* $k = 1, \dots, m-1$, $M_k(t) = \mathbb{E}\left[\|\xi_k\| \left\|\eta_k + \frac{e^{-t^2/2}}{\left(1-e^{-t^2}\right)^{1/2}}\xi_k\right\|\right]$
- *for* $k = m$, $M_m(t) = \mathbb{E}\left[\|\xi_m\| \left\|\eta_m + \frac{\bar{\rho}(t)}{\left(1-\bar{\rho}^2(t)\right)^{1/2}}\xi_m\right\|\right]$,

*Then we have* $V_\infty^2 = \frac{1}{2} + \frac{\kappa_m\kappa_{m-1}}{2(2\pi)^m} \cdot \int_0^\infty t^{m-1}\left[\frac{\bar{\sigma}^4(t)\left(1-\bar{\rho}^2(t)\right)}{1-e^{-t^2}}\right]^{1/2}\left[\prod_{k=1}^m M_k(t) - \prod_{k=1}^m m_{k,1}^2\right]\mathrm{d}t.$

## 3 RANDOM POLYNOMIALS IN ALGEBRAIC STRUCTURES

In this section, we deal with algebraic algorithms that work with univariate polynomials over finite fields. The account on this problem discusses polynomial factorization, irreducibility tests, and constructions of both irreducible polynomials and finite fields. This has led to develop efficient algebraic algorithms that work with polynomials over finite fields. The outline based on generating functions and their asymptotic analysis, permits to analyze the behavior of the algorithms in question. We notice that a complete analysis of algorithms for polynomials reveals the understanding of polynomials over finite fields.

Let $\mathbb{F}_q$ be a finite field. We take univariate monic polynomials over $\mathbb{F}_q$ for discussion. The interesting aspects are as follows:

1. How is a random polynomial in terms of its irreducible factors?

2. Random polynomials in algorithms, and
3. Average-case analysis of algorithms that use polynomials over finite fields.

It is well-known that a polynomial of degree $n$ is irreducible with probability close to $1/n$. One may be interested to know more about the behavior of a random polynomial? For instance, (cf.[59])

- how many irreducible factors should we expect in a random polynomial?
- how often will it be square-free?
- what is the expected largest (smallest) degree among its irreducible factors? and the second largest one?
- how is the degree distribution among its irreducible factors?
- how often a polynomial is $m$ -smooth (all irreducible factors of degree smaller or equal to $m$ )?
- how often are two polynomials $m$-smooth and coprime? and so on.

Let $\mathbb{F}_q$ where $q$ is a prime power, $n \geq 2$ , an integer and $\mathbb{F}_{q^n}$ the finite fields with $q$ and $q^n$ elements, respectively. It is assumed that the arithmetic in $\mathbb{F}_q$ is given.

The extension field $\mathbb{F}_{q^n}$ of degree $n$ over $\mathbb{F}_q$ can be viewed as a vector space of dimension $n$ over the field $\mathbb{F}_q$.

**Theorem 3.1.** *Let $\Omega_n$ be a random variable counting the number of irreducible factors of a random polynomial of degree n over $\mathbb{F}_q$, where each factor is counted with its order of multiplicity.*

1. *The mean value of $\Omega_n$ is asymptotic to $\log n + O(1)$. or more precisely, to the harmonic sum: $H_n = \sum_{i=1}^{n} 1/i$.*
2. *The variance of $\Omega_n$ is asymptotic to $\log n + O(1)$.*
3. *For any two real constants $\lambda < \mu$,*

$$\Pr\left\{\log n + \lambda\sqrt{\log n} < \Omega_n < \log n + \mu\sqrt{\log n}\right\} \to \frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\mu} e^{-t^2/2}dt$$

4. *The distribution of $\Omega_n$ admits exponential tails.*
5. *A local limit theorem holds.*

Theorem 3.1 shows that the average number of irreducible factors of a random polynomial of degree $n$ is asymptotic to $\log n$ with a standard deviation of $\sqrt{\log n}$. A natural variation is to consider the same parameter but for an interval $[a, b] \subseteq [1, n]$.

**Theorem 3.2.** *Let $\Omega_n^{[a,b]}$ be a random variable counting the number of irreducible factors of a random polynomial of degree n over $\mathbb{F}_q$ with degrees belonging to a fixed interval $[a, b]$, where each factor is counted with its order of multiplicity.*

- *The mean value of $\Omega_n^{[a,b]}$ is asymptotic to $\sum_{k=a}^{b} \frac{I_k}{q^k(1-q^{-k})}$.*
- *The variance of $\Omega_n^{[a,b]}$ is asymptotic to $\sum_{k=a}^{b} \frac{I_k}{q^k(1-q^{-k})^2}$.*

The number of irreducible factors of specified degree in polynomials of degree $n$ was given by Williams [72]. See also [56] for low-degree factors of random polynomials.. A detailed analysis including variance appears in Knopfmacher and Knopfmacher [40].

**Theorem 3.3.** *Let $r$ be a positive integer, and let $\Omega_n^r$ be a random variable counting the number of irreducible factors of degree $r$ in a random polynomial of degree $n$ over $\mathbb{F}_q$, where each factor is counted with its order of multiplicity.*

- *The mean value of $\Omega_n^r$ is asymptotic to $\frac{I_r}{q^r(1-q^{-r})}$.*
- *The variance of $\Omega_n^r$ is asymptotic to $\frac{I_r}{a^r(1-q^{-r})^2}$.*

In [59] and the references therein will throw more light on the problems on random polynomials on finite algebraic structures.

## 4 RANDOM POLYNOMIALS ON RANDOM FIELDS AND MANIFOLDS

### 4.1 Random Polynomials on Random Fields

It is well known that there exist two variants of the change of variables formula for multiple integrals used in integral geometry.

- The first one, known as Area Formula, corresponds to smooth, locally bijective functions $G : \mathbb{R}^d \to \mathbb{R}^d$ and
- the second, known as Coarea Formula, applies to smooth functions $G : \mathbb{R}^d \to \mathbb{R}^j$ with $d > j$, having a differential with maximal rank.

**Definition 4.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $T$ a topological space. Then a measurable mapping $f : \Omega \to \mathbb{R}^T$ (the space of all real-valued functions on $T$ ) is called a real-valued random field. Measurable mappings from $\Omega$ to $\left(\mathbb{R}^T\right)^d$, $d > 1$, are called vector-valued random fields. If $T \subset \mathbb{R}^N$, we call $f$ an $(N, d)$ random field, and if $d = 1$, simply an $N$-dimensional random field.

**Note:** we assume that all random fields be *separable*(thanks to Doob[27]). This assumption solves many measurability problems. For instance, without separability, it is not necessarily the case that the supremum of a random field is a well-defined random variable.

Applying the Area and Coarea formulae to trajectories of random fields and taking expectation afterwards, one catches the well-known Kac-Rice formulae. In recent times, the two inspiring resources Adler and Taylor [2] and Azaïs [12], there has been a progressive interest in the application of these formulae in many domains. viz. random algebraic geometry, algorithm complexity for solving large systems of equations, study of zeros of random polynomial systems and finally, engineering applications.

As the ingredients are too technical, we request the interested to reader to refer to Berzin et al.[13] for results and interesting new proofs.

Angst and Poly [3] studied of the volume of the zero-sets (or nodal sets) of Gaussian random fields on $(\mathbb{R}/\mathbb{Z})^n$ and established a set of formulae (of "Kac-Rice type") that compute the volume of the zero-set of a real-valued function $f$ on $(\mathbb{R}/\mathbb{Z})^n$ as an integral of a functional of $f$ and its derivatives. In particular, they gave a general formula in the one-dimensional case and a few specific formulae in higher dimension. Further Jubin [35], established a general Kac-Rice Type formula for functions on compact Riemannian manifolds.

Characterising geometry of a complicated landscape is an important problem motivated by numerous applications in physics, image processing and other fields of applied mathematics [31, 1]. Longuet-Higgins [45, 47], obtained the mean number of all stationary points (minima, maxima and saddles), which is a relevant question in statistical physics of disordered (glassy) systems [46, 33, 20, 41, 19, 21, 22, 58], and more recently in string theory [28].

### 4.2 Random Polynomials on Manifolds

Shiffman and Zelditch[64] studied the limit distribution of zeros of certain sequences of holomorphic sections of high powers $L^N$ of a positive holomorphic Hermitian line bundle $L$ over a compact complex manifold $M$. Their first result concerns 'random' sequences of sections. Using the natural probability measure on the space of sequences of orthonormal bases $\left\{S_j^N\right\}$ of $H^0\left(M, L^N\right)$, they have shown that for almost every sequence $\left\{S_j^N\right\}$, the associated sequence of zero currents $\frac{1}{N}Z_{S_j^N}$ tends to the curvature form $\omega$ of $L$. Thus, the zeros of a sequence of sections $s_N \in H^0\left(M, L^N\right)$ chosen independently and at random become uniformly distributed. Their second result concerns the zeros of quantum ergodic eigenfunctions, where the relevant orthonormal bases $\left\{S_j^N\right\}$ of $H^0\left(M, L^N\right)$ consist of eigensections of a quantum ergodic map. In this case, they also proved that the zeros become uniformly distributed.

On symplectic manifolds, Donaldson[25, 26] and Auroux[9, 10] use analogues of holomorphic sections of an ample line bundle $L$ over a symplectic manifold $M$ to create symplectically embedded zero sections and almost holomorphic maps to various spaces. Their analogues were termed 'asymptotically holomorphic' sequences $\{s_N\}$ of sections of $L^N$. In [63], they studied another analogue $H_J^0\left(M, L^N\right)$ of holomorphic sections, which are called 'almostholomorphic' sections, following a method introduced earlier by Boutet de Monvel - Guillemin [18] in a general setting of symplectic cones.

We note that, by definition, sections in $H_J^0\left(M, L^N\right)$ lie in the range of a Szegö projector $\Pi_N$. Starting almost from scratch, and only using almost complex geometry, Shiffman and Zelditch[63] constructed a simple parametrix for $\Pi_N$ of precisely the same type as the Boutet de Monvel-Sjöstrand parametrix in the holomorphic case [17]. In [63] they proved that $\Pi_N(x, y)$ has precisely the same scaling asymptotics as does the holomorphic Szegö kernel as analyzed in [15]. The scaling asymptotics imply more or less immediately a number of analogues of well-known results in the holomorphic case, e.g. a Kodaira embedding theorem and a Tian almost-isometry theorem. Also in [63], they explained how to modify Donaldson's constructions to prove existence of quantitatively transverse sections in $H_J^0\left(M, L^N\right)$.

Shiffman and Zelditch[64] studied the variance of the number of simultaneous zeros of $m$ IID Gaussian random polynomials of degree $N$ in an open set $U \subset \mathbb{C}^m$ with smooth boundary is asymptotic to $N^{m-1/2}\nu_{mm}\,\mathrm{Vol}(\partial U)$, where $\nu_{mm}$ is a universal constant depending only on the dimension $m$. Also they provided formulae for the variance of the volume of the set of simultaneous zeros in $U$ of $k < m$ random degree$-N$ polynomials on $\mathbb{C}^m$. Infact their results hold more generally for the simultaneous zeros of random holomorphic sections of the $N-$th power of any positive line bundle over any $m-$dimensional compact Kähler manifold. Many results were established by Shiffman and Zelditch[64] in a series of papers.

## 5 RANDOM POLYNOMIALS ON LIE GROUPS

Kazarnovskii and other references in [37] saw a new planet in their journey in cosmos to investigate the behaviour of random polynomials on Lie groups and other algebraic structures. The following results from [37] are thought provoking.

A $\pi$-polynomial on a Lie group $K$ is a finite linear combination of matrix elements of a finite dimensional representation $\pi$ of a Lie group $K$. If $K$ is compact then any $\pi$-polynomial uniquely extends to a holomorphic function on the complexification $K_{\mathbb{C}}$ of $K$. For a system of $n$ $\pi_i$-polynomials, where $n = \dim(K)$, he discussed the proportion of real roots, that is the ratio of the number of roots to the number of roots in $K_{\mathbb{C}}$. It turns out that for growing representations $\pi_i$ and random system of $\pi_i$-polynomials, the expected proportion of real roots converges not to 0 , but to a nonzero constant. He calculated the limit in terms of the volumes of some compact convex sets that determine the growth of representation $\pi$. Notice that for a 1-dimensional torus $K$ the limit is $1/\sqrt{3}$.

## 6 CONCLUSIONS

In this new cosmos, we travelled in many planets. We have just summarised a cross section of results. We have not summarised all results. Our intension is to identify some areas where random polynomials manifest in different directions and we do not claim the originality of results quoted in ths survey. Researchers can probe further to find new paths. Now the stage is open to young minds.

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