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EXISTENCE RESULTS FOR IMPULSIVE CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, the authors establish conditions for the existence of solutions to impulsive Caputo-Hadamard fractional differential equations with integral boundary conditions. The proofs make use of the Banach contraction theorem, Schauder's fixed point theorem, and the nonlinear Leray-Schauder alternative. An example to illustrate the results is given.

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1. INTRODUCTION

In this paper we examine the existence and uniqueness of solutions to the boundary value problem (BVP) for the impulsive fractional differential equation having integral boundary conditions

$$ {}^{H}_{C}D^{r}y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [1, T],\ t \neq t_k,\ k = 1, \dots, m,\ 1 < r \leq 2, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m, \tag{1.2}$$

$$\Delta y'\ |_{t=t_k} = \overline{I}_k(y(t_k^-)),\ k = 1, \dots, m, \tag{1.3}$$

$$y(1) = \int_1^T g(s, y(s))ds, \quad y'(T) = \int_1^T h(s, y(s))ds, \tag{1.4}$$

where $1 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, ${}^{H}_{c}D^{r}$ is the Caputo-Hadamard fractional derivative of order r, f, g, $h : J \times \mathbb{R} \to \mathbb{R}$ and I_k, $\overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, are continuous functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ with $y(t_k^+) = \lim_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim_{\varepsilon \to 0^-} y(t_k + \varepsilon)$, and $\Delta y'$ has a similar meaning for $y'(t)$.

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In the last couple of decades, fractional-order models have been preferred to integer-order ones due to their memory retention properties. Fractional differential equations with integral boundary conditions appear in the mathematical simulation of structures and processes in the fields of physics, chemistry, aerodynamics, electrodynamics, polymer rheology, and many others (see the monographs [1, 2, 4, 23, 26]). Various studies of fractional problems involving the Caputo, Hadamard, and Caputo-Hadamard type fractional derivatives can be found, for example, in the papers [9, 12, 18, 19, 20, 21, 22].

A few preliminary notions will be presented in the next section, and our main results appear in Section 3. The final section of the paper is dedicated to an example to illustrate our results.

2. PRELIMINARIES

In this section, we introduce some notation, definitions, and preliminary facts that will be used in the sequel. We let $C(J,\mathbb{R})$ denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and take $L^1(J,\mathbb{R})$ to be the Banach space of Lebesgue integrable functions $y : J \to \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

In addition, $AC(J,\mathbb{R})$ will denote the space of functions $y : J \to \mathbb{R}$ that are absolutely continuous.

Let $\delta = t\dfrac{d}{dt}$; then we set

$$AC^n_\delta(J,\mathbb{R}) = \{y : J \to \mathbb{R} \mid \delta^{n-1}y(t) \in AC(J,\mathbb{R})\},$$

so that $AC^1(J,\mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$ that are absolutely continuous and have an absolutely continuous first derivative. In what follows, $[r]$ denotes the integer part of the real number r and $\log(\cdot) = \log_e(\cdot)$.

Next, we define the fractional derivatives and integrals to be used in this paper.

Definition 2.1. ([23]) The Hadamard derivative of fractional order q of a C^{n-1} function $y : [1,+\infty) \to \mathbb{R}$ is defined by

$${}^HD^r y(t) = \frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-r-1} y(s)\frac{ds}{s},\ n-1 < r < n,\ n = [r]+1.$$

Definition 2.2. ([23]) The Hadamard fractional integral of order r of a continuous function y is defined as

$${}^HI^r y(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} y(s)\frac{ds}{s},\ r > 0,$$

provided the integral exists.

Definition 2.3. ([23]) For an n-times differentiable function $y : [1, +\infty) \to \mathbb{R}$, the Caputo type Hadamard derivative of fractional order r is defined as

$$^{HC}D^r y(t) = \frac{1}{\Gamma(n-r)} \int_1^t \left(\log \frac{t}{s} \right)^{n-r-1} \delta^n y(s) \frac{ds}{s}, \ n-1 < r < n, \ n = [r] + 1.$$

The following lemma is well known.

Lemma 2.4. ([5]) *Let $y \in AC_\delta^n([a,b], \mathbb{R})$ or $y \in C_\delta^n([a,b], \mathbb{R})$ and $r \in \mathbb{R}$. Then*

$$^{H}I_a^r(^{HC}D_a^r y)(t) = y(t) - \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a} \right)^k.$$

3. MAIN RESULTS

In this section we state and prove our main existence results. Consider the set of functions

$$PC(J, \mathbb{R}) = \left\{ \begin{array}{c} y : J \to \mathbb{R} \mid y \in AC_\delta^2((t_k, t_{k+1}], \mathbb{R}), \ k = 1, \ldots, m, \text{ and } y(t_k^+) \\ \text{and } y(t_k^-), \ k = 1, \ldots, m, \text{ exist with } y(t_k^-) = y(t_k) \end{array} \right\}.$$

This forms a Banach space with the norm

$$\|y\|_{PC} = \sup\{|y(t)| : t \in J\}.$$

For convenience, we set $J' = [a, T] \backslash \{t_1, t_2, ..., t_m\}$. We begin with a lemma relating solutions of a BVP related to (1.1)–(1.4) to the solutions of a fractional integral equation.

Lemma 3.1. *Let $1 < r \leq 2$ and let $\sigma \in AC(J,\mathbb{R})$. A function y is a solution of the fractional integral equation*

(3.1)
$$y(t) = \begin{cases} \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\sigma(s)\frac{ds}{s} + \int_1^T \rho_1(s)ds + T(\log t)\int_1^T \rho_2(s)ds \\ -(\log t)\left[\frac{1}{\Gamma(r-1)}\int_{t_m}^T \left(\log\frac{T}{s}\right)^{r-2}\sigma(s)\frac{ds}{s}\right. \\ \left.+\frac{1}{\Gamma(r-1)}\sum_{i=1}^m \int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\sigma(s)\frac{ds}{s} + \sum_{i=1}^m t_i\bar{I}_i(y(t_i^-))\right],\ t\in[1,t_1], \\ \\ \frac{1}{\Gamma(r)}\int_{t_k}^t \left(\log\frac{t}{s}\right)^{r-1}\sigma(s)\frac{ds}{s} + \int_1^T \rho_1(s)ds + T(\log t)\int_1^T \rho_2(s)ds \\ -(\log t)[\frac{1}{\Gamma(r-1)}\int_{t_m}^T \left(\log\frac{T}{s}\right)^{r-2}\sigma(s)\frac{ds}{s} \\ +\frac{1}{\Gamma(r-1)}\sum_{i=1}^m \int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\sigma(s)\frac{ds}{s} + \sum_{i=1}^m t_i\bar{I}_i(y(t_i^-))] \\ +\frac{1}{\Gamma(r)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\sigma(s)\frac{ds}{s} \\ +\sum_{i=1}^k \frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\sigma(s)\frac{ds}{s} + \sum_{i=1}^k I_i(y(t_i^-)) \\ +\sum_{i=1}^k t_i\left(\log\frac{t}{t_i}\right)\bar{I}_i(y(t_i^-)),\ t\in(t_k,t_{k+1}],\ k=1,...,m, \end{cases}$$

if and only if $y(t)$ is a solution of the fractional BVP

$$^{CH}D^r y(t) = \sigma(t), \quad t\in J=[1,T],\ t\neq t_k,\ 1<r\leq 2, \tag{3.2}$$

$$\Delta y\mid_{t=t_k} = I_k(y(t_k^-)), \quad k=1,\dots,m, \tag{3.3}$$

$$\Delta y\prime\mid_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k=1,\dots,m, \tag{3.4}$$

$$y(1) = \int_1^T \rho_1(s)ds \quad y'(T) = \int_1^T \rho_2(s)ds. \tag{3.5}$$

Proof. Let y be a solution of (3.2)-(3.5). If $t\in[1,t_1]$, then Lemma 2.4 implies that

$$y(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\sigma(s)\frac{ds}{s} + c_1 + c_2(\log t)$$

for some constants $c_1,\ c_2\in\mathbb{R}$. Then we have

$$y'(t) = \frac{1}{t\Gamma(r-1)}\int_1^t \left(\log\frac{t}{s}\right)^{r-2}\sigma(s)\frac{ds}{s} + \frac{c_2}{t}$$

The first condition in (3.5) implies that $c_1 = \int_1^T \rho_1(s)ds$, and so

$$y(t) = c_2(\log t) + \int_1^T \rho_1(s)ds + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s)\frac{ds}{s}$$

and

$$y'(t) = \frac{c_2}{t} + \frac{1}{t\Gamma r - 1)} \int_1^t \left(\log \frac{t}{s}\right)^{r-2} \sigma(s)\frac{ds}{s}.$$

If $t \in (t_1, t_2]$, then again by Lemma 2.4, we have

$$y(t) = \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} + d_1 + d_2(\log \frac{t}{t_1})$$

and

$$y'(t) = \frac{1}{t\Gamma(r-1)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} + \frac{d_2}{t}$$

for some constants d_1, $d_2 \in \mathbb{R}$. From (3.3), $\Delta y \mid_{t=t_1} = y(t_1^+) - y(t_1^-) = I_1(y(t_1^-))$, so

$$d_1 = I_1(y(t_1^-)) + \int_1^T \rho_1(s)ds + c_2(\log t_1) + \frac{1}{\Gamma(r)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s)\frac{ds}{s}$$

and since $\Delta y' \mid_{t=t_1} = y'(t_1^+) - y'(t_1^-) = \bar{I}_1(y(t_1^-))$, we obtain

$$d_2 = t_1\bar{I}_1(y(t_1^-)) + c_2 + \frac{1}{\Gamma(r-1)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \sigma(s)\frac{ds}{s}$$

Thus, for $t \in (t_1, t_2]$, we have

$$\begin{aligned} y(t) = c_2(\log t) &+ \int_1^T \rho_1(s)ds + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} \\ &+ \frac{(\log \frac{t}{t_1})}{\Gamma(r-1)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} + t_1(\log \frac{t}{t_1})\bar{I}_1(y(t_1^-)) + I_1(y(t_1^-)). \end{aligned}$$

Repeating this process, for $t \in J_k$ we obtain

$$\begin{aligned} y(t) = c_2 \log t &+ \int_1^T \rho_1(s)ds + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^k \frac{\left(\log \frac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^k I_i(y(t_i^-)) + \sum_{i=1}^k t_i \left(\log \frac{t}{t_i}\right) \bar{I}_i(y(t_i^-)). \end{aligned}$$

Continuing in the same manner, for $t \in (t_m, T]$,

$$\begin{aligned} y(t) = c_2 \log t &+ \int_1^T \rho_1(s)ds + \frac{1}{\Gamma(r)} \int_{t_m}^t \left(\log \frac{t}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^m I_i(y(t_i^-)) + \sum_{i=1}^m \frac{\left(\log \dfrac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^m t_i \left(\log \frac{t}{t_i}\right) \bar{I}_i(y(t_i^-)) \end{aligned}$$

and

$$\begin{aligned} y'(t) = \frac{c_2}{t} &+ \frac{1}{t\Gamma(r-1)} \int_{t_m}^t \left(\log \frac{t}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^m \frac{1}{t\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^m (\frac{t_i}{t})\bar{I}_i(y(t_i^-)). \end{aligned}$$

Applying the second boundary condition gives

$$\begin{aligned} y'(T) &= \frac{c_2}{T} + \frac{1}{T\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{t}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} \\ &+ \sum_{i=1}^m \frac{1}{T\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} + \sum_{i=1}^m \left(\frac{t_i}{T}\right) \bar{I}_i(y(t_i^-)), \end{aligned}$$

and this implies that

$$\begin{aligned} c_2 &= T\int_1^T \rho_2(s)ds - \left[\frac{1}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{t}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} \right. \\ &\left. + \sum_{i=1}^m \frac{1}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \sigma(s)\frac{ds}{s} + \sum_{i=1}^m (t_i)\bar{I}_i(y(t_i^-)\right], \end{aligned}$$

that is, we obtain (3.1)

It is easy to see that (3.1) satisfies equation (3.2) and conditions (3.3)–(3.5), and this completes the proof of the lemma. □

Our first existence result is based on Banach's fixed point theorem.

Theorem 3.2. *Assume that the following conditions hold.*

(H_1) *There exist constants* l_1, l_2, $l_3 > 0$ *such that*

$|f(t,u)-f(t,v)| \leq l_1|u-v|$, $|g(t,u)-g(t,v)| \leq l_2|u-v|$, *and* $|h(t,u)-h(t,v)| \leq l_3|u-v|$

for $t \in J$ and $u,\, v \in \mathbb{R}$.

(H_2) *There exist constants $l,\, l^* > 0$ such that*

$$|I_k(u) - I_k(v)| \le l|u-v| \quad \text{and} \quad |\overline{I}_k(u) - \overline{I}_k(v)| \le l^*\|u-v\|$$

for $t \in J$ and $u,\, v \in \mathbb{R}$, $k = 1, 2, \dots, m$.

If

$$l_1 \left\{ \frac{(\log T)^r}{\Gamma(r+1)} [(m+1) + (1+2m)r] \right\} + (T-1)(l_2 + l_3 T \log T) + ml + 2m(\log T)Tl^* < 1, \tag{3.6}$$

then the problem (1.1)–(1.4) has a unique solution on $[1, T]$.

Proof. To transform the problem (1.1)–(1.4) into a fixed point problem, we define the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\begin{aligned}
(Fy)(t) = & \int_1^T g(s, y(s))ds + T(\log t) \int_1^T h(s, y(s))ds \\
& - (\log t) \left[\frac{1}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{T}{s} \right)^{r-2} f(s, y(s)) \frac{ds}{s} \right. \\
& \left. + \frac{1}{\Gamma(r-1)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s} \right)^{r-2} f(s, y(s)) \frac{ds}{s} + \sum_{i=1}^m t_i \bar{I}_i(y(t_i^-)) \right] \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s} \right)^{r-1} f(s, y(s)) \frac{ds}{s} \\
& + \sum_{i=1}^k \frac{\left(\log \frac{t}{t_i} \right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s} \right)^{r-2} f(s, y(s)) \frac{ds}{s} + \sum_{i=1}^k I_i(y(t_i^-)) \\
& + \sum_{i=1}^k t_k \left(\log \frac{t}{t_k} \right) \bar{I}_k(y(t_k^-)) + \frac{1}{\Gamma(r)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s} \right)^{r-1} f(s, y(s)) \frac{ds}{s}.
\end{aligned}$$

Clearly, the fixed points of the operator F are solutions of problem (1.1)–(1.4).

Let $x,\, y \in PC(J, \mathbb{R})$; then for each $t \in J$, we have

$$\begin{aligned}
& |(Fx)(t) - (Fy)(t)| \\
& \le \int_1^T |g(s, x(s)) - g(s, y(s))|ds + T(\log t) \int_1^T |h(s, x(s)) - h(s, y(s))|ds \\
& \quad - (\log t) \left[\frac{1}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{T}{s} \right)^{r-2} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \right. \\
& \quad + \frac{1}{\Gamma(r-1)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s} \right)^{r-2} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s}
\end{aligned}$$

$$+\sum_{i=1}^{m} t_i|\bar{I}_i(x(t_i^-)) - \bar{I}_i(y(t_i^-))|\Bigg]$$

$$+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|f(s,x(s)) - f(s,y(s))|\frac{ds}{s}$$

$$+\frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|f(s,x(s)) - f(s,y(s))|\frac{ds}{s}$$

$$+\sum_{i=1}^{k}\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,x(s)) - f(s,y(s))|\frac{ds}{s}$$

$$+\sum_{i=1}^{k}|I_i(x(t_i^-)) - I_i(y(t_i^-))| + \sum_{i=1}^{k} t_k\left(\log\frac{t}{t_k}\right)|\bar{I}_i(x(t_i^-)) - \bar{I}_i(y(t_i^-))|$$

$$\leq \int_1^T l_2\|x-y\|ds + T(\log t)\int_1^T l_3\|x-y\|ds + (\log t)\left[\frac{l_1\|x-y\|}{\Gamma(r-1)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-2}\frac{ds}{s}\right.$$

$$\left.+\frac{l_1\|x-y\|}{\Gamma(r-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\frac{ds}{s} + \sum_{i=1}^{m} t_i l^*|(x(t_i^-)) - (y(t_i^-))|\right]$$

$$+\frac{l_1\|x-y\|}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s} + \frac{l_1\|x-y\|}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s}$$

$$+\sum_{i=1}^{k}\frac{l_1\|x-y\|\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\frac{ds}{s} + \sum_{i=1}^{k} l|(x(t_i^-)) - (y(t_i^-))|$$

$$+\sum_{i=1}^{k} t_k\left(\log\frac{t}{t_k}\right) l^*|(x(t_i^-)) - (y(t_i^-))|$$

$$\leq (T-1)l_2\|x-y\| + T(\log T)(T-1)l_3\|x-y\| + \frac{(\log T)^r}{\Gamma(r)}l_1\|x-y\|$$

$$+\frac{m(\log T)^r}{\Gamma(r)}l_1\|x-y\| + mT(\log T)l^*\|x-y\| + \frac{(\log T)^r}{\Gamma(r+1)}l_1\|x-y\|$$

$$+\frac{m(\log T)^r}{\Gamma(r+1)}l_1\|x-y\| + \frac{mT(\log T)^r}{\Gamma(r)}l_1\|x-y\|$$

$$+ml\|x-y\| + mT(\log T)l^*\|x-y\|$$

$$\leq \left[(T-1)(l_2 + l_3 T\log T) + l_1\left[\frac{(m+1)(\log T)^r}{\Gamma(r+1)} + \frac{(1+2m)(\log T)^r}{\Gamma(r)}\right]\right.$$

$$\left.+ml + 2ml^*(\log T)\,T\right]\|x-y\|\,.$$

We then have

$$\|Fx - Fy\| \leq \left\{(T-1)(l_2 + l_3 T\log T) + l_1\left[\frac{(m+1)(\log T)^r}{\Gamma(r+1)}\right.\right. \tag{3.7}$$

$$+ \frac{(1+2m)(\log T)^r}{\Gamma(r)}\Bigg] + ml + 2m(\log T)Tl^*\Bigg\} \|x-y\|. \tag{3.8}$$

In view of (3.6), F is a contraction, so by Banach's fixed point theorem, F has a fixed point that is a solution of the problem (1.1)–(1.4). □

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.3. *Assume that the following conditions hold:*

(H_3) *There exists a constant $M > 0$ such that $|f(t,u)| \leq M$ for $t \in J$ and $u \in \mathbb{R}$.*

(H_4) *There exists a constant $M^* > 0$ such that*

$$|g(u)| \leq M^* \text{ and } |h(u)| \leq M^* \text{ for } u \in \mathbb{R}.$$

(H_5) *There exist a constant $N > 0$ such that*

$$|I_k(u)| \leq N \quad \text{and} \quad |\bar{I}_k(u)| \leq N \quad \text{for } u \in \mathbb{R},\ k = 1, \dots, m.$$

Then the problem (1.1)–(1.4) has at least one solution on $[1,T]$.

Proof. In order to apply Schaefer's fixed point theorem, the proof will be given in several steps.

Step 1: *F is continuous.* Let y_n be a sequence such that $y_n \to y$ in $PC(J,\mathbb{R})$. Then for each $t \in [1,T]$,

$$\begin{aligned}
&|F(y_n)(t) - F(y)(t)| \\
&\leq \int_1^T |g(s,y_n(s)) - g(s,y(s))|ds + T(\log t)\int_1^T |h(s,y_n(s)) - h(s,y(s))|ds \\
&\quad + \frac{(\log t)}{\Gamma(r-1)}\int_{t_m}^T \left(\log\frac{T}{s}\right)^{r-2} |f(s,y_n(s)) - f(s,y(s))|\frac{ds}{s} \\
&\quad + \frac{(\log t)}{\Gamma(r-1)}\sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log\frac{t_i}{s}\right)^{r-2} |f(s,y_n(s)) - f(s,y(s))|\frac{ds}{s} \\
&\quad + (\log t)\sum_{i=1}^m t_i|\bar{I}(y_n(t_i^-)) - \bar{I}_i(y(t_i^-))| \\
&\quad + \frac{1}{\Gamma(r)}\int_{t_k}^t \left(\log\frac{t}{s}\right)^{r-1} |f(s,y_n(s)) - f(s,y(s))|\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log\frac{t_i}{s}\right)^{r-1} |f(s,y_n(s)) - f(s,y(s))|\frac{ds}{s} \\
&\quad + \sum_{i=1}^k \frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i} \left(\log\frac{t_i}{s}\right)^{r-2} |f(s,y_n(s)) - f(s,y(s))|\frac{ds}{s} \\
&\quad + \sum_{i=1}^k |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|
\end{aligned}$$

$$+\sum_{i=1}^{k} t_k \left(\log \frac{t}{t_k}\right) |\bar{I}(y_n(t_i^-)) - \bar{I}_i(y(t_i^-))|.$$

Due to our continuity assumptions, we have

$$\|F(y_n) - F(y)\|_\infty \to 0 \; as \; n \to \infty,$$

i.e., F is continuous.

Step 2: *F maps bounded sets into bounded sets in $PC(J, \mathbb{R})$.* It suffices to show that for any η^*, there exists a positive constant L such that, for each $y \in B_{\eta^*} = \{y \in PC(J, \mathbb{R}) : \|y\| \leq \eta^*\}$, we have $\|F(y)\| \leq L$. Applying (H_3)–(H_5), for each $t \in J$, we obtain

$$\begin{aligned}
|F(y)(t)| \leq & \int_1^T |g(s, y(s))|ds + T(\log t) \int_1^T |h(s, y(s))|ds \\
& + \frac{(\log t)}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{T}{s}\right)^{r-2} |f(s, y(s)| \frac{ds}{s} \\
& + \frac{\log t}{\Gamma(r-1)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} |f(s, y(s)| \frac{ds}{s} + \sum_{i=1}^{m} t_i |\bar{I}_i(y(t_i^-))| \\
& + \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} |f(s, y(s)| \frac{ds}{s} + \frac{1}{\Gamma(r)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} |f(s, y(s)| \frac{ds}{s} \\
& + \sum_{i=1}^{k} \frac{\left(\log \frac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} |f(s, y(s)| \frac{ds}{s} + \sum_{i=1}^{k} |I_i(y(t_i^-))| \\
& + \sum_{i=1}^{k} t_k \left(\log \frac{t}{t_k}\right) |\bar{I}_k(y(t_k^-))| \\
\leq & M^* \int_1^T ds + T(\log t) M^* \int_1^T ds + \frac{M|(\log t)|}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{T}{s}\right)^{r-2} \frac{ds}{s} \\
& + \frac{M|\log t|}{\Gamma(r-1)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \frac{ds}{s} + m(\log t)TN \\
& + \frac{M}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \frac{ds}{s} + \frac{M}{\Gamma(r)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \frac{ds}{s} \\
& + \sum_{i=1}^{k} \frac{M\left(\log \frac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} \frac{ds}{s} + mN + mT(\log T)N \\
\leq & (T-1)(M^* + T(\log T)M^*) + \frac{M(\log T)^r}{\Gamma(r)} + \frac{Mm(\log T)^r}{\Gamma(r)} + m(\log T)TN \\
& + \frac{M(\log T)^r}{\Gamma(r+1)} + \frac{Mm(\log T)^r}{\Gamma(r+1)} + \frac{Mm(\log T)^r}{\Gamma(r)} + mN + mT(\log T)N
\end{aligned}$$

$$\leq (T-1)(M^* + T(\log T)M^*) + M\frac{(1+2m)(\log T)^r}{\Gamma(r)}$$
$$+ M\frac{(1+m)(\log T)^r}{\Gamma(r+1)} + mN + 2mT(\log T)N.$$

Therefore,

$$\|Fy\| \leq \Bigg\{(T-1)(M^* + T(\log T)M^*) + M\left[\frac{(1+m)}{\Gamma(r+1)} \cdot \frac{(1+2m)}{\Gamma(r)}\right](\log T)^r$$
$$+ mN + 2mT(\log T)N\Bigg\} := L.$$

Step 3: *F maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.* Let τ_1, $\tau_2 \in J$ with $\tau_1 < \tau_2$, let B_{η^*} be a bounded set in $PC(J,\mathbb{R})$ as in Step 2, and let $y \in B_{\eta^*}$. Then,

$$|F(y)(\tau_2) - F(y)(\tau_1)|$$
$$\leq T\left(\log\frac{\tau_2}{\tau_1}\right)\int_1^T |h(s,y(s))|ds + \frac{(\log\frac{\tau_2}{\tau_1})}{\Gamma(r-1)}\int_{t_m}^T \left(\log\frac{T}{s}\right)^{r-2}|f(s,y(s))|\frac{ds}{s}$$
$$+ \frac{(\log\frac{\tau_2}{\tau_1})}{\Gamma(r-1)}\sum_{1<t<\tau_2-\tau_1}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y(s))|\frac{ds}{s} + \sum_{1<t<\tau_2-\tau_1} t_i|\bar{I}_i(y(t_i^-))|$$
$$+ \frac{1}{\Gamma(r)}\int_1^{\tau_1}\left(\left(\log\frac{\tau_2}{s}\right)^{r-1} - \left(\log\frac{\tau_1}{s}\right)^{r-1}\right)|f(s,y(s))|\frac{ds}{s}$$
$$+ \frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{r-1}|f(s,y(s))|\frac{ds}{s}$$
$$+ \frac{1}{\Gamma(r)}\sum_{1<t<\tau_2-\tau_1}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|f(s,y(s))|\frac{ds}{s}$$
$$+ \sum_{1<t<\tau_2-\tau_1}\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y(s))|\frac{ds}{s}$$
$$+ \sum_{1<t<\tau_2-\tau_1}|I_i(y(t_i^-))| + \sum_{1<t<\tau_2-\tau_1} t_k\left(\log\frac{\tau_2}{\tau_1}\right)|\bar{I}_k(y(t_k^-))|.$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero, which shows the equicontinuity.

As a consequence of Steps 1 to 3, together with the Arzelà-Ascoli theorem, we conclude that F is completely continuous.

Step 4: *A priori bounds.* Now it remains to show that the set

$$\varepsilon = \{y \in PC(J,\mathbb{R}) \to PC(J,\mathbb{R}) : y = \lambda F(y)\, for\ \ some\ \ 0 < \lambda < 1\}$$

is bounded. Let $y \in \varepsilon$; then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$\begin{aligned}(Fy)(t) = {}& \lambda \int_1^T g(s, y(s))ds + \lambda T(\log t) \int_1^T h(s, y(s))ds \\ & - \frac{\lambda(\log t)}{\Gamma(r-1)} \int_{t_m}^T \left(\log \frac{T}{s}\right)^{r-2} f(s, y(s) \frac{ds}{s} \\ & - \frac{\lambda(\log t)}{\Gamma(r-1)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} f(s, y(s)) \frac{ds}{s} - \lambda(\log t) \sum_{i=1}^m t_i \bar{I}_i(y(t_i^-)) \\ & + \frac{\lambda}{\Gamma(r)} \int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} f(s, y(s)) \frac{ds}{s} + \frac{\lambda}{\Gamma(r)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} f(s, y(s)) \frac{ds}{s} \\ & + \lambda \sum_{i=1}^k \frac{\left(\log \frac{t}{t_i}\right)}{\Gamma(r-1)} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-2} f(s, y(s)) \frac{ds}{s} + \lambda \sum_{i=1}^k I_i(y(t_i^-)) \\ & + \lambda \sum_{i=1}^k t_k \left(\log \frac{t}{t_k}\right) \bar{I}_k(y(t_k^-)).\end{aligned}$$

For each $t \in J$, by (H9)–(H11), we have

$$\begin{aligned}\|y\| \le {}& (T-1)(M^* + M^* T \log T) + M \left[\frac{(1+m)}{\Gamma(r+1)} + \frac{(1+2m)}{\Gamma(r)}\right] (\log T)^r \\ & + mN + 2mT(\log T)N.\end{aligned}$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, F has a fixed point that in turn is a solution of the problem (1.1)–(1.4). □

Our final existence result is based on the nonlinear Leray-Schauder alternative. Here we are able to weaken conditions (H_3)–(H_5) used above.

Theorem 3.4. *Assume that:*

(H_6) *There exist* ϕ_f, ϕ_g, $\phi_h \in C(J, \mathbb{R}^+)$ *and continuous and non-decreasing functions* ψ_f, ψ_g, $\psi_h : [0, \infty) \to [0, \infty)$ *such that*

$$|f(t, u)| \le \phi_f(t)\psi_f(|u|), \quad |g(t, u)| \le \phi_g(t)\psi_g(|u|), \quad \textit{and} \quad |h(t, u)| \le \phi_h(t)\psi_h(|u|)$$

for $t \in J$ *and* $u \in \mathbb{R}$.

(H_7) *There exist continuous and non-decreasing functions* ω, $\bar{\omega} : [0, \infty) \to [0, \infty)$ *such that*

$$|I_k(u)| \le \omega(|u|) \quad \textit{and} \quad |\bar{I}_k(u)| \le \bar{\omega}(|u|)$$

for $u \in \mathbb{R}$, $k = 1, \ldots, m$.

If there exists a number $\overline{M} > 0$ *such that*

$$\overline{M} \Big\{ \psi_g(\overline{M}) \|\phi_g\|_{L^1} + T(\log T)\psi_h(\overline{M}) \|\phi_h\|_{L^1} \tag{3.9}$$

$$+\phi\psi_f(\overline{M})\left(\frac{(1+m)(\log T)^r}{\Gamma(r+1)}+\frac{(1+2m)(\log T)^r}{\Gamma(r)}\right)$$
$$+m\omega(\overline{M})+2mT(\log T)\bar{\omega}(\overline{M})\Big\}^{-1}>1,$$

where $\phi=\sup\{\phi_f(t):t\in J\}$, *then the problem (1.1)–(1.4) has at least one solution on* J.

Proof. Consider the operator F defined in the proof of Theorems 3.2. As we have seen above, F is continuous and completely continuous. For $\lambda\in(0,1)$ and each $t\in J$, let $y(t)=\lambda(Fy)(t)$. Then from (H$_6$)–(H$_7$),

$$\begin{aligned}
|y(t)| \le & \int_1^T |g(s,y(s))|ds + T(\log t)\int_1^T |h(s,y(s))|ds \\
& + \frac{(\log t)}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}|f(s,y(s))|\frac{ds}{s} \\
& + \frac{(\log t)}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y(s))|\frac{ds}{s} + (\log t)\sum_{i=1}^m t_i|\bar{I}_i(y(t_i^-))| \\
& + \frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}|f(s,y(s))|\frac{ds}{s} + \frac{1}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|f(s,y(s))|\frac{ds}{s} \\
& + \sum_{i=1}^k\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}|f(s,y(s))|\frac{ds}{s} + \sum_{i=1}^k|I_i(y(t_i^-))| \\
& + \sum_{i=1}^k t_k\left(\log\frac{t}{t_k}\right)|\bar{I}_i(y(t_i^-))| \\
\le & \int_1^T \phi_g(s)\psi_g(|y(s)|)ds + T(\log t)\int_1^T \phi_h(s)\psi_h(|y(s)|)ds \\
& + \frac{(\log t)}{\Gamma(r-1)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-2}\phi_f(s)\psi_f(|y(s)|)\frac{ds}{s} \\
& + \frac{(\log t)}{\Gamma(r-1)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\phi_f(s)\psi_f(|y(s)|)\frac{ds}{s} + (\log t)\sum_{i=1}^m t_i\bar{\omega}(|y|) \\
& + \frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}\phi_f(s)\psi_f(|y(s)|)\frac{ds}{s} \\
& + \frac{1}{\Gamma(r)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\phi_f(s)\psi_f(|y(s)|)\frac{ds}{s} \\
& + \sum_{i=1}^k\frac{\left(\log\frac{t}{t_i}\right)}{\Gamma(r-1)}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-2}\phi_f(s)\psi_f(|y(s)|)\frac{ds}{s} + \sum_{i=1}^k\omega(|y|)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k} t_k\left(\log\frac{t}{t_k}\right)\bar{\omega}(|y|)\\
&\leq \psi_g(\|y\|)\int_1^T \phi_g(s)ds + T(\log T)\psi_h(\|y\|)\int_1^T \phi_h(s)ds + \frac{(\log T)^r}{\Gamma(r)}\phi\psi_f(\|y\|)\\
&+\frac{m(\log T)^r}{\Gamma(r)}\phi\psi_f(\|y\|) + mT(\log T)\bar{\omega}(\|y\|) + \frac{(\log T)^r}{\Gamma(r+1)}\phi\psi_f(\|y\|)\\
&+\frac{m(\log T)^r}{\Gamma(r+1)}\phi\psi_f(\|y\|) + \frac{m(\log T)^r}{\Gamma(r)}\phi\psi_f(\|y\|) + m\omega(\|y\|) + mT(\log T)\bar{\omega}(\|y\|)\\
&\leq \psi_g(\|y\|)\|\phi_g\|_{L^1} + T(\log T)\psi_h(\|y\|)\|\phi_h\|_{L^1}\\
&+\phi\psi_f(\|y\|)\left(\frac{(1+m)(\log T)^r}{\Gamma(r+1)} + \frac{(1+2m)(\log T)^r}{\Gamma(r)}\right)\\
&+m\omega(\|y\|) + 2mT(\log T)\bar{\omega}(\|y\|).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|y\|\Bigg\{&\psi_g(\|y\|)\|\phi_g\|_{L^1} + T(\log T)\psi_h(\|y\|)\|\phi_h\|_{L^1}\\
&+\phi\psi_f(\|y\|)\left(\frac{(1+m)(\log T)^r}{\Gamma(r+1)} + \frac{(1+2m)(\log T)^r}{\Gamma(r)}\right)\\
&+m\omega(\|y\|) + 2mT(\log T)\bar{\omega}(\|y\|)\Bigg\}^{-1} \leq 1.
\end{aligned}
$$

Then by condition (3.9), there exists $\overline{M}$ such that $\|y\| \neq \overline{M}$.
Let

$$U = \{y \in PC(J,\mathbb{R}) : \|y\| \leq \overline{M}\}.$$

The operator $F : \overline{U} \to PC(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda F(y)$ for some $\lambda \in (0,1)$. As a consequence of the Leray-Schauder nonlinear alternative, F has a fixed point $y \in \overline{U}$ that is a solution of the problem (1.1)–(1.4). This completes the proof of the theorem. □

4. EXAMPLE

Consider the boundary value problem

$$ {}^{CH}D^{\frac{3}{2}}y(t) = \frac{1}{(20+t)^2}|y(t)|, \quad t \in J = [1,e], \quad t \neq \frac{7}{4}, \tag{4.1}$$

$$ \Delta y(7/4) = \frac{1}{10 + |y(\frac{7}{4}^-)|}, \quad \Delta y'(7/4) = \frac{1}{25 + |y(\frac{7}{4}^-)|}, \tag{4.2}$$

$$ y(1) = \int_1^e \frac{|y(s)|}{15+|y(s)|}ds, \quad y'(e) = \int_1^e \frac{|y(s)|}{13+|y(s)|}ds. \tag{4.3}$$

For $(t,y) \in J \times \mathbb{R}$, we have

$$f(t,y) = \frac{1}{(20+t)^2} y, \quad g(t,y) = \frac{y}{15+y}, \quad \text{and} \quad h(t,y) = \frac{y}{13+y}.$$

Also,

$$I_k(y) = \frac{1}{10+y} \quad \text{and} \quad \overline{I_k}(y) = \frac{1}{25+y} \quad \text{for } y \in \mathbb{R}.$$

Here $r = \frac{3}{2}$, $m = 1$, $t_1 = \frac{7}{4}$, and $T = e$. It is easy to see that $l_1 = \frac{1}{400}$, $l_2 = \frac{1}{15}$, $l_3 = \frac{1}{13}$, $l = \frac{1}{10}$, and $l^* = \frac{1}{25}$. Finally, we see that (3.6) becomes

$$\frac{1}{400}\left[\frac{4}{3\sqrt{\pi}}\left(2+\frac{9}{2}\right)\right] + (e-1)\left(\frac{1}{15}+\frac{e}{13}\right) + \frac{1}{10} + \frac{2e}{25} = 0.82971 < 1.$$

All the conditions of Theorem 3.2 are satisfied and so problem (4.1)–(4.3) has a unique solution y on $[1, e]$.

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