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# APPLICATIONS OF FIXED POINT THEOREMS TO THE PROBLEM OF THE EQUILIBRIUM STRATEGY EXISTENCE PRESERVATION IN A PARAMETRIC FAMILY OF ANTAGONISTIC GAMES

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**ABSTRACT.** For a parametric family of antagonistic games with two players, the problem is considered of the equilibrium strategy existence preservation, under the changing of the parameter. This issues are considered both in the case when the players' strategy spaces are ordered sets, and in the case of metric players' strategy spaces. Some previous results are used concerning fixed point existence preservation in metric spaces, as well as similar results in ordered sets.

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## 1. SETTINGS AND PRELIMINARIES

The paper is devoted to the problem of the equilibrium strategy existence preservation in a parametric family of antagonistic games with two players. We consider this problem from the point of view of the fixed point theory of multivalued mappings. We investigate here two different situations concerning strategy spaces of the players. The first one is the case when the strategy spaces are ordered sets, and the second one is the case when the strategy spaces are metric spaces.

As it is known, the game theory considers mathematical models of conflict situation. Several applications of the fixed point theory in the game theory are described in the book [1] (see also [2] ). Recall that the participants of a conflict situation are called players. On each step of the game, their behaviour is totally defined by the strategy choice, that is by choosing a point from some set of available strategies. In the case of the game with two participants, let $X$ be the set of available strategies of the Player 1, and $Y$ be the set of available strategies of the Player 2. The game rule for Player 1 is the mapping which takes any selected strategy $y \in Y$ of Player

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2 to the set $A(y) \subseteq X$ of that best strategies from which Player 1 will choose his strategy in this case. Similarly, the game rule for Player 2 is the mapping which takes any selected strategy $x \in X$ of Player 1 to the set $B(x) \subseteq Y$ of the best answers. These rules can be mathematically described as multivalued mappings $A : Y \rightrightarrows X$, $B : X \rightrightarrows Y$.

So, we consider the antagonistic game with two players. Suppose on the product $X \times Y$ a game function $f : X \times Y \to \mathbb{R}$ is given. In this case, if Player 1 chooses a strategy $x \in X$ and Player 2 chooses a strategy $y \in Y$, the gain of Player 1 will be equal to $f(x, y)$, and the gain of Player 2 will be equal to $-f(x, y)$ where $f(x, y)$ is the game function. The game rules of Players 1 and 2 can be described in the form of the following multivalued mappings (if the corresponding minima and maxima exist):

$$B(x) = \{y | y \in Y, f(x, y) = \min_{\tilde{y} \in Y} f(x, \tilde{y})\}; \tag{1.1}$$

$$A(y) = \{x | x \in X, f(x, y) = \max_{\tilde{x} \in X} f(\tilde{x}, y)\}. \tag{1.2}$$

**Definition 1.1.** In the described situation the pair $(x_0, y_0)$ is called *an equilibrium strategy* if the following conditions hold:

$$\begin{cases} x_0 \in A(y_0); \\ y_0 \in B(x_0). \end{cases} \tag{1.3}$$

It follows from the condition 1.3 of the definition of an equilibrium strategy that the pair $(x_0, y_0)$ is equilibrium if and only if the point $(x_0, y_0)$ is a fixed point of the multivalued mapping $\mathcal{P} = A \times B : X \times Y \rightrightarrows X \times Y$, where

$$\mathcal{P}(x, y) = (A \times B)(x, y) := A(y) \times B(x). \tag{1.4}$$

It should be noticed that the described concept of an equilibrium strategy in the antagonistic game is a partial case of the well-known more general concept of *Nash equilibrium* in the game theory (see, for example, [7, 8, 10, 11]).

Now, one can see that the equilibrium strategy existence problem is equivalent to the fixed point existence problem for a multivalued mapping. Therefore, to solve this problem, people use methods of the fixed point theory.

In the next section, we consider this problem from the point of view of the fixed point theory of multivalued self-mappings of ordered sets. We use some constructions and results obtained recently in [9].

## 2. THE CASE OF ORDERED STRATEGY SPACES

We suppose that in an antagonistic game with two players, the strategy sets of the players are ordered sets.

So, let ordered strategy sets $(X, \preceq_X)$, $(Y, \preceq_Y)$ be given. We define an order $\preceq$ on the product $X \times Y$ by the following rule. For any $(x, y), (u, v) \in X \times Y$, put

$$(x, y) \preceq (u, v) \Longleftrightarrow (x \preceq_X u) \wedge (y \preceq_Y v).$$

Suppose the game rules for Players 1 and 2 are described with multivalued mappings $A : Y \rightrightarrows X$, $B : X \rightrightarrows Y$ defined by formulas 1.1 and 1.2.

We recall several definitions (see [9]).

**Definition 2.1.** A multivalued mapping $F : X \rightrightarrows Y$ between two ordered sets $(X, \preceq_X$ $), (Y, \preceq_Y)$ is called *isotone* if for any $x \in X$, $y \in F(x)$, and for any $x' \in X, x' \preceq_X x$, there is an element $y' \in F(x')$ such that $y' \preceq_Y y$.

It should be noticed that, for a single-valued mapping $f : X \to Y$, the isotone property with respect to the given orders $\preceq_X, \preceq_Y$ is equivalent to the isotone property relative to the dual orders $\preceq^*_X, \preceq^*_Y$. But for multivalued mappings these properties are clearly not equivalent. We recall that given an order $\preceq$ on some set $Z$, the order $\preceq^*$ is called dual to the order $\preceq$ if $a \preceq b \Longleftrightarrow b \preceq^* a$

Let $(Z, \preceq)$ be an ordered set. Given a multivalued mapping $G : Z \rightrightarrows Z$, following [9] we define below *special sets of chains* $\mathcal{S}(G; \preceq_X)$*,* $\mathcal{S}^*(G; \preceq_X)$*,* $\mathcal{S}(x_0, G; \preceq_X)$*, and* $\mathcal{S}^*(x_0, G; \preceq_X)$.

We denote by $\mathcal{S}(G; \preceq_X)$ the set of all pairs of the form $(S, g)$ where $S \subseteq Z$ is a chain in $Z$, $g : S \to Z$ be a single-valued selection of the mapping $G$ on the chain $S$, that is for any $z \in S$ it is true that $g(z) \in G(z)$, and the following conditions hold:

1) for any $z \in S$ it is true that $g(z) \preceq z$;

2) $\forall u, v \in S, u \prec v \Longrightarrow u \preceq g(v)$.

Similarly, we denote by $\mathcal{S}^*(G; \preceq_X)$ the set of pairs of the form $(S, g)$ where $S \subseteq X$ is a chain in $X$, $g : S \to X$ is a single-valued selection of the multivalued mapping $G$ on the chain $S$, that is for any $x \in S$ it is true that $g(x) \in G(x)$, and the following conditions hold:

1) for any element $x \in S$, $g(x) \preceq^*_X x$;

2) $\forall x, y \in S, x \prec^*_X y \Longrightarrow x \preceq^*_Y g(y)$.

Notice that $\mathcal{S}^*(G; \preceq_X) = \mathcal{S}(G; \preceq^*_X)$.

Given $x_0 \in X$, we denote $\mathcal{S}(x_0, G; \preceq_X) := \{(S, g) \in \mathcal{S}(G; \preceq_X) | S \subseteq T_{(X, \preceq_X)}(x_0)\}$, where $T_{(X, \preceq_X)}(x_0) := \{y \in X | y \preceq_X x_0\}$ Similarly, we denote $\mathcal{S}^*(x_0, G; \preceq_X) := \{(S, g) \in \mathcal{S}^*(G; \preceq_X) | S \subseteq T^*_{(X, \preceq_X)}(x_0)\}$, where $T^*_{(X, \preceq_X)}(x_0) := \{y \in X | y \preceq^*_X x_0\}$

In addition, we consider the following denotations (introduced in [9]):

$\widehat{\mathcal{S}}(x_0, G; \preceq_X) := \{(S, g) \in \mathcal{S}^*(G; \preceq_X) | g(S) \subseteq T_{(X, \preceq_X)}(x_0)\}$.

Similarly $\widehat{\mathcal{S}}(x_0, G; \preceq^*_X) := \{(S, g) \in \mathcal{S}(G; \preceq_X) | g(S) \subseteq T^*_{(X, \preceq_X)}(x_0)\}$,

The following statement on the equilibrium strategy existence is true.

**Theorem 2.2.** [9, theorem 5] *Let antagonistic game with two players be given, with ordered strategy spaces* $(X, \preceq_X)$ *and* $(Y, \preceq_Y)$*, the game rules of the players be defined by the above conditions 1.1, 1.2, and the game multivalued mapping* $\mathcal{P} : X \times Y \rightrightarrows X \times Y$ *be defined by the above formulae 1.4. Suppose that the mapping* $\mathcal{P}$ *is isotone with respect to the order* $\preceq$ *on* $X \times Y$*, and for some pair* $(a, b) \in X \times Y$*, the following conditions hold.*

*(i) There exists a pair* $(a', b') \in \mathcal{P}(a, b)$ *such that* $(a', b') \preceq (a, b)$*;*

*(ii) for any pair* $(S, P) \in \mathcal{S}((a, b), \mathcal{P}, \preceq)$ *the chain* $S$ *has a low bound* $(u, w) \in X \times Y$*, and there exists an element* $(v, q) \in \mathcal{P}(u, w)$*, such that* $(v, q) \preceq (u, w)$*, and* $(v, q)$ *is a low boundary of the chain* $P(S)$*.*

*Then the mapping* $\mathcal{P}$ *has a fixed point* $(\xi, \eta) \in T_{(X \times Y, \preceq)}(a, b) \cap Fix(\mathcal{P})$*, which is minimal in the indicated set. In other words, there exists an equilibrium strategy* $(\xi, \eta)$ *in the given game which is minimal relative to the order* $\preceq$*, among all strategies which are not greater than* $(a, b)$*.*

**Definition 2.3.** We say a multivalued mapping $G : X \rightrightarrows X$ *(orderly) covers the identical mapping* $Id_X$ on $X$ relative to the order $\preceq_X$ iff for any point $x \in X$, such that $\exists u \in G(x), x \preceq u$, there exists a point $x' \in X$, $x' \preceq x$, such that $x \in G(x')$.

The next statement is one more variant of the equilibrium strategy existence theorem.

**Theorem 2.4.** [9] *Let an antagonistic game with two players be given, with ordered strategy spaces* $(X, \preceq_X)$ *and* $(Y, \preceq_Y)$*, the game rules of the players be defined by the above conditions 1.1, 1.2, and the game multivalued mapping* $\mathcal{P} : X \times Y \rightrightarrows X \times Y$ *be defined by the above formulae 1.4. Suppose that the mapping* $\mathcal{P}$ *covers the identical mapping with respect to the order* $\preceq$ *on* $X \times Y$*, and for some initial pair* $(a, b) \in X \times Y$ *the following conditions hold.*

*(i') There exists a pair* $(a', b') \in \mathcal{P}(a, b)$ *such that* $(a, b) \preceq (a', b')$*;*

*(ii') for any pair* $(S, P) \in \widehat{\mathcal{S}}((a, b), \mathcal{P}, \preceq)$ *(* $(S, P) \in \widehat{\mathcal{S}}((a, b), \mathcal{P}, \preceq^*)$*, respectively) the chain* $S$ *has a low bound* $(u, w) \in X \times Y$ *(relative to the corresponding order), and there exists an element* $(v, q) \in \mathcal{P}(u, w)$*,* $(u, w) \preceq (v, q)$ *(respectively,* $(u, w) \preceq^* (v, q)$*), and* $(v, q)$ *is a low boundary of the chain* $P(S)$ *(relative to the corresponding order).*

*Then the mapping* $\mathcal{P}$ *has a fixed point* $(\xi, \eta) \in T_{(X \times Y, \preceq)}(a, b) \cap Fix(\mathcal{P})$ *(respectively,* $(\xi, \eta) \in T^*_{(X \times Y, \preceq)}(a, b) \cap Fix(\mathcal{P})$*) which is minimal in the indicated set. In other words, there exists an equilibrium strategy* $(\xi, \eta)$ *in the given game which is minimal relative to the order* $\preceq$*, among all strategies which are not greater than* $(a, b)$ *(relative to the corresponding dual order).*

Now, suppose that the game function $f(x, y) : X \times Y \to \mathbb{R}$ is changing. A change of the game function $f$ and the corresponding changes of the game rules of

Player 1 and Player 2 defined by the mappings $A : Y \rightrightarrows X$, $B : X \rightrightarrows Y$, imply the corresponding changes of the multivalued mapping $\mathcal{P} : X \times Y \rightrightarrows X \times Y$. In order to obtain sufficient conditions for the equilibrium strategy existence preservation, under changes of the mapping $\mathcal{P}$, one can use results of [9] concerning multivalued homotopy of mappings between ordered sets.

We need some definitions more.

**Definition 2.5.** [9] Let $(X_1, \preceq_1)$, $(X_2, \preceq_2)$ be two ordered sets. Given mappings $F, F' : X_1 \rightrightarrows X_2$, we say that $F'$ *majorizes (minorizes)* $F$ *at a point* $x \in X_1$ (with respect to the order $\preceq_2$) and write $F \nearrow F'$ ($F \searrow F'$) at $x$ if, for each $v \in F(x)$, there exists an element $u \in F'(x)$ such that $v \preceq_2 u$ ($u \preceq_2 v$). We say that $F \nearrow F'$ ($F \searrow F'$) on a subset $D \subseteq X_1$ if $F \nearrow F'$ ($F \searrow F'$) at each point $x \in D$.

Note that each of these partial binary relations $\nearrow$ and $\searrow$ (with respect to the given order $\preceq_2$ on $X_2$) on the totality of pairs of subsets (such as pairs $(F(x), F'(x)), x \in X_1,$) of $X_2$ is obviously reflexive and transitive but in general is not symmetric or antisymmetric. In addition, it is clear that the binary relations $\nearrow$ and $\searrow$ are not dual to each other. It should be also noticed that the assertion $F'$ *majorizes* $F$ *at a point* $x \in X_1$ *(on a set* $D \subseteq X_1$*) with respect to the order* $\preceq_2$ is equivalent to the assertion $F'$ *minorizes* $F$ *at the point* $x \in X_1$ *(on the set* $D \subseteq X_1$*) with respect to the dual order* $\preceq^*$, and vice versa.

**Definition 2.6.** [9] Let $(X_1, \preceq_1)$, $(X_2, \preceq_2)$ be two ordered sets. Given multivalued mappings $F, F' : X_1 \rightrightarrows X_2$, *a multivalued homotopy* connecting the mappings $F$ and $F'$ is a finite family of multivalued mappings of the form $\mathcal{H} = \{H_0, H_1, ..., H_n\}$ where $H_k : X_1 \rightrightarrows X_2, k = 0, 1, ..., n$, $H_0 = F$, $H_n = F'$, and $H_k \nearrow H_{k+1}$ either relative to the order $\preceq_{X_2}$, or relative to the dual order $\preceq^*_{X_2}$.

In order to proceed, we need to adapt some previous concepts and definitions to the game situation described above.

So, let the game function $f(x, y)$ go through discrete changes, and we have a finite sequence $F := \{f_k\}_{1 \le k \le n}$ of game functions $f_k : X \times Y \to \mathbb{R}, 1 \le k \le n$, where $f_1 = f$, $f_n = \tilde{f}$. In this sense, we have *a discrete parametric family (with parameter* $k$, $1 \le k \le n$*) of antagonistic games.*

Let the corresponding game rules of the game function $f_k$ for Players 1 and 2 be described with multivalued mappings $A_k : Y \rightrightarrows X$, $B_k : X \rightrightarrows Y$ defined by formulas 1.1 and 1.2, respectively, $1 \le k \le n$. Consequently, we have the following sequence of multivalued game mappings $\mathcal{P}_k = A_k \times B_k : X \times Y \rightrightarrows X \times Y$, where $\mathcal{P}_k(x, y) = (A_k \times B_k)(x, y) := A_k(y) \times B_k(x)$, $1 \le k \le n$.

Take a strategy $(a, b) \in X \times Y$. Denote $T_{(X \times Y, \preceq)}((a, b)) := \{(c, d) \in X \times Y | (c, d) \preceq (a, b)\}$, $T^*_{(X \times Y, \preceq)}((a, b)) := \{(u, v) \in X \times Y | (u, v) \preceq^* (a, b)\}$. One can notice that for any $(a,b) \in X \times Y$ it is true that $T^*_{(X\times Y,\preceq)}((a,b)) = T_{(X\times Y,\preceq^*)}((a,b))$ where $\preceq^*$ is the dual order relative to $\preceq$.

The next theorems 2.7 and 2.8 represent some development of the paper [9].

**Theorem 2.7.** *In the described situation, let $(x_1,y_1) \in Fix(\mathcal{P}_1) \neq \emptyset$, that is $(x_1,y_1)$ is an equilibrium strategy for the game function $f_1 = f$. In addition, suppose the following conditions (a),(b),(c) simultaneously hold either relative to the given orders $\preceq_X, \preceq_Y, \preceq_{X\times Y}$, or relative to the dual orders $\preceq^*_X, \preceq^*_Y, \preceq^*_{X\times Y}$.*

*(a) $A_k, B_k$ are isotone mappings, $1 \leq k \leq n$;*

*(b) $A_{k-1} \searrow A_k, B_{k-1} \searrow B_k$, $1 < k \leq n$;*

*(c) for any pair $(S,\mathcal{P}_k) = ((S_X,S_Y),(a_k,b_k)) \in \mathcal{S}((x_k,y_k),(A_k,B_k),\preceq_{X\times Y})$ (or, in the dual case, $(S,\mathcal{P}_k) =$*
*$= ((S_X,S_Y),(a_k,b_k)) \in \mathcal{S}((x_k,y_k),(A_k,B_k),\preceq^*_{X\times Y})$) the chain $S = (S_X,S_Y)$ has a low boundary $(u,v) \in X \times Y$, and there is an element $(w,h), w \in A_k(v), h \in B_k(u)$, $w \preceq_X u, h \preceq_Y v$ (or, in the dual case, $w \preceq^*_X u, h \preceq^*_Y v$.) In addition, the elements $w,h$ are low boundaries of the chains $a_k(S_Y), b_k(S_X)$ that is $w \preceq_X a_k(y)$, $\forall y \in S_Y$, $h \preceq_Y b_k(x)$, $\forall x \in S_X$ (or, in the dual case, $w \preceq^*_X a_k(y)$, $\forall y \in S_Y$, $h \preceq^*_Y b_k(x)$, $\forall x \in S_X$. Here $a_k : S_Y \to X$, $b_k : S_X \to Y$ stand for single-valued selections of the mappings $A_k, B_k$, respectively.*

*Then every multivalued mapping $\mathcal{P}_k$ has a fixed point $x_k, 1 \leq k \leq n$, that is every game function $f_k$, $1 \leq k \leq n$, and in particular $f_n = \tilde{f}$, has an equilibrium strategy $(x_k,y_k)$. So, the property of having equilibrium strategy is preserved.*

*Proof.* The statement is implied by theorem 2.2. The reasonings are quite similar to that of the proof of the first part of [9, theorem 4].

Let the conditions (a)–(c) of the theorem be fulfilled relative to the initial orders $\preceq_X, \preceq_Y, \preceq_{X\times Y}$.

It is not difficult to see that the condition (a) of the theorem provides that the mapping $\mathcal{P}_k$ is isotone, $1 \leq k \leq n$. As $(x_1,y_1) \in Fix(\mathcal{P}_1)$, the condition (b) implies that there is an element $(x_2,y_2) \in \mathcal{P}_2((x_1,y_1))$ such that $(x_2,y_2) \preceq (x_1,y_1)$. As the mapping $\mathcal{P}_2$ is isotone, there exists an element $(x_3,y_3) \in \mathcal{P}_2((x_2,y_2))$ such that $(x_3,y_3) \preceq (x_2,y_2)$. So, we have a chain $S = \{(x_1,y_1),(x_2,y_2)\}$ with a single-valued selection $P_2 = \{(x_2,y_2),(x_3,y_3)\}$ of the mapping $\mathcal{P}_2$ on $S$. It is easy to see that a pair $(S,P_2) \in \mathcal{S}((x_1,y_1),\mathcal{P}_2,\preceq) \neq \emptyset$.

Elements of the set $\mathcal{S}((x_1,y_1),\mathcal{P}_2,\preceq)$ are arranged by inclusion, that is we write $(S,P) \sqsubseteq (S',P')$ for two pairs $(S,P),(S',P') \in \mathcal{S}((x_1,y_1),\mathcal{P}_2,\preceq)$ if and only if the chain $S$ is an initial segment of the chain $S'$ and the selection $P'$ is a continuation of the section $P$ that is, $P'|_S = P$. By Zorn lemma applied to the partially ordered set $(\mathcal{S}((x_1,y_1),P_2,\preceq)),\sqsubseteq)$, there exists a maximal pair $(\hat{S},\hat{P}) \in \mathcal{S}((x_1,y_1),P_2,\preceq))$.

According to condition (c), the chain $\hat{S}$ has a lower bound $(u,v) \in X \times Y$, and there exists an element $(w,h) \in \mathcal{P}_2((u,v))$ such that $(w,h) \preceq (u,v)$. In addition, $(w,h)$ is a low bound of the chain $\hat{P}(\hat{S})$. If $(w,h) = (u,v)$, it means that $(u,v) \in Fix(\mathcal{P}_2)$. If $(w,h) \prec (u,v)$, then according to the condition (a), one can find a pair $(\xi,\eta) \in \mathcal{P}_2((w,h))$, such that $(\xi,\eta) \preceq (w,h)$. If $(\xi,\eta) = (w,h)$, it means again that $(w,h) \in Fix(\mathcal{P}_2)$. If $(\xi,\eta) \prec (w,h)$, then $(\hat{S},\hat{P}) \sqsubset (\bar{S},\bar{P})$, where $\bar{S} = \hat{S} \cup (\xi,\eta)$, $\bar{P}((w,h) := (\xi,\eta)$. And $(\bar{S},\bar{P}) \in \mathcal{S}((x_1,y_1),P_2,\preceq)$. This contradicts the maximality of the pair $(\hat{S},\hat{P})$.

So, one can see that the conditions of the theorem 2.7 (relative to the initial orders $\preceq_X, \preceq_Y, \preceq_{X\times Y}$) imply all conditions of theorem 2.2. By virtue of theorem 2.2, the mapping $\mathcal{P}_2$ has a fixed point. Repeating the above reasonings with respect to the mappings $\mathcal{P}_k, 2 \le k \le n$, we obtain the fixed point existence property for every mapping $\mathcal{P}_k, 2 \le k \le n$. So, for the concerned case, the theorem is proved.

For the case when the conditions (a)–(c) of the theorem are fulfilled relative to the dual orders $\preceq^*_X, \preceq^*_Y, \preceq^*_{X\times Y}$, the reasonings are quite similar. □

Basing on theorem 2.4, we present one more variant of a fixed point existence preservation theorem.

**Theorem 2.8.** *Let in the situation described above $(x_1,y_1) \in Fix(\mathcal{P}_1) \neq \emptyset$, that is $(x_1,y_1)$ is an equilibrium strategy for the game function $f_1 = f$. In addition, suppose the following conditions (a'), (b'), (c') simultaneously hold relative either to the initial orders on $X, Y, X \times Y$, or to the dual orders.*

*(a') the mapping $\mathcal{P}_k = (A_k, B_k)$ covers the identical mapping $Id_{X\times Y}$, that is for any point $(x,y) \in X \times Y$ such that $\exists u \in A(y), x \preceq u$, $\exists v \in B(x), y \preceq v$, there exists a point $(x',y') \in X \times Y$, $(x',y') \preceq (x,y)$, such that $x \in A(y'), y \in B(x')$, $1 \le k \le n$;*

*(b') $A_{k-1} \nearrow A_k, B_{k-1} \nearrow B_k$, $1 < k \le n$;*

*(c') $\forall (S,P_k) = ((S_X,S_Y),P_k) \in \widehat{\mathcal{S}}((x_1,y_1),\mathcal{P}_k,\preceq_{X\times Y})$ $(\forall (S,P_k) \in \widehat{\mathcal{S}}((x_1,y_1)\mathcal{P}_k,\preceq^*_{X\times Y}))$ the chain $S$ has a low (relative to the corresponding order) bound $(u,v) \in X \times Y$, and there is an element $(w,h), w \in A_k(v), h \in B_k(u)$, $w \preceq_X u, h \preceq_Y v$ (or $w \preceq^*_X u, h \preceq^*_Y v$.) In addition, the elements $w,h$ are low bounds of the chains $a_k(S_Y), b_k(S_X)$ that is $w \preceq_X a_k(y), \forall y \in S_Y$, $h \preceq_Y b_k(x), \forall x \in S_X$ (or, in the dual case, $w \preceq^*_X a_k(y), \forall y \in S_Y$, $h \preceq^*_Y b_k(x), \forall x \in S_X$.)*

*Then every multivalued mapping $\mathcal{P}_k$ has a fixed point $x_k, 1 \le k \le n$, that is every game function $f_k$, $1 \le k \le n$, and in particular $f_n = \tilde{f}$, has an equilibrium strategy $(x_n,y_n)$. So, the property of having equilibrium strategy is preserved.*

*Proof.* Similarly to the previous theorem, we have the equilibrium strategy $(x_1,y_1)$ of the game function $f_1$ which is a fixed point of the mapping $\mathcal{P}_1$. Let all conditions of theorem 2.8 be fulfilled relative to the initial orders $\preceq_X, \preceq_Y, \preceq_{X\times Y}$. Then, the condition (a') of theorem 2.8 clearly coincides with the corresponding condition of theorem 2.4, for the mapping $\mathcal{P}_2$. The condition (b') of theorem 2.8 provides the condition (i') of theorem 2.4, for $\mathcal{P}_2$. And finally, it is easy to see that the condition (c') of theorem 2.8 coincides with the condition (ii') of theorem 2.4, for $\mathcal{P}_2$. So, by virtue of theorem 2.4, the mapping $\mathcal{P}_2$ has a fixed point $(x_2, y_2)$. Repeating the above reasonings with respect to every mapping $\mathcal{P}_k, 2 \leq k \leq n$, we prove that every mapping $\mathcal{P}_k$ has a fixed point $(x_k, y_k)$, $1 \leq k \leq n$. In means that in the case of initial orders, theorem is proved.

As for the case when all conditions of theorem 2.8 hold relative to the dual orders, one can similarly obtain that all conditions of theorem 2.4 are also fulfilled relative to the dual orders, for $\mathcal{P}_k$, $1 \leq k \leq n$. Consequently, by virtue of theorem 2.4, every mapping $\mathcal{P}_k$ has a fixed point $(x_k, y_k)$ that is every game function $f_k(x, y)$ has an equilibrium strategy $(x_k, y_k)$. $\square$

**Remarks.**

1. Using remarks after [9, theorem 4], one can specify the minimality (maximality) property of the fixed points $(x_k, y_k)$ obtained in theorems 2.7 and 2.8, with respect to the whole set of fixed points of the mappings $\mathcal{P}_k$ contained in $T_X(x_{k-1})$ (or respectively, in $T^*_X(x_{k-1})$), $1 \leq k \leq n$.

2. The conditions of theorems 2.7 and 2.8 may be combined, for different values of $k$. Therefore, it follows that the property of having an equilibrium strategy is preserved for a game function $f(x, y)$, under any order homotopy which meets the conditions of theorems 2.7 and 2.8 or their combinations.

In the next section we shall consider the same problem of the equilibrium strategy existence preservation in an antagonistic game with two players, but in quite another situation.

## 3. THE CASE OF METRIC STRATEGY SPACES

Below we shall suppose that the players' strategy spaces are metric spaces. To investigate such a game situation, we shall use some constructions and results obtained in [12], basing on the zero search principle for multivalued $(\alpha, \beta)$-search functionals introduced earlier by the author.

At first, we present some necessary definitions and the formulation of the above-mentioned search principle for zeros (see [3, 4, 5, 6]).

**Definition 3.1.** Let $(X, d)$ be a metric space, and let $\varphi : X \rightrightarrows \mathbb{R}_+$ be a set-valued functional on $X$. The graph $Graph(\varphi) := \{(x, c) \in X \times \mathbb{R}_+ | c \in \varphi(x)\}$ of the functional $\varphi$ is said to be 0-*complete* iff any Cauchy sequence $(x_n, c_n) \subseteq Graph(\varphi)$ such that $c_n \to 0$, converges to some element $(\xi, 0) \in Graph(\varphi)$, i.e., $0 \in \varphi(\xi)$.

Now, let us consider the notion of an $(\alpha, \beta)$-search functional.

**Definition 3.2.** Let $(X, d)$ be a metric space, and let $0 \leq \beta < \alpha$. A set-valued functional $\varphi : X \rightrightarrows \mathbb{R}_+$ is called $(\alpha, \beta)$-*search on* $X$ if and only if, for any point $x \in X$ and any $c \in \varphi(x)$, there exists a point $x' \in X$ and a number $c' \in \varphi(x')$ such that $d(x, x') \leq \frac{c}{\alpha}$, and $c' \leq \frac{\beta}{\alpha}$.

Here we need the more general notion of a functional which is $(\alpha, \beta)$-search on a given open subset of the metric space $(X, d)$.

**Definition 3.3.** [12] Let $(X, d)$ be a metric space, $U \subset X$ be an open subset, and $0 \leq \beta < \alpha$. A set-valued functional $\varphi : U \rightrightarrows \mathbb{R}_+$ is called an $(\alpha, \beta)$-*search on* $U$ iff, for any point $x \in U$ and any $r > 0, c \in \varphi(x)$ such that $\overline{B(x; r)} \subset U$, $c \leq (\alpha - \beta)r$, there exist a point $x' \in B(x; \frac{c}{\alpha})$ and a value $c' \in \varphi(x')$ with $c' \leq \frac{\beta}{\alpha}c$.

**Theorem 3.4.** [12] *Let $(X, d)$ be a metric space, $U \subset X$ be an open subset, and $\varphi : U \rightrightarrows \mathbb{R}_+$ be a set-valued functional, which is an $(\alpha, \beta)$-search on $U$, with $0$-complete graph, $0 \leq \beta < \alpha$. Let there be given $x_0 \in U, c_0 \in \varphi(x_0)$, and $r > 0$ such that $\overline{B(x_0, r)} \subset U$; $c_0 \leq (\alpha - \beta)r$.*

*Then there exists a point $\xi \in \overline{B(x_0, r)}$ for which $0 \in \varphi(\xi)$.*

**Definition 3.5.** [12]. Let $(X, d)$ be a metric space, $U \subset X$. Let $\theta : [0; 1] \to \mathbb{R}$ be a continuous increasing function. A one-parameter family $\Phi = \{\Phi_t : U \rightrightarrows \mathbb{R}_+\}_{t \in [0;1}$, of set-valued functionals is said to be $\theta$-*continuous on* $U$ iff, for each $x \in U$, any $t', t'' \in [0; 1]$, and any $c' \in \Phi_{t'}(x)$, there exists a value $c'' \in \Phi_{t''}(x)$ for which $|c' - c''| \leq |\theta(t') - \theta(t'')|$.

We introduce the following notation, for any subsets $Z$ and $Y$, $Z \subseteq Y \subseteq X$, and any family $\Phi = \{\Phi_t : Y \rightrightarrows \mathbb{R}_+\}_{t \in [0;1}$ of set-valued functionals:

$$M_Z(\Phi) := \{(x, t) \in Z \times [0; 1] | 0 \in \Phi_t(x)\}.$$

On the space $X \times [0; 1]$ (in particular, on $U \times [0; 1]$), we consider the metric $D : (X \times [0; 1])^2 \to \mathbb{R}_+$ defined by the rule:

$$D((x', t'), (x'', t'')) = d(x', x'') + |t' - t''|,$$

for all $x', x'' \in X$ and any $t', t'' \in [0; 1]$. Convergence in this metric is obviously equivalent to component-wise convergence.

**Theorem 3.6.** [12] *Let $(X, d)$ be a metric space, $U \subset X$ be an open subset of $X$, and $\theta : [0; 1] \to \mathbb{R}$ be a continuous increasing function. Let there be given a parametric family $\Phi = \{\Phi_t : \overline{U} \rightrightarrows \mathbb{R}_+\}_{t \in [0;1]}$ of set-valued functionals with $0$-complete graphs, $\theta$-continuous on the set $M_U(\Phi)$, which are $(\alpha, \beta)$-search on $U$, for any $t \in [0; 1]$. Let also the set $M_U(\Phi)$ be closed.*

*Then, if there exists an element of the form* $(x_0, 0) \in M_U(\Phi)$*, then there exists an element of the form* $(x_1, 1) \in M_U(\Phi)$*.*

Now, we can apply theorem 3.6 to the problem under consideration.

Recall that we consider antagonistic games with two players and suppose now that the strategy spaces of Player 1 and Player 2 are metric spaces.

Let a parametric family of antagonistic game functions $F = \{f_t\}_{t\in[0;1]}$ be given, where $f_t(x, y) : X \times Y \to \mathbb{R}$. So, we have a parametric family of game mappings $\mathcal{P} = \{\mathcal{P}_t\}_{t\in[0;1]}$, where the mapping $\mathcal{P}_t$ is defined by the above condition 1.4, $\forall t \in [0; 1]$.

Let us notice that as opposed to the previous case (with a discrete parameter) considered in section 2, here we consider *a parametric family of antagonistic games with the continuous real parameter* $t \in [0; 1]$

Let $U \subset X \times Y$ be an open subset. We are to find sufficient conditions for the fixed point existence preservation in the given family $\mathcal{P} = \{\mathcal{P}_t\}_{t\in[0;1]}$ of multivalued game mappings, under changing the parameter, on the open subset $U$.

We recall that the game rules of the players can be described in the form of the multivalued mappings $A : Y \rightrightarrows X$, $B : X \rightrightarrows Y$, defined by the above formulas 1.1, 1.2. An equilibrium strategy of the game is a point $(x_0, y_0)$ which is a fixed point of the multivalued game mapping $\mathcal{P} = A \times B : X \times Y \rightrightarrows X \times Y$ defined by the above formulas 1.3, 1.4.

Below we denote $M_U(\Phi) := \{((x, y), t) \in X \times Y \times [0; 1] | (x, y) \in \Phi_t(x, y)\}$.

The following statement is true.

**Theorem 3.7.** *Let the strategy spaces* $(X, d_X), (Y, d_Y)$ *be complete,* $U \subset X \times Y$ *be an open subset. Suppose a parametric family of game mappings* $\mathcal{P} = \{\mathcal{P}_t\}_{t\in[0;1]}$ *be given where* $\mathcal{P}_t : \overline{U} \to CB(X \times Y)$ *(here* $CB(X \times Y)$ *stands for the totality of closed bounded subsets of* $X \times Y$*). Let, for some* $\alpha, \beta$, $0 \le \beta < \alpha < 1$*, and* $1 < q < \alpha/\beta$*, and some continuous increasing function* $\theta : [0; 1] \to \mathbb{R}$*, the following conditions hold (below* $H(U, V)$ *stands for the Hausdorff distance between subsets* $U$ *and* $V$*):*

*(1) for any* $t \in [0; 1]$ *and any* $(x', y'), (x'', y'') \in \overline{U}$*,*

$$H(\mathcal{P}_t(x', y'), \mathcal{P}_t(x'', y'')) \le \frac{\beta}{\alpha\gamma q} D((x', y'), (x'', y''));$$

*(2) for any pair* $((x, y), t) \in M_U(\Phi)$*, and any* $t' \in [0; 1]$*,*

$$H(\mathcal{P}_t(x, y), \mathcal{P}_{t'}(x, y)) \le |\theta(t') - \theta(t)|;$$

*(3) for any* $t \in [0; 1]$*, there are no fixed points of the mapping* $\mathcal{P}_t$ *on the boundary of* $U$ *that is* $Fix(\mathcal{P}_t) \cap \partial U = \emptyset$*, where* $Fix(\mathcal{P}_t) := \{(x, y) \in \overline{U} | (x, y) \in \mathcal{P}_t((x, y))\}$*.*

*Then,* $Fix(\mathcal{P}_0) \cap U \ne \emptyset \Longrightarrow Fix(\mathcal{P}_1) \cap U \ne \emptyset$*.*

*Proof.* for each $t \in [0;1]$, we consider on $\overline{U}$ the set-valued functional $\Phi_t : \overline{U} \rightrightarrows \mathbb{R}_+$ defined by the rule: $\Phi_t((x,y)) = \{D((x,y),(u,v)) | (u,v) \in \mathcal{P}_t((x,y)))\}$. The choice of a concrete value $c \in \Phi_t((x,y))$ means the choice of an element $(u,v) \in \mathcal{P}_t((x,y))$. Then, $0 \in \Phi_t((x,y))$ if and only if $(x,y) \in \mathcal{P}_t((x,y))$.

Let us show that, for any $t \in [0;1]$, the functional $\Phi_t$ has 0-complete graph. Let $((x_n,y_n),c_n) \subseteq Graph(\Phi_t)$ be an arbitrary Cauchy sequence, where $c_n \to 0$. Each $c_n$ corresponds to some $(u_n,v_n) \in \mathcal{P}_t((x_n,y_n))$ such that $c_n = D((x_n,y_n),(u_n,v_n))$. As $c_n \to 0$ the sequences $\{(x_n,y_n)\}$ and $\{(u_n,v_n)\}$ draw together, and it follows that the sequence $\{(u_n,v_n)\}$ is also Cauchy. As the space $(X \times Y, D)$ is complete, the sequence $\{(x_n,y_n)\}$ converges to some element $(\xi,\eta) \in X \times Y$. Consequently $(u_n,v_n) \to (\xi,\eta)$, too.

It remains to show that $(\xi,\eta) \in \mathcal{P}_t((\xi,\eta))$ that is $(\xi,\eta) \in Fix(\mathcal{P}_t)$.

One can estimate the distance $D((\xi,\eta),\mathcal{P}_t((\xi,\eta)))$. Using the triangle inequality and condition (1) of the theorem we obtain

$$D((\xi,\eta),\mathcal{P}_t((\xi,\eta))) \leq D((\xi,\eta),(u_n,v_n)) + D((u_n,v_n),\mathcal{P}_t((x_n,y_n))+$$

$$+H(\mathcal{P}_t((x_n,y_n)),\mathcal{P}_t((\xi,\eta))) \leq D((\xi,\eta),(u_n,v_n)) + \frac{\beta}{\alpha\gamma q} D((x_n,y_n),(\xi,\eta)) \to 0.$$

So, $D((\xi,\eta),\mathcal{P}_t((\xi,\eta))) = 0$. As the images of $\mathcal{P}_t$ are closed, it means that $(\xi,\eta) \in \mathcal{P}_t((\xi,\eta))$. Thus, for any $t \in [0;1]$, the graph of the functional $\Phi_t$ is $\{0\}$-complete.

Let us show that, for any $t \in [0;1]$, the functional $\Phi_t$ is an $(\alpha,\beta)$-search on $\overline{U}$. Let $(x,y) \in \overline{U}, c \in \Phi_t((x,y)), r > 0$ be such that $\overline{B((x,y),r)} \subset U$ and $c \leq (\alpha-\beta)r$. The value $c$ corresponds to the element $(x,y)$ and some element $(x',y') \in \mathcal{P}_t((x,y))$, for which $c = D((x,y),(x',y'))$. We notice that the point $(x',y')$ (trivially) meets the condition $D((x,y),(x',y')) \leq \frac{1}{\alpha}c$. In addition, $(x',y') \in U$ because $D((x,y),(x',y')) \leq \frac{1}{\alpha}c < r$ and $\overline{B((x,y),r)} \subset U$.

Further, it is not difficult to show that as the sets $\mathcal{P}_t((x,y)), \mathcal{P}_t(x',y'))$ are closed and bounded, there exists an element $(x'',y'') \in \mathcal{P}_t((x',y'))$ such that

$$D((x',y'),(x'',y'')) \leq qH(\mathcal{P}_t((x,y)),\mathcal{P}_t((x',y'))) \leq q\frac{\beta}{\alpha q} D((x,y),(x',y')) = \frac{\beta}{\alpha}c.$$

So, taking $c' = D((x',y'),(x'',y''))$, we have $c' \leq (\beta/\alpha)c$.

Thus, for any $t \in [0;1]$, the functional $\Phi_t$ is $(\alpha,\beta)$-search on $\overline{U}$.

Now we verify that the condition (2) of the theorem implies that the functional $\Phi_t$ is $q\theta$-continuous on the set $M_U(\Phi)$. Take any pair $((x,y),t) \in M_U(\Phi)$ and any $t' \in [0;1]$. Then there exists an element $(x',y') \in \mathcal{P}_{t'}(x,y)$ such that $D((x,y),(x',y'))) \leq qH(\mathcal{P}_t(x,y),\mathcal{P}_{t'}(x,y))$. Using the condition (2) of the theorem and taking $c' = D((x,y),(x',y'))$, we have

$$c' \leq qH(\mathcal{P}_t(x,y),\mathcal{P}_{t'}(x,y)) \leq q|\theta(t') - \theta(t)|.$$

It means that the family $\Phi = \{\Phi_t\}_{t\in[0;1]}$ is $q\theta$-continuous on the set $M_U(\Phi)$.

Finally, one can show that as there are no fixed points of $\Phi_t$, $t \in [0;1]$, on the boundary $\partial U$, the set $M_U(\Phi)$ is closed (see [12] for the proof).

Thus, all conditions of theorem 3.6 are fulfilled. By virtue of theorem 3.6, if there exists an element of the form $((x_0, y_0), 0) \in M_U(\Phi)$, then there exists an element of the form $((x_1, y_1), 1) \in M_U(\Phi)$. As the set of zeros of the functional $\Phi_t$ coincides with the fixed point set of the mapping $\mathcal{P}_t$, $\forall t \in [0;1]$, it completes the proof. $\square$

It should be noticed that in fact theorem 3.7 may be considered as a consequence of the more general statements (see [12, theorem 7, corollary 1, theorem 8]). But, such a consequence was not explicitly formulated in [12]. Therefore we presented it here with the detailed proof.

## 4. CONCLUSION

In this paper, we considered the problem of the equilibrium strategy existence preservation, under the changing the parameter, in a parametric antagonistic game with two players. We investigated this problem from the point of view of fixed point theory of multivalued mappings. These issues were considered both in the case when the players' strategy spaces are ordered sets and the parameter possesses the values in the discrete finite set $\{1, 2, ..., n\}$, and in the case of metric players' strategy spaces when the parameter possesses any value $t \in [0;1]$. We essentially used some previous results. In the case of ordered strategy spaces, the main results are above theorems 2.3 and 2.4. For the case of metric strategy spaces our main result is theorem 3.7. In all these main theorems we present sufficient conditions for the equilibrium strategy existence preservation, under changing the parameter, in a parametric antagonistic game with two players.

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