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# ON SOME RETARDED INTEGRAL INEQUALITIES OF GRONWALL TYPE: APPLICATION TO INTEGRAL AND DIFFERENTIAL EQUATIONS

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**ABSTRACT.** We present in this paper some new inequalities of Gronwall-type. As an application, we consider the problem of asymptotic behaviors of a class of retarded Volterra equations. We establish bounds on the solutions and, by means of examples, we show and discuss the usefulness of our results in investigating the asymptotic stability of the solutions. Moreover, we study via integral inequalities the stability of certain dynamical systems.

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**Key Words and Phrases.** Integral inequalities of Gronwall type, integral equations, differential equations, stability.

## 1. INTRODUCTION

Recently, there has been a growing interest in Volterra integral equations for the stability problem, as they represent a powerful instrument for the mathematical representation of memory dependent phenomena in population dynamic, economy, and so forth. A pioneering research on this subject in [10] and [11], where the main results concerning the existence, uniqueness, and boundedness of solutions are presented. After that, a Volterra theory has been developed and it is still evolving; see, for example [1], [24], [25] and the bibliography therein. The general theory of the stability of motion, [23], is presented in monographs ([14], [19], [28]). There are different methods for the stability analysis for differential or integral equations ([2]-[5], [7], [9], [12], [15]-[18]). The integral inequalities occupy privileged position in the theory of differential and integral equations. In the recent years nonlinear integral inequalities have received considerable attention because of the important applications to a variety of problems in diverse fields of nonlinear differential and integral equations [6], [18].

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Some integral inequalities for differential and integral equations are established by Gronwall [13], Bellman and pachpatte ([8], [11], [20], [27]) that can be used as handy tools to study the qualitative properties of solutions of some integral equations [26], [29]. In this paper we are basically interested on retarded Gronwall like inequalities, we will give generalizations of those done in [2] and integral inequalities for retarded Volterra equations generalizing those done in [21], [22]. Some applications are also given to convey the importance of our results.

## 2. LINEAR INTEGRAL INEQUALITIES

The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of the various types. Some applications of this result can be used to the study of existence, uniqueness theory of differential equations and the stability of the solution of linear and nonlinear differential equations. During the past few years, several authors have established several Gronwall type integral inequalities in one or two independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations. Here we start proving the following results required for stability purpose.

**Theorem 2.1.** *Let $k$, $b \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $a \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $(t,s) \mapsto \partial_t a(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. Assume in addition that $\alpha$ is nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)u(s)ds + \int_0^t b(s)u(s)ds, \qquad t \geq 0, \tag{1}$$

*then*

$$u(t) \leq k(t) + e^{h(t)} \int_0^t e^{-h(r)} \partial_r \left( \int_0^{\alpha(r)} a(r,s)k(s)ds + \int_0^r k(s)b(s)ds \right) dr, \qquad t \geq 0, \tag{2}$$

*where $h(t) = \int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds$.*

*Proof.* Denote $z(t) = \int_0^{\alpha(t)} a(t,s)u(s)ds + \int_0^t b(s)u(s)ds$. Our assumptions on the functions $a$, $b$ and $\alpha$ imply that $z$ is nondecreasing on $\mathbb{R}_+$. Hence, for $t \geq 0$, we have

$$\begin{aligned} z^{'}(t) &= a(t,\alpha(t))u(\alpha(t))\alpha^{'}(t) + \int_0^{\alpha(t)} \partial_t a(t,s)u(s)ds + b(t)u(t) \\ &\leq a(t,\alpha(t))[k(\alpha(t)) + z(\alpha(t))]\alpha^{'}(t) + \\ &\quad \int_0^{\alpha(t)} \partial_t a(t,s)[k(s)+z(s)]ds + b(t)[k(t)+z(t)] \\ &\leq a(t,\alpha(t))[k(\alpha(t)) + z(t)]\alpha^{'}(t) + \int_0^{\alpha(t)} \partial_t a(t,s)k(s)ds + \\ &\quad z\int_0^{\alpha(t)} \partial_t a(t,s)ds + b(t)[k(t)+z(t)], \end{aligned}$$

or, equivalently,

$$z^{'}(t) - z(t)\frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds\right) \leq \frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)k(s)ds + \int_0^t k(s)b(s)ds\right).$$

Multiplying the above inequality by $e^{-h(t)}$, we get

$$\frac{d}{dt}\left(z(t)e^{-h(t)}\right) \leq e^{-h(t)}\frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)k(s)ds + \int_0^t k(s)b(s)ds\right).$$

Consider now the integral on the interval $[0,t]$ to obtain

$$z(t) \leq e^{h(t)}\int_0^t e^{-h(r)}\partial_r\left(\int_0^{\alpha(r)} a(r,s)k(s)ds + \int_0^r k(s)b(s)ds\right)dr, \qquad t \geq 0.$$

Combine the above inequality with $u(t) \leq k(t) + z(t)$ to get (2) and, with this, the proof is complete.

□

**Corollary 2.2.** *Assume $a$, $k$, $\alpha$ are as in Theorem 2.1 and $b(t) \equiv 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)u(s)ds, \qquad t \geq 0,$$

*then,*

$$(3)\quad u(t) \leq k(t) + e^{\int_0^{\alpha(t)} a(t,s)ds}\int_0^t e^{-\int_0^{\alpha(r)} a(r,s)ds}\partial_r\left(\int_0^{\alpha(r)} a(r,s)k(s)ds\right)dr, \qquad t \geq 0.$$

**Corollary 2.3.** *Assume $a$, $b$ and $\alpha$ are as in Theorem 2.1 and $k(t) \equiv k > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k + \int_0^{\alpha(t)} a(t,s)u(s)ds + \int_0^t b(s)u(s)ds, \qquad t \geq 0,$$

*then,*

$$u(t) \leq ke^{\int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds}, \qquad t \geq 0.$$

*Proof.* Apply Theorem 2.1 to obtain

$$\begin{aligned} u(t) &\leq k + ke^{h(t)} \int_0^t e^{-h(r)} \partial_r \left( \int_0^{\alpha(r)} a(r,s)ds + \int_0^r b(s)ds \right) dr \\ &= k + ke^{\int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds} (1 - e^{-(\int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds)}) \\ &= ke^{\int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds}, \qquad t \geq 0. \end{aligned}$$

□

Notice that all these integral inequalities are linear on $u$.

**Application to Volterra integral equation**

**Corollary 2.4.** *Assume $a$, $b$ and $\alpha$ are as in Theorem 2.1 and $k(t) \equiv k > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the Volterra integral equation*

$$u(t) = k + \int_0^{\alpha(t)} a(t,s)u(s)ds + \int_0^t b(s)u(s)ds, \qquad t \geq 0. \tag{4}$$

*If $\lim_{t\to\infty} \int_0^{\alpha(t)} a(t,s)ds < \infty$ and $\lim_{t\to\infty} \int_0^t b(s)ds < \infty$, then $u$ is bounded on $\mathbb{R}_+$.*

*Proof.* The conclusion follows immediately from Corollary 2.3.

Note that the limits $\lim_{t\to\infty} \int_0^{\alpha(t)} a(t,s)ds$ and $\lim_{t\to\infty} \int_0^t b(s)ds$ always exists since the functions $t \mapsto \lim_{t\to\infty} \int_0^{\alpha(t)} a(t,s)ds$ and $t \mapsto \lim_{t\to\infty} \int_0^t b(s)ds$ are nondecreasing on $\mathbb{R}_+$.

□

**Example 2.5.** The functions $b(t) = e^{-t}$ and $a(t,s) = t/(1+2t+(1+t)s^2)$, $t$, $s \geq 0$, satisfies the hypothesis in Corollary 2.4 for any nondecreasing $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\alpha(t) \leq t$ for $t \geq 0$. In this case, all solutions $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ of (4) are bounded.

**Theorem 2.6.** *Let $c$, $k$, $b \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $a \in C^1(\mathbb{R}_+, \mathbb{R}_+)$. Assume in addition that $\alpha$ is nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s)ds + \int_0^t c(s)u(s)ds, \qquad t \geq 0,$$

*then,*

$$u(t) \leq k(t) + e^{h(t)} \int_0^t e^{-h(r)} \left( a(r) \int_0^{\alpha(r)} b(s)k(s)ds + \int_0^r c(s)k(s)ds \right) dr, \quad t \geq 0, \tag{5}$$

*where* $h(t) = a(t) \int_0^{\alpha(t)} b(s)ds + \int_0^t c(s)ds.$

*Proof.* Denote $z(t) = a(t) \int_0^{\alpha(t)} b(s)u(s)ds + \int_0^t c(s)u(s)ds$. Our assumptions on the functions $a$, $b$ and $\alpha$ imply that $z$ is nondecreasing on $\mathbb{R}_+$. Hence, for $t \geq 0$, we have

$$\begin{aligned} z^{'} &= a(t)u(\alpha(t))b(\alpha(t))\alpha^{'}(t) + a^{'}(t) \int_0^{\alpha(t)} b(s)u(s)ds + c(t)u(t) \\ &\leq a(t)b(\alpha(t))[k(\alpha(t)) + z(\alpha(t))]\alpha^{'}(t) + \\ &\quad a^{'}(t) \int_0^{\alpha(t)} b(s)[k(s) + z(s)]ds + c(t)[k(t) + z(t)] \\ &\leq a(t)b(\alpha(t))[k(\alpha(t)) + \\ &\quad z(t)]\alpha^{'}(t) + a^{'}(t) \int_0^{\alpha(t)} b(s)k(s)ds + a^{'}(t)z \int_0^{\alpha(t)} b(s)ds + c(t)[k(t) + z(t)] \end{aligned}$$

or equivalently,

$$\begin{aligned} z^{'} - z\left(a(t)b(\alpha(t))\alpha^{'}(t) + a^{'}(t) \int_0^{\alpha(t)} b(s)ds + c(t)\right) &\leq a(t)b(\alpha(t))k(\alpha(t))\alpha^{'}(t) + \\ &\quad a^{'} \int_0^{\alpha(t)} b(s)k(s)ds + c(t)k(t). \end{aligned}$$

Multiplying the above inequality by $e^{-h(t)}$, we get

$$\frac{d}{dt}\left(z(t)e^{-h(t)}\right) \leq e^{-h(t)} \frac{d}{dt}\left(a(t) \int_0^{\alpha(t)} b(s)k(s)ds + \int_0^t c(s)k(s)ds\right).$$

Consider now the integral on the interval $[0, t]$ to obtain

$$z(t) \leq e^{h(t)} \int_0^t e^{-h(r)} \left(a(r) \int_0^{\alpha(r)} b(s)k(s)ds + \int_0^r c(s)k(s)ds\right) dr, \qquad t \geq 0.$$

Combine the above inequality with $u(t) \leq k(t) + z(t)$ to get (5) and, with this, the proof is complete.

□

**Corollary 2.7.** *Let $k$, $b$, $\alpha$, $a$ be as in Theorem 2.6 and suppose that $c \equiv 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s)ds, \qquad t \geq 0,$$

*then,*

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s)b(s)ds} b(r)k(r)dr, \qquad t \geq 0.$$

**Corollary 2.8.** *Let $a$, $b$, $c$, $k$ and $\alpha$ be as in Theorem 2.6. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation*

$$u(t) = k(t) + a(t)\int_0^{\alpha(t)} b(s)u(s)ds + \int_0^t c(s)u(s)ds, \qquad t \geq 0.$$

*If the functions $a$, $k$ are bounded on $\mathbb{R}_+$, $\int_0^{\alpha(\infty)} b(s)ds < \infty$ and $\int_0^{\infty} c(s)ds < \infty$, then $u$ is bounded on $\mathbb{R}_+$.*

**Corollary 2.9.** *Let $a$, $b$, $c$, $k$ and $\alpha$ be as in Theorem 2.6 with $k(t) \to 0$ as $t \to \infty$. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation*

$$u(t) = k(t) + a(t)\int_0^{\alpha(t)} b(s)u(s)ds + \int_0^t c(s)u(s)ds, \qquad t \geq 0.$$

*If $\int_0^{\alpha(\infty)} a(s)b(s)ds < \infty$, $\int_0^{\infty} c(s)ds < \infty$, and $\lim_{t\to\infty} a(t)\int_0^{\alpha(t)} b(r)k(r)dr = 0$, then $u(t) \to 0$ as $t \to \infty$. In particular, if $a(t)$, $k(t) \to 0$ as $t \to \infty$, $\int_0^{\alpha(\infty)} b(s)ds < \infty$ and $\int_0^{\infty} c(s)ds < \infty$, then $u(t) \to 0$ as $t \to \infty$.*

**Example 2.10.** We consider a particular case by supposing $\alpha(t) = t$, $c(t) \equiv 0$. The integral equation

$$u(t) = k(t) + a(t)\int_0^{\alpha(t)} b(s)u(s)ds, \qquad t \geq 0,$$

has the exact solution

$$u(t) = k(t) + a(t)\int_0^t e^{\int_r^{\alpha(t)} a(s)b(s)ds} b(r)k(r)dr, \qquad t \geq 0.$$

For $a(t) = k(t) = t^{-2}$ and $b(t) = t^2$, for the conditions imposed on corollary 2.9, we have $\lim_{t\to\infty} k(t) = 0$, $\quad \lim_{t\to\infty} a(t)\int_0^{\alpha(t)} b(r)k(r)dr = 0 \quad$ and $\quad \int_0^{\infty} a(s)b(s)ds = \infty$. Notice that the solution equals $u(t) = (t+1)^{-2} + (e^t - 1)(t+1)^{-2} \to \infty$ as $t \to \infty$. This example shows the importance of the conditions imposed by Corollary 2.9.

**Theorem 2.11.** *Let $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$, $f$ continuous on $(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g$, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq a(t) + \int_0^t h(s)u(s)ds + \int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)u(\tau)d\tau ds, \qquad t \geq 0,, \tag{6}$$

*then*

$$u(t) \leq a(t)\exp\left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)d\tau ds\right], \qquad t \geq 0.$$

*Proof.* Since $a(t)$ is positive and nondecreasing, from (6), we have

$$\begin{aligned}\frac{u(t)}{a(t)} &\le 1+\int_0^t h(s)\frac{u(s)}{a(t)}ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(t)}d\tau ds\\ &\le 1+\int_0^t h(s)\frac{u(s)}{a(s)}ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds.\end{aligned}$$

We define a function $z(t)$ on $\mathbb{R}_+$ by

$$z(t)=1+\int_0^t h(s)\frac{u(s)}{a(s)}ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds$$

then $z(t)$ is positive and nondecreasing. $z(0)=1$, $\frac{u(t)}{a(t)}\le z(t)$, $t\in\mathbb{R}_+$ and

$$\begin{aligned}z'(t) &= h(t)\frac{u(t)}{a(t)}+f(t,\alpha(t))\alpha'(t)\int_0^{\alpha(t)} g(s)\frac{u(s)}{a(s)}ds+\int_0^{\alpha(t)}\partial_t f(t,s)\int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds\\ &\le h(t)z(t)+f(t,\alpha(t))\alpha'(t)\int_0^{\alpha(t)} g(s)z(s)ds+\int_0^{\alpha(t)}\partial_t f(t,s)\int_0^s g(\tau)z(\tau)d\tau ds\\ &\le z(t)\left[h(t)+f(t,\alpha(t))\alpha'(t)\int_0^{\alpha(t)} g(s)ds+\int_0^{\alpha(t)}\partial_t f(t,s)\int_0^s g(\tau)d\tau ds\right].\end{aligned}$$

by integration, we get

$$z(t)\le z(0)\exp\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)d\tau ds\right].$$

Since $\frac{u(t)}{a(t)}\le z(t)$, it comes that

$$u(t)\le a(t)\exp\left[\int_0^t h(s)ds+\int_0^{\alpha(t)} f(t,s)\int_0^s g(\tau)d\tau ds\right].$$

□

## 3. NONLINEAR INTEGRAL INEQUALITIES

**Theorem 3.1.** *Let $b\in C(\mathbb{R}_+,\mathbb{R}_+)$, $a\in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$ and $\alpha\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, assume in addition that $\alpha$ is nondecreasing with $\alpha(t)\le t$ for $t\ge 0$. Assume $k$, $\omega\in C(\mathbb{R}_+,\mathbb{R}_+)$ are nondecreasing functions with $k(0)>0$, $\omega(t)>0$ for $t>0$ and $\int_1^\infty \frac{dt}{\omega(t)}<\infty$. If $u\in C(\mathbb{R}_+,\mathbb{R}_+)$ satisfies*

$$u(t)\le k(t)+\int_0^{\alpha(t)} a(t,s)\omega(u(s))\,ds+\int_0^t b(s)\omega(u(s))\,ds,\qquad t\ge 0,$$

*then*

$$u(t) \leq G^{-1}\left(G\left(k(t)\right) + \int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds\right), \qquad t \geq 0, \tag{7}$$

*where* $G(t) = \int_1^t \frac{ds}{\omega(s)}$, $t \geq 0$.

*Proof.* Let $T \geq 0$ be fixed and denote $z(t) = \int_0^{\alpha(t)} a(t,s)\omega\left(u(s)\right) ds + \int_0^t b(s)\omega\left(u(s)\right) ds$, $t \geq 0$. Our assumptions on $a$, $b$, $\alpha$ imply that $z$ is nondecreasing on $\mathbb{R}_+$. Hence for $t \in [0,T]$, we have

$$\begin{aligned}
z^{'}(t) &= a(t,\alpha(t))\omega\left(u(\alpha(t))\right)\alpha^{'}(t) + \int_0^{\alpha(t)} \partial_t a(t,s)\omega\left(u(s)\right)ds + b(t)\omega\left(u(t)\right) \\
&\leq a(t,\alpha(t))\omega\left[k(\alpha(t)) + z(\alpha(t))\right]\alpha^{'}(t) + \\
&\quad \int_0^{\alpha(t)} \partial_t a(t,s)\omega\left[k(s)+z(s)\right]ds + b(t)\omega\left[k(t)+z(t)\right] \\
&\leq a(t,\alpha(t))\alpha^{'}(t)\omega[k(\alpha(T)) + z(t)] + \\
&\quad \omega[k(\alpha(T)) + z(t)] \int_0^{\alpha(t)} \partial_t a(t,s)ds + b(t)\omega[k(T)+z(t)] \\
&\leq \left(a(t,\alpha(t))\alpha^{'}(t) + \int_0^{\alpha(t)} \partial_t a(t,s)ds + b(t)\right)\omega[k(T)+z(t)]
\end{aligned}$$

and then

$$\frac{z^{'}(t)}{\omega[k(T)+z(t)]} \leq \frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)ds + b(t)\right), \qquad t \in [0,T]. \tag{8}$$

Integrating both sides of (8) on $[0,t]$, we get

$$G\left(k(T)+z(t)\right) \leq G(k(T)) + \int_0^{\alpha(t)} a(t,s)ds + b(t), \qquad t \in [0,T],$$

or, equivalently,

$$k(T) + z(t) \leq G^{-1}\left[G(k(T)) + \int_0^{\alpha(t)} a(t,s)ds + b(t)\right], \qquad t \in [0,T]. \tag{9}$$

Note that the right-hand side of (9) is well defined as $G(\infty) = \infty$. Letting $t = T$ in the above relation, we obtain

$$u(T) \leq k(T) + z(T) \leq G^{-1}\left[G(k(T)) + \int_0^{\alpha(T)} a(t,s)ds + b(t)\right],$$

and since $T \geq 0$ was arbitrarily chosen, we get (7).

□

**Corollary 3.2.** *Assume $a$, $\alpha$, $k$ are as in Theorem 3.1 and $b(t) \equiv 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)\omega\left(u(s)\right)ds, \qquad t \geq 0,$$

*then*

$$u(t) \leq G^{-1}\left(G\left(k(t)\right) + \int_0^{\alpha(t)} a(t,s)ds\right), \qquad t \geq 0, \tag{10}$$

*where $G(t) = \displaystyle\int_1^t \frac{ds}{\omega(s)}$.*

**Corollary 3.3.** *Let $a$, $b$, $k$, $\omega$ and $\alpha$ be as in Theorem 3.1. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the nonlinear Volterra integral equation*

$$u(t) = k(t) + \int_0^{\alpha(t)} a(t,s)\omega(u(s))ds + \int_0^t b(s)\omega(u(s))ds, \qquad t \geq 0. \tag{11}$$

*If $k$ is bounded, $\displaystyle\lim_{t\to\infty}\int_0^{\alpha(t)} a(t,s)ds < \infty$ and $\displaystyle\lim_{t\to\infty}\int_0^t b(s)ds < \infty$, then $u$ is bounded on $\mathbb{R}_+$.*

**Example 3.4.** The functions $\omega(t) = (t+1)\ln(t+1)$, $k(t) \equiv k > 0$, $b(t) = e^{-t}$ and $a(t,s) = t/(1+2t+(1+t)e^s)$, $t$, $s \geq 0$, satisfy the hypothesis in Corollary 3.3 for any nondecreasing $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\alpha(t) \leq t$ for $t \geq 0$. In this case all solutions $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ of (11) are bounded.

**Theorem 3.5.** *Let $b \in C(\mathbb{R}_+, \mathbb{R}_+)$, $a \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, assume in addition that $\alpha$ is nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. Assume $k$, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $k(0) > 0$, $\omega(t) > 0$ for $t > 0$, $\omega(t) \geq t$ and $\displaystyle\int_1^\infty \frac{dt}{\omega(t)} < \infty$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)\omega\left(u(s)\right)ds + \int_0^t b(s)u(s)ds, \qquad t \geq 0,$$

*then*

$$u(t) \leq G^{-1}\left(G\left(k(t)\right) + \int_0^{\alpha(t)} a(t,s)ds + \int_0^t b(s)ds\right), \qquad t \geq 0, \tag{12}$$

*where $G(t) = \displaystyle\int_1^t \frac{ds}{\omega(s)}$, $t \geq 0$.*

*Proof.* Using that $\omega(t) \geq t$, it comes that

$$\int_0^t b(s)u(s)ds \leq \int_0^t b(s)\omega\left(u(s)\right)ds,$$

and we can conclude using Theorem 3.1.

□

**Theorem 3.6.** *Let $a$, $b$, $c$, $k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that $a$, $k$, $\alpha$ are nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. Let also $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function such that $\omega(t) > 0$ for $t > 0$, $\omega(t) \geq t$ and $\int_1^\infty \frac{dt}{\omega(t)} < \infty$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + a(t)\int_0^{\alpha(t)} b(s)\omega\left(u(s)\right)ds + \int_0^t c(s)u(s)ds, \qquad t \geq 0,$$

*then*

$$u(t) \leq G^{-1}\left(G\left(k(t)\right) + a(t)\int_0^{\alpha(t)} b(s)ds + \int_0^t c(s)ds\right), \qquad t \geq 0, \tag{13}$$

*where $G(t) = \int_1^t \frac{ds}{\omega(s)}$, $t \geq 0$.*

**Corollary 3.7.** *Let $a$, $b$, $c$, $k$, $\alpha$, $\omega$ be as in Theorem 3.6. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation*

$$u(t) = k(t) + a(t)\int_0^{\alpha(t)} b(s)\omega\left(u(s)\right)ds + \int_0^t c(s)u(s)ds, \qquad t \geq 0.$$

*If $a$, $k$ are bounded, $\int_0^\infty c(s)ds < \infty$ and $\int_0^{\alpha(\infty)} b(s)ds < \infty$, then $u$ is bounded on $\mathbb{R}_+$.*

**Theorem 3.8.** *Assume that $a \geq 0$, $p \geq 1$, then for any $k > 0$ we have*

$$a^{\frac{1}{p}} \leq \frac{1}{p}k^{\frac{1-p}{p}}a + \frac{p-1}{p}k^{\frac{1}{p}}. \tag{14}$$

*or equivalently $a^{\frac{1}{p}} \leq m_1 a + m_2$ where $m_1 = \frac{1}{p}k^{\frac{1-p}{p}}$ and $m_2 = \frac{p-1}{p}k^{\frac{1}{p}}$.*

*Proof.* Using that the function : $t \mapsto e^t$ is convex, we can write $u^\alpha v^\beta \leq \alpha u + \beta v$ for $\alpha + \beta = 1$. Taking $\alpha = \frac{1}{p}$, $\beta = \frac{p-1}{p}$, $u = k^{\frac{1-p}{p}}a$ and $v = k^{\frac{1}{p}}$ we get the desired result.

□

**Theorem 3.9.** *Let $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$, $f$, $g$, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p \geq 1$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u^p(t) \leq a(t) + \int_0^t h(s)u^p(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)u(\tau)d\tau ds, \qquad t \geq 0,$$

*then*

$$u(t) \leq \left(a(t) + \int_0^{\alpha(t)} m_2 f(s) \int_0^s g(\tau)d\tau ds\right)^{\frac{1}{p}} \exp\frac{1}{p}\left(\int_0^t h(s)ds + \int_0^{\alpha(t)} m_1 f(s) \int_0^s g(\tau)d\tau ds\right).$$

*Proof.* Let $z(t) = u^p(t)$, then $z$ satisfies

$$z(t) \leq a(t) + \int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)z^{\frac{1}{p}}(\tau)d\tau ds,$$

using (14), we get

$$\begin{aligned} z(t) &\leq a(t) + \int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)\left[m_1 z(\tau) + m_2\right] d\tau ds \\ &\leq a(t) + \int_0^t h(s)z(s)ds + \int_0^{\alpha(t)} f(s) \left[m_1 \int_0^s g(\tau)z(\tau)d\tau + m_2 \int_0^s g(\tau)d\tau\right] ds \\ &\leq a(t) + \int_0^{\alpha(t)} m_2 f(s) \int_0^s g(\tau)d\tau ds + \int_0^t h(s)z(s)ds + \\ &\quad \int_0^{\alpha(t)} m_1 f(s) \int_0^s g(\tau)z(\tau)d\tau ds. \end{aligned}$$

using Theorem 2.11, we get

$$z(t) \leq \left(a(t) + \int_0^{\alpha(t)} m_2 f(s) \int_0^s g(\tau)d\tau ds\right) \exp\left(\int_0^t h(s)ds + \int_0^{\alpha(t)} m_1 f(s) \int_0^s g(\tau)d\tau ds\right).$$

Finally, it comes that

$$u(t) \leq \left(a(t) + \int_0^{\alpha(t)} m_2 f(s) \int_0^s g(\tau)d\tau ds\right)^{\frac{1}{p}} \exp\frac{1}{p}\left(\int_0^t h(s)ds + \int_0^{\alpha(t)} m_1 f(s) \int_0^s g(\tau)d\tau ds\right).$$

□

**Theorem 3.10.** *Let $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$, $f$, $g$, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p \geq 1$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u^p(t) \leq a(t) + \int_0^t h(s)u(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)u^p(\tau)d\tau ds, \qquad t \geq 0,$$

*then*

$$u(t) \leq \left(a(t) + m_2 \int_0^t h(s)ds\right)^{\frac{1}{p}} \exp\frac{1}{p}\left(m_1 \int_0^t h(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds\right).$$

*Proof.* Let $z(t) = u^p(t)$, then $z$ satisfies

$$z(t) \leq a(t) + \int_0^t h(s) z^{\frac{1}{p}}(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) z(\tau) d\tau ds$$

using (14), we get

$$\begin{aligned} z(t) &\leq a(t) + \int_0^t h(s)[m_1 z(s) + m_2] ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) z(\tau) d\tau ds \\ &\leq a(t) + m_2 \int_0^t h(s) ds + m_1 \int_0^t h(s) z(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) z(\tau) d\tau ds. \end{aligned}$$

using Theorem 2.11, we get

$$z(t) \leq \left( a(t) + m_2 \int_0^t h(s) ds \right) \exp \left( m_1 \int_0^t h(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) d\tau ds \right)$$

finally it comes that

$$u(t) \leq \left( a(t) + m_2 \int_0^t h(s) ds \right)^{\frac{1}{p}} \exp \frac{1}{p} \left( m_1 \int_0^t h(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) d\tau ds \right).$$

□

**Theorem 3.11.** *Let $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$, $f$, $g$, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p \geq q \geq 1$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u^p(t) \leq 1 + \int_0^t h(s) u^p(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) u^q(\tau) d\tau ds, \qquad t \geq 0,$$

*then*

$$u(t) \leq \exp \frac{1}{p} \left[ \int_0^t h(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) d\tau ds \right].$$

*Proof.* Define a function $z(t)$ by

$$z^p(t) = 1 + \int_0^t h(s) u^p(s) ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) u^q(\tau) d\tau ds, t \in \mathbb{R}_+ \tag{15}$$

then $u^p(t) \leq z^p(t)$ and $z^p(0) = 1$. Differentiating (15) and using that $u(t) \leq z(t)$ and $z(t)$ is monotone nondecreasing for $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
pz'(t)z^{p-1}(t) &= h(t)u^p(t) + \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(s)u^q(s)ds \\
&\leq h(t)z^p(t) + \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(s)z^q(s)ds \\
&\leq h(t)z^p(t) + \alpha'(t)f(\alpha(t))z^q(t) \int_0^{\alpha(t)} g(s)ds \\
&\leq z^p(t)\left[h(t) + \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(s)ds\right].
\end{aligned}$$

then

$$pz'(t) \leq z(t)\left[h(t) + \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(s)ds\right]$$

by integration we get

$$z(t) \leq \exp\frac{1}{p}\left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

Finally it comes that,

$$u(t) \leq \exp\frac{1}{p}\left[\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right].$$

□

## 4. APPLICATION TO DIFFERENTIAL EQUATIONS

The new inequalities, derived in this paper, are useful in many applications in particular to the stability of dynamical systems. We propose new sufficient conditions to ensure the global uniform asymptotic stability of time-varying differential equations described by the following form:

$$\dot{x} = f(t,x) + g(t,x) \tag{16}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are piecewise continuous in $t$ and locally Lipschitz in $x$ on $\mathbb{R}_+ \times \mathbb{R}^n$, and the associated nominal system is given by

$$\dot{x} = f(t,x) \tag{17}$$

For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$, we will denote by $x(t;t_0,x_0)$, or simply by $x(t)$, the unique solution at time $t_0$ starting from the point $x_0$. Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation, $\|.\|$ stands for the Euclidean norm vectors. We recall now some standard concepts from stability and practical stability theory, any book on Lyapunov stability can be consulted for these, particularly good references are [12]. $\mathcal{K}$ is the class of functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. We denote $\mathcal{KL}$ the class of functions $\mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument.

A positive definite function $\mathbb{R}_+ \to \mathbb{R}_+$ is one that is zero at the origin and positive otherwise.

We define the closed ball $B_r := \Big\{x \in \mathbb{R}^n : \ \|x\| \leq r\Big\}$.

We begin by giving the definition of uniform boundedness and uniform stability (see [10]-[11]).

**Definition 4.1.** (uniform boundedness) A solution of (16) is said to be globally uniformly bounded if for every $\alpha > 0$ there exists $c = c(\alpha)$ such that, for all $t_0 \geq 0$,

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\| \leq c(\alpha), \quad \forall\, t \geq t_0.$$

**Definition 4.2.** (uniform stability)

$(i)$ The origin $x = 0$ is uniformly stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that for all $t_0 \geq 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\, t \geq t_0.$$

$(ii)$ The origin $x = 0$ is globally uniformly stable if it is uniformly stable and the solutions of the system (16) are globally uniformly bounded.

We recall in the following definition the notion of practical stability ( see [24]).

**Definition 4.3.** (practical stability) The system (16) is said to be

$(PS1)$ uniformly practically stable if given $(\lambda, A)$ with $0 < \lambda < A$, we have

$$\|x_0\| < \lambda \Rightarrow \|x(t)\| < A, \quad t \geq t_0, \quad \forall t_0 \in \mathbb{R}_+.$$

$(PS2)$ quasi-uniformly asymptotically stable (in the large) if $\forall \varepsilon > 0$, $\alpha > 0$, $t_0 \in \mathbb{R}_+$, there exists a positive number $T = T(\varepsilon, \alpha)$ such that

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\| < \varepsilon, \quad t \geq t_0 + T.$$

$(PS3)$ uniformly practically asymptotically stable if $(PS1)$ and $(PS2)$ hold at the same time.

As application to stability, let us consider the nonlinear dynamical system

$$\dot{x} = A(t)x + g(t, x) \tag{18}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, the matrix $A(.)$ is continuous and bounded, $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t, x)$, locally Lipschitz in $x$ such that $g(t, 0) = 0$. We suppose that $x = 0$ is globally uniformly asymptotic stable equilibrium point for the nominal system $\dot{x} = A(t)x$, this is equivalent to say that,

$$\|\Phi(t, t_0)\| \leq k \exp -\gamma(t - t_0), \forall t \geq t_0, k > 0, \gamma > 0 \tag{19}$$

where $\Phi(t, t_0)$ is the state transition matrix associated to $A(t)$. The solution of this system which the initial condition $(t_0, x_0)$ is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)g(s, x(s))ds. \tag{20}$$

We have

$$\|x(t)\| \leq k \exp -\gamma(t - t_0)\|x(t_0)\| + \int_{t_0}^{t} ke^{-\gamma(t-s)}\|g(s, x(s))\|ds. \tag{21}$$

It follows that,

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\|g(s, x(s))\|ds. \tag{22}$$

We will impose a restriction on $g$ to study the practical stability. If we suppose that for all $(t, x)$,

$$\|g(t, x)\| \leq \rho(\alpha(t))\alpha'(t),$$

with $\rho$ is a nonnegative continuous function satisfying $\int_0^\infty \rho(t)e^{\gamma\alpha^{-1}(t)}dt < \infty$, then (22) becomes

$$\begin{aligned}
e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\rho(\alpha(s))\alpha'(s)ds \\
&\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{\alpha(t_0)}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\rho(s)ds \\
&\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{0}^{\infty} ke^{\gamma\alpha^{-1}(s)}\rho(s)ds,
\end{aligned}$$

or equivalently,

$$\|x(t)\| \leq ke^{-\gamma(t-t_0)}\|x(t_0)\| + e^{-\gamma t}\int_0^\infty ke^{\gamma\alpha^{-1}(s)}\rho(s)ds.$$

Seen that the function : $t \mapsto ke^{-\gamma(t-t_0)}\|x(t_0)\|+e^{-\gamma t}\displaystyle\int_0^{\infty} ke^{\gamma\alpha^{-1}(s)}\rho(s)ds$ vanishes, then it fulfills that

$$\|x(t)\| \leq \varepsilon, \quad \forall t \geq t_0 + T,$$

for a certain $T > 0$, this shows the practical stability of the system. An other approach is to study the asymptotic behavior of the system in a small neighbourhood of the origin. For the rest, we need the following definitions which are related to stability.

**Definition 4.4.** (uniform stability of $B_r$)
$(i)$ $B_r$ is uniformly stable if for all $\epsilon > r$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $t_0 \geq 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\, t \geq t_0.$$

$(ii)$ $B_r$ is globally uniformly stable if it is uniformly stable and the solutions of system (16) are globally uniformly bounded.

**Definition 4.5.** (uniform attractivity) The origin $x = 0$ is globally uniformly attractive if for all $\epsilon > 0$ and $c > 0$, there exists $T(\epsilon, c) > 0$, such that for all $t_0 \geq 0$,

$$\|x(t)\| < \epsilon, \quad \forall\, t \geq t_0 + T(\epsilon, c), \quad \|x_0\| < c.$$

**Definition 4.6.** (Class $\mathcal{K}$ function) A continuous function $\alpha : [0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

**Definition 4.7.** (Class $\mathcal{KL}$ function) A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$, if for each fixed point $s$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$ and for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to +\infty$.

The following result provides a characterization of global uniform attractivity and global uniform stability.

**Theorem 4.8.** *If there exists a class $\mathcal{KL}$ function $\beta$, a class $\mathcal{K}_\infty$ $\alpha$, a constant $r > 0$ such that, given any initial state $x_0$, the solution satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + r, \quad \forall t \geq 0,$$

*then $B_r$ is globally uniformly attractive and globally uniformly stable.*

Note that, if the class $\mathcal{KL}$-function $\beta$ on the above relation is of the form $\beta(r, s) = kre^{-\lambda t}$, with $\lambda, k > 0$ we say that the ball $B_r$ is globally uniformly exponentially stable. It is also worth to notice that if, in the above definitions, we take $r = 0$, then one deals with the standard concept of GUAS and GUES of the origin (see [12] for more details). Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| x(t) \| - r$, so that if $r = 0$ we find the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point. Other applications to stability will be done in the following example by considering the system (16), we keep the same assumptions.

**Example 4.9.** Suppose that the folowing condition holds for all $(t, x)$,

$$\|g(t,x)\| \leq \eta(\alpha(t))\|x(\alpha(t))\|\alpha^{'}(t),$$

with $\eta$ is an integrable function, then (22) becomes

$$\begin{aligned} e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} k\eta(\alpha(s))\alpha^{'}(s)e^{\gamma s}\|x(\alpha(s))\|ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{\alpha(t_0)}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\eta(s)\|x(s)\|ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{0}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\eta(s)\|x(s)\|ds. \end{aligned}$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}e^{-\gamma s}\eta(s)u(s)ds,$$

using Corollary 2.3 we get

$$u(t) \leq ku(t_0)\exp\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}e^{-\gamma s}\eta(s)ds,$$

then

$$u(t) \leq kMu(t_0), \quad where \quad M = \exp\int_0^{\infty} ke^{\gamma\alpha^{-1}(t)}e^{-\gamma t}\eta(t)dt.$$

One can obtain an estimation on the trajectories as follows, for all $t \geq t_0$,

$$\|x(t)\| \leq kM\|x(t_0)\|e^{-\gamma(t-t_0)}.$$

Then the origin is globally uniformly exponentially stable equilibrium point for the system.

In the following example $g(t, 0)$ is not necessarily zero, in a such situation $x = 0$ is no longer an equilibrium point.

**Example 4.10.** If we suppose that for all $(t, x)$,

$$\|g(t,x)\| \leq \eta(\alpha(t))\|x(\alpha(t))\|\alpha^{'}(t) + \mu,$$

with $\eta$ is an integrable function and $\mu > 0$, then (22) becomes

$$\begin{aligned} e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\{\eta(\alpha(s))\|x(\alpha(s))\|\alpha^{'}(s)+\mu\}ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{\alpha(t_0)}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\{\eta(s)\|x(s)\|+\mu\}ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{0}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\{\eta(s)\|x(s)\|+\mu\}ds. \end{aligned}$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$\begin{aligned} u(t) &\leq ku(t_0) + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\{\eta(s)e^{-\gamma s}u(s)+\mu\}ds \\ &\leq ku(t_0) + \mu\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}ds + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}u(s)ds \\ &\leq k(t) + \int_0^{\alpha(t)} b(s)u(s)ds, \end{aligned}$$

where $k(t) = ku(t_0) + \mu\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}ds$ and $b(s) = ke^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}$.
Using Corollary 2.7 we get

$$u(t) \leq k(t) + \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} b(s)ds}k(r)b(r)dr,$$

then

$$u(t) \leq ku(t_0) + \mu\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}ds + \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} b(s)ds}k(r)b(r)dr,$$

or equivalently

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\gamma(t-t_0)} + \mu e^{-\gamma t}\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}ds + e^{-\gamma t}\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}ds.$$

By supposing $e^{-\gamma t}\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}ds \longrightarrow 0$ and $\int_0^{\infty} e^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}ds < \infty$, one can obtain an estimation on the trajectories as follows, for all $t \geq t_0$,

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\gamma(t-t_0)} + r.$$

Then, $B_r$ is globally uniformly exponentially stable.

**Example 4.11.** We suppose that for all $(t, x)$,

$$\|g(t,x)\| \leq \eta(\alpha(t))\|x(\alpha(t))\|\alpha^{'}(t) + \mu(\alpha(t))\alpha^{'}(t),$$

with $\eta$ is integrable and $\eta'$ is a piecewise continuous function, then (22) becomes

$$\begin{aligned} e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\{\eta(\alpha(s))\|x(\alpha(s))\|\alpha^{'}(s) + \mu(\alpha(s))\alpha^{'}(s)\}ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{\alpha(t_0)}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\{\eta(s)\|x(s)\| + \mu(s)\}ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{0}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\{\eta(s)\|x(s)\| + \mu(s)\}ds. \end{aligned}$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$\begin{aligned} u(t) &\leq ku(t_0) + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\{\eta(s)e^{-\gamma s}u(s) + \mu(s)\}ds \\ &\leq ku(t_0) + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\mu(s)ds + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}u(s)ds \\ &\leq k(t) + \int_0^{\alpha(t)} b(s)u(s)ds, \end{aligned}$$

where $k(t) = ku(t_0) + \int_0^{\alpha(t)} k\mu(s)e^{\gamma\alpha^{-1}(s)}ds$ and $b(s) = ke^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}$. Using Corollary 2.7 we get

$$u(t) \leq k(t) + \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} b(s)ds}k(r)b(r)dr.$$

Let $M = \int_0^{\infty} k\mu(s)e^{\gamma\alpha^{-1}(s)}ds$ and $M' = \int_0^{\infty} ke^{\gamma\alpha^{-1}(s)}\eta(s)e^{-\gamma s}ds$, then

$$u(t) \leq ku(t_0) + M + e^{M'}(ku(t_0) + M)M'.$$

Finally, we get for all $t \geq t_0$,

$$\|x(t)\| \leq k(1 + M'e^{M'})\|x(t_0)\|e^{-\gamma(t-t_0)} + e^{-\gamma t}M(1 + M'e^{M'}).$$

Using that the function : $t \mapsto e^{-\gamma t}M(1 + M'e^{M'})$ vanishes, it comes that the system (16) is uniformly practically asymptotically stable.

**Example 4.12.** We suppose that for all $(t, x)$,

$$\|g(t, x)\| \leq \alpha^{'}(t)\int_0^{\alpha(t)} \eta(s)\|x(s)\|ds,$$

with $\eta$ is a continuous nonnegative function, then (22) becomes

$$\begin{aligned} e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\alpha^{'}(s)\int_0^{\alpha(s)} \eta(\tau)\|x(\tau)\|d\tau ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{\alpha(t_0)}^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\int_0^{s} \eta(\tau)\|x(\tau)\|d\tau ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)}\int_0^{s} \eta(\tau)\|x(\tau)\|d\tau ds. \end{aligned}$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)} \int_0^s \eta(\tau)e^{-\gamma\tau}u(\tau)d\tau ds.$$

Using Theorem 2.11, we get

$$u(t) \leq ku(t_0) \exp\left(\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)} \int_0^s \eta(\tau)e^{-\gamma\tau}d\tau ds\right).$$

Finally, we get for all $t \geq t_0$,

$$\|x(t)\| \leq kM\|x(t_0)\|e^{-\gamma(t-t_0)},$$

where $M$ is the upper bound of the function : $t \mapsto \exp\left(\int_0^{\alpha(t)} ke^{\gamma\alpha^{-1}(s)} \int_0^s \eta(\tau)e^{-\gamma\tau}d\tau ds\right)$.

**Example 4.13.** We suppose that for all $(t, x)$,

$$\|g(t, x)\| \leq \omega(\|x(t)\|),$$

with $\omega$ is a function as in Theorem 3.1, then (21) becomes

$$\|x(t)\| \leq k \exp -\gamma(t - t_0)\|x(t_0)\| + \int_{t_0}^t ke^{-\gamma(t-s)}\omega(\|x(s)\|)ds.$$

Using Theorem 3.1, we get

$$\begin{aligned}
\|x(t)\| &\leq G^{-1}\left(G\left(ke^{-\gamma(t-t_0)}\|x(t_0)\|\right) + \int_{t_0}^t ke^{-\gamma(t-s)}ds\right)\\
&\leq G^{-1}\left(G\left(ke^{-\gamma(t-t_0)}\|x(t_0)\|\right) + \int_0^t ke^{-\gamma(t-s)}ds\right)\\
&\leq G^{-1}\left(G\left(ke^{-\gamma(t-t_0)}\|x(t_0)\|\right) + k\right)\\
&\leq G^{-1}\left(G\left(ke^{\gamma t_0}\|x(t_0)\|\right) + k\right).
\end{aligned}$$

**Conclusion** In this paper, some new retarded inequalities of Gronwall-type are obtained. As applications, we have considered the problem of asymptotic behaviors of a class of retarded Volterra equations and the stability of a class of dynamical systems.

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