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WELL-POSEDNESS AND STABILITY OF A RETARDED IMPULSIVE WAVE EQUATION WITH A FINITE MEMORY

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ABSTRACT. Of concern is a wave equation which takes into account a discrete time delay in the state itself (and not in its time derivative). It is also subject to impulses and a control given by a finite memory term. We prove the well-posedness and exponential stability of our system. Our stability result shows that the damping effect of the finite memory term is not destroyed neither by the impulses nor by the delay.

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1. Introduction

Many researchers are devoting time and efforts in understanding the impact of impulses in different processes. As a matter of fact, these impacts cannot be anticipated or predicted. They may be the cause of stability as they may be the cause of instability. They may arise naturally and they may be used as controls to drive the system to a desirable terminal state. The applications are numerous [2, 3, 11, 16] and the mathematical challenges are considerable. Therefore, the study of these processes are of the upmost importance nowadays. They are studied, together with many similar problems dealing with short perturbations, in the context of Impulsive Differential Equations.

Let $N \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary Γ and a closure $\overline{\Omega}$, a_0, $\tau > 0$, $a_1 \in \mathbb{R}$, $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+ := [0, +\infty)$ such that

$$(1.1) \qquad 0 < t_0 < t_1 < \cdots < t_k < \cdots, \quad \inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} > \tau \quad \text{and} \quad \lim_{k\to+\infty} t_k = +\infty,$$

$g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying, for some $\xi > 0$,

$$(1.2) \qquad g \in C^1(\mathbb{R}_+), \quad 0 < g_0 := \int_0^{+\infty} g(s)ds < a_0 \quad \text{and} \quad g' \leq -\xi g,$$

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and $g_k,\ f_k : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N}$, satisfying, for some $\xi_k,\ \tilde{\xi}_k > 0$,

$$g_k \in C^1(\mathbb{R}_+), \quad \sup_{k\in\mathbb{N}} \xi_k < +\infty, \quad g_k(0) = 0 \quad \text{and} \quad |g_k'| \le \xi_k \tag{1.3}$$

and

$$f_k \in C(\mathbb{R}_+), \quad \sup_{k\in\mathbb{N}} \tilde{\xi}_k < +\infty \quad \text{and} \quad |f_k(s)| \le \tilde{\xi}_k |s|, \quad s \in \mathbb{R}. \tag{1.4}$$

We consider the following system:

$$\begin{cases} u_{tt}(x,t) - a_0\Delta u(x,t) + a_1 u(x,t-\tau) \\ \quad + \displaystyle\int_0^t g(s)\Delta u(x,t-s)ds = 0, & x \in \Omega,\ t \in \mathbb{R}_+^* \backslash \{t_k\}_{k\in\mathbb{N}}, \\ u(x,t) = 0, & x \in \Gamma,\ t > 0, \\ u(x,s) = u_0(x,s),\ u_t(x,0) = u_1(x), & x \in \overline{\Omega},\ s \in [-\tau, 0], \\ u(x,t_k) = g_k(u(x,t_k^-)),\ u_t(x,t_k) = f_k(u_t(x,t_k^-)), & x \in \overline{\Omega},\ k \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $u(x,t)$ is the unknown function, u_0 and u_1 are given functions (initial data), Δ is the classical Laplacian operator, the subscript t denotes the derivative with respect to t, and t_k^- denotes the limit when t converges to t_k from the left. The function g is the kernel of the considered finite memory term, which plays the role of control for (1.5). The constant τ represents the considered discrete time delay with a size a_1. The impulses are taken in consideration thanks to the functions g_k and f_k. In case of continuity and $g_k(s) = f_k(s) = s$, there will be no impulses.

So far as we know, this problem has not been investigated in the literature. We cite below the very few papers on this subject but without impulsive conditions. We start by the work of Nicaise and Pignotti [13] who proved exponential stability for the problem

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) + \eta(x)\left[a_1 w_t(x,t) + a_2 w_t(x,t-\tau)\right] = 0, & x \in \Omega,\ t > 0 \\ w(x,t) = 0, & x \in \Gamma_D,\ t > 0, \\ \frac{\partial w}{\partial \nu}(x,t) = 0, & x \in \Gamma_N,\ t > 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in \Omega, \\ w_t(x,t-\tau) = w_2(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases}$$

and also for the problem where the control

$$a_1 w_t(x,t) + a_2 w_t(x,t-\tau)$$

acts on the boundary. The main condition was that the amplitude of the delayed term must be strictly smaller than the one of the frictional damping. Otherwise, the system is shown to be unstable. By this work, they extended an earlier work of Xu et al. [17] from 1-d to any dimension. Time-varying delays have been treated in [14, 15]. There are other works dealing with the case where one of the terms is in the equation and the other one in the boundary as in [13], we mention only the work of Datko et al. [6]. The authors considered the internal terms

$$2bw_t(x,t) + b^2 w(x,t)$$

and the boundary delayed control

$$w_x(1,t) + \eta w_t(1,t-\tau) = 0$$

and proved an exponential stability result when η is small enough.

Now we pass to the case of a memory damping. We mention here that Kirane and Said-Houari [10] added a viscoelastic term to the equation

$$w_{tt}(x,t) - \Delta w(x,t) + a_1 w_t(x,t) + a_2 w_t(x,t-\tau) + \int_0^t g(s)\Delta u(x,t-s)ds = 0$$

and proved a stability result when $0 \leq a_2 < a_1$ under a Dirichlet boundary condition. Moreover, they showed that the exponential stability continues to hold when $a_2 = a_1$ because of the memory dissipation. A couple of years later, Guesmia [8] investigated a similar problem but without the frictional damping and with infinite memory in the abstract setting

$$\begin{cases} w_{tt}(t) + Aw(t) = \displaystyle\int_0^{+\infty} h(s)Aw(t-s)ds - aw_t(t-\tau), & t > 0, \\ w(-t) = w_0(t), & t \in \mathbb{R}_+, \\ w_t(0) = w_1,\ w_t(t-\tau) = w_2(t-\tau), & t \in (0,\tau). \end{cases}$$

He proved the existence of a positive number λ such that the system is exponentially stable when $|a| < \lambda$. Then, Alabau-Boussouira et al. [1] considered the problem

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t) - \displaystyle\int_0^{\infty} h(s)\Delta w(x,t-s)ds - aw_t(x,t-\tau), & x \in \Omega,\ t > 0, \\ w(x,t) = 0, & x \in \Gamma,\ t > 0, \\ w(x,t) = w_0(x,t), & x \in \Omega,\ t \in (-\infty, 0] \end{cases}$$

and proved a similar result with an explicit estimate for the value of λ. In [9], Guesmia and Tatar proved an exponential stability (and also arbitrary stability depending on the kernel of the viscoelasticity) for the abstract problem

$$\begin{cases} w_{tt}(t) + Aw(t) = \displaystyle\int_0^{\infty} h(s)Bw(t-s)ds - \int_0^{\infty} k(s)w_t(t-s)ds, & t > 0, \\ w(-t) = w_0(t), & t \in \mathbb{R}_+, \\ w_t(-t) = w_1(t), & t \in \mathbb{R}_+. \end{cases}$$

This is established under certain conditions on the operators A and B and the kernels h and k, and provided that

$$\int_0^{\infty} k(s)ds < \mu,$$

for some positive constant μ. Here a distributed delay is considered generalizing the discrete delay.

We note here that the above problems without delays have been treated earlier in a good number of papers. We refer the readers to the references in the above papers where many of them are cited there. All the above results have been shown despite the damaging and harmful character of delays in general [4, 5, 6, 12].

In this work, for problem (1.5), we assume the growth conditions (1.3) and (1.4) on the impulses g_k and f_k, respectively. An exponential stability result will be proved for a certain range of values of a_1. This means that the dissipation effect of the memory damping resists to both the impulses as well as the delay occuring in the state. To this end, we shall combine the multiplier technique with an impulsive version of the Halanay inequality (see Lemma 3.3 below).

In the next section, we prove the well-posedness of our problem. Section 3 is devoted to the statement and proof of our stability result which relies crucially on an impulsive version of the well-known Halanay inequality. We end up the paper by few remarks.

2. Well-posedness

In this paper, we shall state and prove our stability result together with the crucial impulsive version of the Halanay inequality. Frequently in the sequel, C (resp. C_ϵ) denotes a generic positive constant (resp. depending on some $\epsilon > 0$), which may be different from step to step. We use $\|\cdot\|$ to denote both $L^2(\Omega)$ and $\left(L^2(\Omega)\right)^N$ norms.

This section is devoted to the well-posedness issue. We shall start by proving the existence and uniqueness of a solution to the problem without impulses, then use this result to establish a similar result for the impulsive case. We introduce the traditional function

$$z(x,\rho,t) = u(x,t-\rho\tau),\ x\in\Omega,\ \rho\in(0,1),\, t>0.$$

Obviously, this new function satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x\in\Omega,\ \rho\in(0,1),\, t>0.$$

At $t=0$, we denote

$$z_0(x,\rho) = z(x,\rho,0) = u(x,-\rho\tau).$$

We first consider the problem without impulses

(2.1)
$$\begin{cases} w_{tt}(x,t) - a_0\Delta w(x,t) + a_1 w(x,t-\tau) + \displaystyle\int_0^t g(s)\Delta w(x,t-s)ds = 0, & x\in\Omega,\, t>0,\\ w(x,t)=0, & x\in\Gamma,\, t>0,\\ w(x,s)=w_0(x,s),\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x), & x\in\overline{\Omega},\, s\in[-\tau,0]. \end{cases}$$

Theorem 2.1. *Let $w_0\in H_0^1(\Omega)$, $w_1\in L^2(\Omega)$ and $z_0\in L^2(\Omega\times(0,1))$. For any $T>0$, there exists a unique weak solution (w,z) of (2.1) fulfilling*

$$w\in C\left([0,T],H_0^1(\Omega)\right)\cap C^1\left([0,T],H_0^1(\Omega)\right). \tag{2.2}$$

Proof. We give here the sketch of the proof. We shall adopt the Faedo-Galerkin method with the appropriate changes and modifications in line with the new features of our problem. We denote by $\{v_\kappa\}_{\kappa\in\mathbb{N}^*}$ a basis of the space $H_0^1(\Omega)$ and

$$V_n = span\{v_1,...,v_n\},\quad n\in\mathbb{N}^*.$$

For the space $L^2(\Omega\times(0,1))$, we select a basis whose first n elements $\chi_1(x,\rho),...,\chi_n(x,\rho)$ in $L^2(\Omega\times(0,1))$ span the space Z_n and such that $\chi_j(x,0)=v_j(x)$, $j=1,...,n$.

Let $\{w_{0n}\}_{n\in\mathbb{N}^*}$ and $\{w_{1n}\}_{n\in\mathbb{N}^*}$ be two sequences in V_n, and $\{z_{0n}\}_{n\in\mathbb{N}^*}$ be a sequence in Z_n such that

$$\begin{cases} w_{0n}\to w_0 \text{ strongly in } H_0^1(\Omega),\\ w_{1n}\to w_1 \text{ strongly in } L^2(\Omega),\\ z_{0n}\to z_0 \text{ strongly in } L^2(\Omega\times(0,1)). \end{cases}$$

We consider the expressions

$$w_n(x,t)=\sum_{j=1}^n \omega_{jn}(t)v_j(x)\quad\text{and}\quad z_n(x,\rho,t)=\sum_{j=1}^n \xi_{jn}(t)\chi_j(x,\rho)$$

solutions of the finite dimensional problems

$$\begin{cases} \displaystyle\int_\Omega w_{ntt}v_j dx + a_0\int_\Omega \nabla w_n\nabla v_j dx + a_1\int_\Omega z_n(x,1,t)v_j dx - \int_0^t g(t-s)\int_\Omega \nabla w_n\nabla v_j dx ds = 0,\\ w_n(0)=w_{0n},\ w_{nt}(0)=w_{1n} \end{cases}$$

and

$$\begin{cases} \int_\Omega (\tau z_{nt} + z_{n\rho})\,\chi_j dx = 0, \\ z_n(x,\rho,0) = z_{0n}(x,\rho),\ z_n(x,0,t) = w_n(x,t), \end{cases}$$

respectively. Obviously, these problems admit $(\omega_{jn}(t), \xi_{jn}(t))$ as solutions over the intervals $[0, T_n)$. Next, it will be shown that in fact these T_n are equal to T.

Using a suitable multiplier, the identity

$$(u * v)_t\,(t) = -\frac{1}{2}\frac{d}{dt}\left[(u\Box v)\,(t) - \left(\int_0^t u(s)ds\right)|v(t)|^2\right] - \frac{1}{2}u(t)\,|v(t)|^2 + \frac{1}{2}\,(u_t\Box v)\,(t),$$

where $*$ is for the usual convolution and

$$(u\Box v)\,(t) = \int_0^t u(t-s)\,|v(t)-v(s)|^2\,ds,$$

and integration by parts, we end up with

(2.3)
$$\frac{1}{2}\left[\|w_{nt}\|^2 + \left(a_0 - \int_0^t g(s)ds\right)\|\nabla w_n\|^2 + \int_\Omega (g\Box\nabla w_n)\,dx\right] + a_1\int_0^t\int_\Omega z_n(x,1,s)w_{nt}(x,s)dxds$$
$$+\frac{1}{2}\int_0^t g(s)\,\|\nabla w_n(s)\|^2\,ds - \frac{1}{2}\int_0^t\int_\Omega (g'\Box\nabla w_n)\,dxds = \frac{1}{2}\left[a_0\,\|\nabla w_{0n}\|^2 + \|w_{1n}\|^2\right]$$

and

$$\frac{\tau}{2}\int_0^1\int_\Omega z_n^2(x,\rho,t)dxd\rho + \int_0^1\int_0^t\int_\Omega z_{n\rho}(x,\rho,s)z_n(x,\rho,s)dxdsd\rho = \frac{\tau}{2}\,\|z_{0n}\|^2_{L^2(\Omega\times(0,1))}. \tag{2.4}$$

We denote by $E_n(t)$ the expression

$$E_n(t) = \frac{1}{2}\left[\|w_{nt}\|^2 + \left(a_0 - \int_0^t g(s)ds\right)\|\nabla w_n\|^2 + \int_\Omega (g\Box\nabla w_n)\,dx\right] + \frac{1}{2}\,\|z_n\|^2_{L^2(\Omega\times(0,1))}.$$

Then, by virtue of the two identities (2.3) and (2.4) and the remark

$$\int_0^1\int_0^t\int_\Omega z_{n\rho}(x,\rho,s)z_n(x,\rho,s)dxdsd\rho = \frac{1}{2}\int_0^t\int_\Omega \left[z_n^2(x,1,s) - z_n^2(x,0,s)\right]dxds,$$

we may write

$$E_n(t) + \frac{1}{2}\int_0^t g(s)\,\|\nabla w_n(s)\|^2\,ds - \frac{1}{2}\int_0^t\int_\Omega (g'\Box\nabla w_n)\,dxds - \frac{1}{2\tau}\int_0^t \|w_n\|^2\,ds$$
$$+\frac{1}{2\tau}\int_0^t\int_\Omega z_n^2(x,1,s)dxds + a_1\int_0^t\int_\Omega z_n(x,1,s)w_n(x,s)dxds \le E_n(0).$$

Young inequality implies that

$$E_n(t) + \frac{1}{2}\int_0^t g(s)\,\|\nabla w_n(s)\|^2\,ds - \frac{1}{2}\int_0^t\int_\Omega (g'\Box\nabla w_n)\,dxds - \left(\frac{1}{2\tau} - |a_1|\delta\right)\int_0^t\int_\Omega z_n^2(x,1,s)dxds$$
$$\le E_n(0) + \left(\frac{1}{2\tau} + \frac{|a_1|}{4\delta}\right)\int_0^t \|w_n\|^2\,ds,$$

for $\delta > 0$ which we choose satisfying $\frac{1}{2\tau} - |a_1|\delta > 0$. To fix ideas, take $\delta = \frac{1}{4\tau|a_1|}$. It is clear now that we can appeal to Gronwall inequality to deduce that, for some $C_1 > 0$,

$$E_n(t) \le E_n(0)e^{C_1 t}, \quad t \ge 0.$$

If $E_n(0) \le r$, for some $r > 0$, then we can find $C > 0$ depending on T, $\tau, a_1,...$ such that

$$\|w_{nt}\| \le C.$$

The rest of the proof is standard (see, for instance Theorem 3.1 in [10]). □

Let H be a Hilbert space. We introduce the spaces

$$PC(\mathbb{R}_+, H) = \left\{ \begin{array}{c} v \in C(\mathbb{R}_+ \backslash \{t_k\}_{k\in\mathbb{N}}, H),\ v(x,t_k^+),\ v(x,t_k^-)\,\text{exist} \\ \text{and}\, v(x,t_k) = v(x,t_k^+) = g_k(v(x,t_k^-)),\ k \in \mathbb{N} \end{array} \right\}$$

and

$$PC^1(\mathbb{R}_+, H) = \left\{ \begin{array}{c} v \in C^1(\mathbb{R}_+ \backslash \{t_k\}_{k\in\mathbb{N}}, H),\ v(x,t_k^+),\ v(x,t_k^-),\ v_t(x,t_k^+),\ v_t(x,t_k^-)\,\text{exist,} \\ v(x,t_k) = v(x,t_k^+) = g_k(v(x,t_k^-))\,\text{and}\, v_t(x,t_k) = v_t(x,t_k^+) = f_k(v_t(x,t_k^-)),\ k \in \mathbb{N} \end{array} \right\}.$$

Theorem 2.2. *For any*

$$(u_0(\cdot,0), u_1, z_0) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)), \tag{2.5}$$

the problem (1.5) *admits a unique weak solution*

$$u \in PC(\mathbb{R}_+, H_0^1(\Omega)) \cap PC^1(\mathbb{R}_+, L^2(\Omega)). \tag{2.6}$$

Proof. We proceed in several steps.

Step 1: Let w_0 be the restriction on $[0, t_1)$ of the solution of (2.1) corresponding to the initial data (2.5). Thanks to (2.2), it is easy to see that

$$\left(w_0(\cdot, t_1^-), \frac{\partial w_0}{\partial t}(\cdot, t_1^-), w_0(\cdot, t_1 - \tau p) \right) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)).$$

Step 2: According to (1.3) and (1.4), we have

$$\left(g_1(w_0(\cdot, t_1^-)), f_1\left(\frac{\partial w_0}{\partial t}(\cdot, t_1^-) \right), z_1 \right) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)), \tag{2.7}$$

where $z_1(x,p) = w_0(x, t_1 - \tau p)$. So let u_1 be the restriction on $[t_1, t_1 + \tau]$ of the solution of (2.1) corresponding to the initial data (2.7) instead of $(w_0(\cdot,0), w_1, z_0)$. In fact, the equation reads

$$\begin{aligned} w_{tt}(x,t) - a_0 \Delta w(x,t) + a_1 w_0(x, t-\tau) + \int_0^{t_1} g(s) \Delta w_0(x, t-s) ds \\ + \int_{t_1}^{t} g(s) \Delta w(x, t-s) ds = 0. \end{aligned}$$

Notice that for $t \in [t_1, t_1 + \tau)$, we have $0 < t_1 - \tau \leq t - \tau < t_1$ and therefore $w(x, t-\tau)$ is well-defined, continuous and equal to $w_0(x, t-\tau)$. The limit of the solution and its time-derivative exist at $(t_1 + \tau)^-$.

Step 3: Consider the interval $[t_1 + \tau, t_1 + 2\tau)$. We may assume, without loss of generality, that $t_2 \in (t_1 + \tau, t_1 + 2\tau]$, otherwise, we perform an extension of the solution w_1 with the help of Theorem 2.1 from $t_1 + \tau$ up to $t_1 + 2\tau$ and consider the next intervals $[t_1 + 2\tau, t_1 + 3\tau)$, ... untill we reach the one containing t_2. So, if $t_1 + \tau < t_2 \leq t_1 + 2\tau$, we discuss two cases

(a) On the interval $[t_1 + \tau, t_2)$, we extend normally the solution to $\tilde{w}_1$ (also denoted simply by w_1). This is possible as $t_1 + \tau \leq t < t_2$ implies $t_1 \leq t - \tau < t_2 - \tau < t_1 + 2\tau - \tau = t_1 + \tau$. Therefore, $w(x, t-\tau)$ is well-defined, continuous and equal to $w_1(x, t-\tau)$. Clearly

$$\left(w_1(\cdot, t_2^-), \frac{\partial w_1}{\partial t}(\cdot, t_2^-), w_1(\cdot, t_2 - \tau p) \right) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)).$$

(b) Next, we start from

$$\left(g_2(w_0(\cdot, t_2^-)), f_2\left(\frac{\partial w_0}{\partial t}(\cdot, t_2^-) \right), z_2 \right) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)),$$

where $z_2(x,p) = w_1(x, t_1 - \tau p)$ and construct our solution w_2 on $[t_2, t_1 + 2\tau)$ for the problem with equation

$$w_{tt}(x,t) - a_0 \Delta w(x,t) + a_1 w_1(x, t-\tau) + \int_0^{t_1} g(s) \Delta w_0(x, t-s) ds$$
$$+ \int_{t_1}^{t_2} g(s) \Delta w_1(x, t-s) ds + \int_{t_2}^{t} g(s) \Delta w(x, t-s) ds = 0.$$

Again this is possible via Theorem 2.1 as $t_2 \le t < t_1 + 2\tau$ implies $t_1 = t_1 + \tau - \tau < t_2 - \tau \le t - \tau < t_1 + \tau$.

Step 4: An induction argument leads to the existence of a unique solution

$$u(x,t) = \begin{cases} w_0(x,t), \ t \in [0, t_1), \\ w_1(x,t), \ t \in [t_1, t_2), \\ \dots \\ w_k(x,t), \ t \in [t_k, t_{k+1}), \\ \dots \end{cases}$$

of (1.5) having the regularity (2.6). The proof is complete. □

3. **Stability**

The energy E corresponding to (1.5) is equal to

(3.1)
$$E(t) = \tfrac{1}{2}(a_0 - \tilde{g}(t)) \|\nabla u\|^2 + \tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2} \int_0^t g(s) \left\| \nabla(u(t) - u(t-s)) \right\|^2 ds$$
$$+ \tfrac{|a_1|}{2} \int_{t-\tau}^{t} g(t-s) \|\nabla u(s)\|^2 ds, \quad t \in \mathbb{R}_+,$$

where

(3.2)
$$\tilde{g}(t) = \int_0^t g(s) ds, \quad t \in \mathbb{R}_+.$$

Because $0 \le \tilde{g}(t) \le g_0 < a_0$ (see (1.2)), the following relations (equivalence) hold:

(3.3)
$$\alpha_2 E_0(t) \le E(t) \le \alpha_1 E_0(t), \quad t \in \mathbb{R}_+,$$

where $\alpha_1 = \frac{1}{2} \max\{a_0, 1\}$, $\alpha_2 = \frac{1}{2} \min\{a_0 - g_0, 1\}$ (α_1 and α_2 are positive) and

(3.4)
$$E_0(t) = \|\nabla u\|^2 + \|u_t\|^2 + \int_0^t g(s) \left\| \nabla(u(t) - u(t-s)) \right\|^2 ds + |a_1| \int_{t-\tau}^{t} g(t-s) \|\nabla u(s)\|^2 ds, \quad t \in \mathbb{R}_+.$$

We start by estimating the derivative of E.

Lemma 3.1. *The energy functional E satisfies*

(3.5)
$$E'(t) \le \frac{c_0 g(\tau)}{2} |a_1| \|u_t\|^2 + \frac{|a_1| g(0)}{2} \|\nabla u\|^2 + \frac{1}{2} \int_0^t g'(s) \left\| \nabla(u(t) - u(t-s)) \right\|^2 ds, \quad t \in \mathbb{R}_+ \backslash \{t_k\}_{k \in \mathbb{N}},$$

where c_0 is the Poincar's constant defined by

(3.6)
$$\|v\|^2 \le c_0 \|\nabla v\|^2, \quad v \in H_0^1(\Omega).$$

Proof. Let $t \in \mathbb{R}_+ \backslash \{t_k\}_{k \in \mathbb{N}}$. It is clear that

(3.7)
$$\frac{d}{dt}\left(\int_{t-\tau}^{t} g(t-s) \|\nabla u(s)\|^2 ds \right) = g(0)\|\nabla u\|^2 - g(\tau) \|\nabla u(t-\tau)\|^2 + \int_{t-\tau}^{t} g'(t-s) \|\nabla u(s)\|^2 ds.$$
$$\le g(0)\|\nabla u\|^2 - g(\tau) \|\nabla u(t-\tau)\|^2.$$

By differentiating with respect to t, integrating by parts and using $(1.5)_1$ and $(1.5)_2$, we arrive at

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\left(a_0\|\nabla u\|^2+\|u_t\|^2\right)&=\int_\Omega\left(a_0\nabla u\cdot\nabla u_t+u_tu_{tt}\right)dx=\int_\Omega u_t(u_{tt}-a_0\Delta u)dx\\&=-a_1\int_\Omega u_tu(t-\tau)dx+\int_\Omega\nabla u_t\cdot\int_0^t g(s)\nabla u(t-s)dsdx.\end{aligned}\tag{3.8}$$

Similarly, we have

$$-\frac{1}{2}\frac{d}{dt}\left(\tilde{g}(t)\|\nabla u\|^2\right)=-\frac{1}{2}g(t)\|\nabla u\|^2-\tilde{g}(t)\int_\Omega\nabla u\cdot\nabla u_tdx\tag{3.9}$$

and, using a change of variable,

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^t g(s)\left\|\nabla(u(t)-u(t-s))\right\|^2ds\right)=\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)\left\|\nabla(u(t)-u(s))\right\|^2ds\right)\tag{3.10}$$

$$\begin{aligned}&=\frac{1}{2}\int_0^t g'(t-s)\left\|\nabla(u(t)-u(s))\right\|^2ds+\int_\Omega\nabla u_t\cdot\int_0^t g(t-s)\nabla(u(t)-u(s))dsdx\\&=\frac{1}{2}\int_0^t g'(s)\left\|\nabla(u(t)-u(t-s))\right\|^2ds+\int_\Omega\nabla u_t\cdot\int_0^t g(s)\nabla(u(t)-u(t-s))dsdx\\&=\frac{1}{2}\int_0^t g'(s)\left\|\nabla(u(t)-u(t-s))\right\|^2ds+\tilde{g}(t)\int_\Omega\nabla u_t\cdot\nabla udx-\int_\Omega\nabla u_t\cdot\int_0^t g(s)\nabla u(t-s)dsdx.\end{aligned}$$

Summing (3.7)-(3.10), we find

$$\begin{aligned}E'(t)&=-a_1\int_\Omega u_tu(t-\tau)dx-\frac{1}{2}g(t)\|\nabla u\|^2\\&+\tfrac{1}{2}\int_0^t g'(s)\left\|\nabla(u(t)-u(t-s))\right\|^2ds+\frac{|a_1|}{2}\left(g(0)\|\nabla u\|^2-g(\tau)\|\nabla u(t-\tau)\|^2\right).\end{aligned}$$

It suffices to take into account (1.2) and apply Young's and Poincar's inequalities to $u_tu(t-\tau)$ and $\|u(t-\tau)\|^2$, respectively, to achieve (3.5). □

Lemma 3.2. *Let* $\lambda_0,\ \lambda_1,\lambda_2>0$ *and*

$$\begin{aligned}L(t)&=\lambda_0E(t)+\lambda_1\int_\Omega u_tudx+\lambda_2\int_0^t g(t-s)\left\|\nabla u(s))\right\|^2ds\\&-\int_\Omega u_t\int_0^t g(s)(u(t)-u(t-s))dsdx,\quad t\in\mathbb{R}_+.\end{aligned}\tag{3.11}$$

Then, there exist $M_1,\ M_2,\ c_1,\ c_2>0$ *(independent of* a_1*) such that*

$$M_1E(t)\le L(t)\le M_2E(t),\quad t\in\mathbb{R}_+\tag{3.12}$$

and

$$L'(t)\le-c_1L(t)+c_2|a_1|\sup_{t-\tau\le s\le t}L(s),\quad t\in[\tau,+\infty)\backslash\{t_k\}_{k\in\mathbb{N}}.\tag{3.13}$$

Proof. Let $t\in[\tau,+\infty)\backslash\{t_k\}_{k\in\mathbb{N}}$. A differentiation with respect to t, integration by parts and use of $(1.5)_1$ and $(1.5)_2$, leads to

$$\frac{d}{dt}\left(\int_\Omega u_tudx\right)=\|u_t\|^2+\int_\Omega u\left[a_0\Delta u-a_1u(t-\tau)-\int_0^t g(s)\Delta u(t-s)ds\right]dx\tag{3.14}$$

$$=\|u_t\|^2-\left[a_0-\tilde{g}(t)\right]\|\nabla u\|^2-a_1\int_\Omega u(t)u(t-\tau)dx-\int_\Omega\nabla u\cdot\int_0^t g(s)\nabla(u(t)-u(t-s))dsdx.$$

Next, an application of Young's, Poincar's and Hlder's inequalities to the last two integrals in the above relation and observing that $\tilde{g}(t) \leq g_0$, we find, for any $\epsilon > 0$,

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u_t u dx\right) \leq & -\left(a_0 - g_0(1+\epsilon) - \epsilon|a_1|\right)\|\nabla u\|^2 + \|u_t\|^2 \\ & + \frac{C}{\epsilon}|a_1|\|\nabla u(t-\tau)\|^2 + \frac{C}{\epsilon}\int_0^t g(s)\|\nabla(u(t)-u(t-s))\|^2 ds.\end{aligned} \tag{3.15}$$

Similarly, we have

$$\frac{d}{dt}\int_0^t g(t-s)\left\|\nabla u(s)\right\|^2 ds = g(0)\left\|\nabla u\right\|^2 + \int_0^t g'(t-s)\left\|\nabla u(s)\right\|^2 ds \leq g(0)\left\|\nabla u\right\|^2 \tag{3.16}$$

and

$$\begin{aligned} & -\frac{d}{dt}\left(\int_\Omega u_t \int_0^t g(s)(u(t)-u(t-s))dsdx\right) \\ = & \int_\Omega \left(a_1 u(t-\tau) + \int_0^t g(s)\Delta(u(t-s) - u(t) + u(t))ds\right)\int_0^t g(s)(u(t)-u(t-s))dsdx \\ & + a_0\int_\Omega \nabla u \int_0^t g(s)\nabla(u(t)-u(t-s))dsdx - \int_\Omega u_t \frac{d}{dt}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)dx.\end{aligned}$$

Since

$$\frac{d}{dt}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right) = \int_0^t g'(t-s)(u(t)-u(s))ds + \tilde{g}(t)u_t,$$

and $\tilde{g}(t) \geq \tilde{g}(\tau) > 0$, for $t \geq \tau$, it is clear that in view of (1.2), for any $\epsilon > 0$,

$$\begin{aligned} -\frac{d}{dt}\left(\int_\Omega u_t \int_0^t g(s)(u(t)-u(t-s))dsdx\right) \leq & \epsilon\|\nabla u\|^2 - (\tilde{g}(\tau)-\epsilon)\|u_t\|^2 + \epsilon|a_1|\|\nabla u(t-\tau)\|^2 \\ + C\left(\frac{|a_1|+1}{\epsilon}+1\right)\int_0^t g(s)\|\nabla(u(t)-u(t-s))\|^2 ds & - \frac{C}{\epsilon}\int_0^t g'(s)\|\nabla(u(t)-u(t-s))\|^2 ds.\end{aligned}$$

Now, when summing up (3.5), (3.14), (3.15), (3.16) and (3.17), we find

$$\begin{aligned} L'(t) \leq & \frac{\lambda_0 c_0 g(\tau)}{2}|a_1|\|u_t\|^2 + \lambda_0\frac{|a_1|g(0)}{2}\|\nabla u\|^2 + \frac{\lambda_0}{2}\int_0^t g'(s)\left\|\nabla(u(t)-u(t-s))\right\|^2 ds \\ & + \lambda_2 g(0)\left\|\nabla u\right\|^2 - \lambda_1\left[a_0 - g_0(1+\epsilon) - \epsilon|a_1|\right]\|\nabla u\|^2 + \lambda_1\|u_t\|^2 + \frac{\lambda_1 C}{\epsilon}|a_1|\|\nabla u(t-\tau)\|^2 \\ & + \frac{\lambda_1 C}{\epsilon}\int_0^t g(s)\|\nabla(u(t)-u(t-s))\|^2 ds + \epsilon\|\nabla u\|^2 - (\tilde{g}(\tau)-\epsilon)\|u_t\|^2 + \epsilon|a_1|\|\nabla u(t-\tau)\|^2 \\ & + C\left(\frac{|a_1|+1}{\epsilon}+1\right)\int_0^t g(s)\|\nabla(u(t)-u(t-s))\|^2 ds - \frac{C}{\epsilon}\int_0^t g'(s)\|\nabla(u(t)-u(t-s))\|^2 ds\end{aligned}$$

or

$$\begin{aligned} L'(t) \leq & \left[\frac{\lambda_0 c_0 g(\tau)}{2}|a_1| + \lambda_1 - \tilde{g}(\tau) + \epsilon\right]\|u_t\|^2 + |a_1|\left(\frac{\lambda_1 C}{\epsilon}+\epsilon\right)\|\nabla u(t-\tau)\|^2 \\ & + \left[\frac{\lambda_0|a_1|g(0)}{2} + \lambda_2 g(0) + \epsilon - \lambda_1\left[a_0 - g_0(1+\epsilon) - \epsilon|a_1|\right]\right]\|\nabla u\|^2 \\ & + C\left(\frac{\lambda_1}{\epsilon} + \frac{|a_1|+1}{\epsilon} + 1\right)\int_0^t g(s)\|\nabla(u(t)-u(t-s))\|^2 ds + \left(\frac{\lambda_0}{2} - \frac{C}{\epsilon}\right)\int_0^t g'(s)\|\nabla(u(t)-u(t-s))\|^2 ds.\end{aligned}$$

Exploiting (1.2), we get, for all $\lambda_0 \geq \frac{2C}{\epsilon}$,

$$\begin{aligned} L'(t) \leq & -I_1\|\nabla u\|^2 - I_2\|u_t\|^2 - I_3\int_0^t g(s)\|\nabla(u(t)-u(t-s))\|^2 ds \\ & + |a_1|C_{\epsilon,\lambda_j}\sup_{t-\tau\leq s\leq t}\left(\|\nabla u(s)\|^2 + \|u_t(s)\|^2\right),\end{aligned} \tag{3.17}$$

where

$$I_1 = \lambda_1\left[a_0 - g_0(1+\epsilon)\right] - \lambda_2 g(0) - \epsilon, \quad I_2 = \tilde{g}(\tau) - \epsilon - \lambda_1 \quad \text{and} \quad I_3 = \frac{\lambda_0\xi}{2} - C\left(\frac{\lambda_1}{\epsilon} + \frac{|a_1|+1}{\epsilon} + 1\right).$$

In order to get $I_j > 0$, we choose ϵ such that

$$0 < \epsilon < \min\left\{\tilde{g}(\tau), \frac{a_0}{g_0} - 1\right\} \text{ and } \frac{\epsilon}{a_0 - g_0(\epsilon+1)} < \tilde{g}(\tau) - \epsilon,$$

which is possible according to (1.2). This choice of ϵ guarantees the existence of λ_1 satisfying

$$\frac{\epsilon}{a_0 - g_0(\epsilon+1)} < \lambda_1 < \tilde{g}(\tau) - \epsilon.$$

After, we observe that the choice of λ_1 allows us to pick λ_2 sufficiently small so that

$$0 < \lambda_2 < \frac{\lambda_1 \left[a_0 - g_0(1+\epsilon)\right] - \epsilon}{g(0)}.$$

Finally, using the property

$$\int_{t-\tau}^{t} \|\nabla u(s)\|^2 ds \leq \tau \sup_{t-\tau \leq s \leq t} \|\nabla u(s)\|^2,$$

we see that, for any λ_0 satisfying

$$\lambda_0 > \max\left\{\frac{2C}{\epsilon}, \frac{2C}{\xi}\left(\frac{\lambda_1}{\epsilon} + \frac{|a_1|+1}{\epsilon} + 1\right)\right\}, \tag{3.18}$$

one can conclude from (3.3) and (3.17) that there exists $M_0 > 0$ (not depending on a_1) such that

$$L'(t) \leq -M_0 E(t) + C|a_1| \sup_{t-\tau \leq s \leq t} E(t), \quad t \in [\tau, +\infty)\backslash\{t_k\}_{k\in\mathbb{N}}. \tag{3.19}$$

On the other hand, as

$$\begin{aligned} \int_0^t g(t-s)\left\|\nabla u(s))\right\|^2 ds &= \int_0^t g(t-s)\left\|\nabla u(s) - \nabla u(t) + \nabla u(t)\right\|^2 ds \\ &\leq 2\int_0^t g(t-s)\left\|\nabla u(s) - \nabla u(t)\right\|^2 ds + 2g_0\left\|\nabla u(t)\right\|^2 \\ &\leq CE_0(t), \end{aligned} \tag{3.20}$$

then, using (3.3) and applying Hlder's, Young's and Poincar's inequalities, we see that

$$|L(t) - \lambda_0 E(t)| \leq C(\lambda_1 + \lambda_2 + 1)E_0(t) \leq C(\lambda_1 + \lambda_2 + 1)E(t), \quad t \in \mathbb{R}_+.$$

Therefore, by fixing λ_0 satisfying (3.18) and, if needed, $\lambda_0 > C(\lambda_1 + \lambda_2 + 1)$, we obtain (3.12) with

$$M_1 = \lambda_0 - C(\lambda_1 + \lambda_2 + 1) \quad \text{and} \quad M_2 = \lambda_0 + C(\lambda_1 + \lambda_2 + 1).$$

Consequently, (3.12) and (3.19) lead to (3.13) with $c_1 = \frac{M_0}{M_2}$ and $c_2 = \frac{C}{M_1}$. □

Lemma 3.3. [18] *Let $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that*

$$0 < t_0 < t_1 < \cdots < t_k < \cdots \quad \text{and} \quad \lim_{k\to+\infty} t_k = +\infty.$$

Let $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $\{b_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $a > 0$, $b \geq 0$, $\delta > 1$ and $\tau > 0$ such that

$$a > b \quad \text{and} \quad \inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} > \delta\tau. \tag{3.21}$$

Let η be the unique solution of the equation $\eta = a - be^{\eta\tau}$ and $r_k = \max\{1, a_k + b_k e^{\eta\tau}\}$ such that there exist M, $\rho > 0$ satisfying

$$r_0 r_1 \cdots r_{k+1} e^{k\eta\tau} \leq M e^{\rho(t_k - t_0)}, \quad k \in \mathbb{N}. \tag{3.22}$$

Let h be a nonnegative function continuous except at the jump discontinuity points $\{t_k\}_{k\in\mathbb{N}}$ solution of

$$\begin{cases} h'(t) \leq -ah(t) + b \sup\limits_{t-\tau \leq s \leq t} h(s), & t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}}, \\ \\ h(t_k) \leq a_k h(t_k^-) + b_k \sup\limits_{t_k-\tau \leq s \leq t_k} h(s), & k \in \mathbb{N}. \end{cases}$$

Then

$$h(t) \leq M \sup_{t_0-\tau \leq s \leq t_0} h(s) e^{-(\eta-\rho)(t-t_0)}, \quad t \in \mathbb{R}_+.$$

Moreover, for $\omega = \sup_{k\in\mathbb{N}}\{1, a_k + b_k e^{\eta\tau}\}$, *we have*

$$h(t) \le \omega \sup_{t_0-\tau\le s\le t_0} h(s) e^{-\left(\eta - \frac{\ln[\omega e^{\eta\tau}]}{\delta\tau}\right)(t-t_0)}, \quad t \in \mathbb{R}_+.$$

Theorem 3.4. *Assume that* (3.22) *holds and*

$$|a_1| < \frac{c_1}{c_2}, \tag{3.23}$$

where c_1 *and* c_2 *are given in Lemma 3.2,* $a = c_1$, $b = c_2|a_1|$, $a_k = C_k$, $b_k = 0$ *and* C_k *is defined in* (3.25) *below. Then, there exist* c_3, $c_4 > 0$ *such that*

$$E(t) \le c_3 e^{-c_4 t}, \quad t \in [\tau, +\infty). \tag{3.24}$$

Proof. By virtue of (1.3), (1.4) and $(1.5)_4$, it is clear that, for any $k \in \mathbb{N}$,

$$\|\nabla u(t_k)\|^2 = \|\nabla(g_k(u(t_k^-)))\|^2 \le \|g_k'\|_\infty^2 \|\nabla u(t_k^-)\|^2 \le \xi_k^2 \|\nabla u(t_k^-)\|^2,$$

$$\|u_t(t_k)\|^2 = \|f_k(u_t(t_k^-))\|^2 \le \tilde{\xi}_k^2 \|u_t(t_k^-)\|^2,$$

$$\int_{t_k-\tau}^{t_k} g(t_k - s)\|\nabla u(s)\|^2 ds = \int_{t_k^- -\tau}^{t_k^-} g(t_k^- - s)\|\nabla u(s)\|^2 ds,$$

and

$$\int_0^{t_k} g(s)\,\|\nabla(u(t_k) - u(t_k - s))\|^2\,ds \le 2g_0\|\nabla u(t_k)\|^2 + 2\int_0^{t_k} g(s)\,\|\nabla u(t_k - s))\|^2\,ds$$

$$\le 2g_0\xi_k^2\|\nabla u(t_k^-)\|^2 + 2\int_0^{t_k^-} g(t_k^- - s)\,\|\nabla u(s))\|^2\,ds.$$

Therefore, using (3.20), we deduce that

$$E_0(t_k) \le \tilde{C}_k E_0(t_k^-).$$

Then, using (3.3) and (3.12), we get

$$L(t_k) \le C_k L(t_k^-). \tag{3.25}$$

Consequently, according to Lemma 3.3 (with $h = L$), (3.13) and (3.25) lead to (3.24). □

4. **General remarks**

1. The argument above in the previous section may be employed for other problems as well such as for different kinds of beams and in thermoelasticity.

2. It can also be adapted for other types of kernels which may be found in the literature. Here we assumed (1.2) only for simplicity (see [9] in case of absence of impulses).

3. Other types of time delays can be considered for which our approach can be also adapted like, for example, variable discrete time delays, multiple time delays and distributed time delays (in u or in u_t).

4. It is also possible to adapt our arguments to the case where impulses are present in both

$$(u(x, t_k^-), u_t(x, t_k^-)) \quad \text{and} \quad (u(x, t_k^- - \tau_1), u_t(x, t_k^- - \tau_2)).$$

5. Our stability result (3.24) holds true if we replace the finite memory by an infinite one; that is, $\int_0^t$ and $[-\tau, 0]$ in (1.5) are replaced by $\int_0^{+\infty}$ and $(-\infty, 0]$, respectively. The well-posedness question in case of absence of impulses can be treated using the semigroups approach (see, for

example, [7]) instead of the arguments used for the proof of Theorem 2.1. The well-posedness and stability problems for (1.5) can be solved using similar arguments as in the proofs of Theorem 2.2 and Theorem 3.4.

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