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SOME QUALITATIVE PROPERTIES OF SOLUTIONS FOR MULTI-DELAY NONAUTONOMOUS STOCHASTIC LIÉNARD EQUATION

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ABSTRACT. In this study, we analyse the nonautonomous multi-delay stochastic Liénard equation and develop sufficient conditions for the equation's stability and boundedness. The Lyapunov functional (LF) technique is used as the primary instrument in the proofs and we extend and enhance some stability and boundedness conclusions from the literature through our work. Two examples are used to support the correctness and efficacy of the results as an application. Finally, using the Euler-Maruyama method (EM-method), numerical simulations are provided in the final part to illustrate the approximative numerical solutions for the Liénard system under consideration.

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Key Words and Phrases. Multi-delay, Stochastic Liénard equation, Stability, Boundedness, Lyapunov functional approach, EM-method.

1. INTRODUCTION

The basic theory of stochastic delay differential equations (SDDEs) has been systematically established in [20] and there are many interesting results on the qualitative properties (QP) of solutions for SDDEs in the literature, see, for example, [7, 8, 10, 11, 14, 18, 19, 21, 22, 24, 26].

Liénard equations, which have been applied to describe fluid mechanical and nonlinear elastic mechanical phenomena, have been the subject of extensive research.

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The Liénard equation is currently used in applied sciences to solve practical issues in mechanics, engineering method domains, economy, control theory, physics, chemistry, biology, medicine, automated energy, information theory, etc.

By this point, the literature has extensively explored and is still investigating the QP of the Liénard equation with and without delays such that stability, boundedness, convergence, existence and uniqueness of periodic solutions, for instance, [9, 12, 13, 27, 32, 42], and others.

It is worth noting that the problem of constructing IFs in the case of delay and stochastic differential equations remains an intriguing and important area of research in and of itself.

In addition, outstanding papers on QP of solutions for second-order differential equations using the technique of LF, have been discussed by researchers, see foe example, [28, 30, 31, 34, 35, 36, 38, 39, 40], and the references cited therein.

Due to its numerous applications to issues in control theory, chemistry, physics, information theory and mechanics, second-order linear and nonlinear delay differential equations (DDEs) have been the subject of extensive study about their QP, for example, [29, 33, 37], etc.

To the best of our knowledge, we only find a few papers in the literature on the QP of second and third-order stochastic differential equations (SDEs) with or without delay, such as $[1-6, 15-17, 25, 41]$, and the references cited therein.

The QP of solutions for the multi-delay stochastic Liénard equation, on the other hand, has not yet been discussed in the literature.

As a result, the goal of this research is to investigate the stability and boundedness of solutions to the stochastic Liénard equation with multiple variable delays $\tau_i(t)$ $(i = 1, 2, 3, ..., n)$ as follows:

$$\ddot{x}+\phi(t)f(x,\dot{x})\dot{x}+\sum_{i=1}^{n}g_i(x)+\sum_{i=1}^{n}\psi_i(t)h_i(x(t-\tau_i(t)))+\Delta x(t)\dot{\vartheta}(t)=e(t,x,\dot{x}), \tag{1.1}$$

in which Δ is a positive constant and $\tau_i(t)$ are positive bounded delays with $0 \le \tau(t) = \max_{1\le i\le n} \tau_i(t) \le \gamma$, γ is a positive constant that will be determined later, $\dot{\tau}_i(t) \le \bar{\omega}$, $\bar{\omega} \in (0, 1)$.
The functions f, g_i, h_i and e are continuous in their respective arguments with $g_i(0) = h_i(0) = 0$, for all $(i = 1, 2, 3, ..., n)$, and $\vartheta(t) \in \mathbb{R}^m$ is typical Brownian motion. The functions $\phi(t)$ and $\psi_i(t)$ $(i = 1, 2, 3, ..., n)$ are positive and continuously differentiable on $[0, \infty)$.

Equation (1.1) can be expressed in the following system form

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -\phi(t)f(x,y)y - \sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{n} \psi_i(t)h_i(x) - \Delta x(t)\dot{\vartheta}(t) \\
&\quad + \sum_{i=1}^{n} \psi_i(t) \int_{t-\tau_i(t)}^{t} h_i'(x(s))y(s)ds + e(t,x,y).
\end{aligned} \tag{1.2}
$$

Further, it is supposed existence and continuity of the derivatives $\dot{\phi}(t) = \frac{d\phi(t)}{dt}$, $h_i'(x) = \frac{dh_i'(x)}{dx}$, $g_i'(x) = \frac{dg_i'(x)}{dx}$ and $f_y(x,y) = \frac{\partial f(x,y)}{\partial y}$, for all x, y and $i = 1, 2, 3, ..., n$.

2. STOCHASTIC STABILITY RESULT

We introduce the following hypotheses before proving our main results.
For all $i = 1, 2, 3, ..., n$, we assume that there are positive constants $\phi_0, \phi_1, a_0, \alpha, \alpha_i$, $\beta_i, \delta_i, \mu_i, \rho_i, b_i$ and l_i. So that the following conditions are met:

(H_1) $\frac{1}{2} < \phi_0 \leq \phi(t) \leq \phi_1$, $\dot{\phi}(t) \leq \alpha$, for all $t \in \mathbb{R}^+$.
(H_2) $1 \leq f(x,y) \leq a_0$, $xf_y(x,y) \geq 0$, for all $x, y \in \mathbb{R}$.
(H_3) $g_i(0) = 0$, $\frac{g_i(x)}{x} \geq \alpha_i$, $(x \neq 0)$ and $0 < g'(x) \leq b_i$, for all $x \in \mathbb{R}$.
(H_4) $0 < \delta_i \leq \psi_i(t) \leq l_i$ and $\dot{\psi}_i(t) \leq \mu_i$, for all $x \in \mathbb{R}^+$.
(H_5) $h_i(0) = 0$, $\frac{h_i(x)}{x} \geq \beta_i$, $(x \neq 0)$ and $0 < h_i'(x) \leq \rho_i$, for all $x \in \mathbb{R}$.
(H_6) $0 \leq \tau_i(t) \leq \gamma$ and $0 \leq \dot{\tau}_i(t) \leq \bar{\omega}$, $\bar{\omega} \in (0,1)$, for each $i = 1, 2, 3, ..., n$.
(H_7) $\delta_i\beta_i + \alpha_i + \frac{1}{2}\phi_0 > 1$, for each $i = 1, 2, 3, ..., n$.

The following theorem is the first result of this paper.

Theorem 2.1. *Let the conditions* $(H_1) - (H_7)$ *be hold. If*

$$
\gamma < \min\left\{ \frac{A_0 - \frac{1}{2}\alpha a_0 - \Delta^2}{M}, \frac{2\phi_0 - 1}{B_0 M} \right\},
$$

with

$$
M = \sum_{i=1}^{n} \rho_i l_i > 0, \quad \sum_{i=1}^{n} (\delta_i\beta_i + \alpha_i) + \frac{1}{2}\phi_0 > 1, \quad A_0 = \sum_{i=1}^{n} (\alpha_i + \delta_i\beta_i - \mu_i\rho_i) > 0,
$$

$$
B_0 = 1 + \frac{3}{2(1-\bar{\omega})} \quad \text{and} \quad 2A_0 - \alpha a_0 > 2\Delta^2.
$$

Then, the zero solution of stochastic Liénard equation (1.1) *with* $e(t,x,y) \equiv 0$ *is stochastically asymptotically stable.*

Proof. To demonstrate the preceding theorem, we define an LF as follows:

$$
\begin{aligned}
V_1(t, x_t, y_t) =& 2\sum_{i=1}^{n} \psi_i(t) \int_0^x h_i(\xi)d\xi + 2\sum_{i=1}^{n} \int_0^x g_i(\xi)d\xi + \phi(t) \int_0^x f(\xi, 0)\xi d\xi \\
&+ xy + y^2 + \sum_{i=1}^{n} \lambda_i \int_{-\tau_i(t)}^{0} \int_{t+s}^{t} y^2(u)duds,
\end{aligned} \tag{2.1}
$$

where $\lambda_i,\ i = 1, 2, 3, ..., n$ are positive scalars, which will be specified later. Using Itô formula, the time derivative of the functional $V_1(t, x_t, y_t)$ along any solution $(x(t), y(t))$ of stochastic Liénard delay differential system (1.2) with $e \equiv 0$ gives as

$$\begin{aligned}\mathcal{L}V_1(t, x_t, y_t) = y^2 + \sum_{i=1}^{n} \lambda_i \tau_i(t) y^2 - \sum_{i=1}^{n} \lambda_i (1 - \dot{\tau}_i(t)) \int_{t-\tau_i(t)}^{t} y^2(s)ds \\ + \Delta^2 x^2 + \sum_{j=1}^{5} U_j,\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}U_1 &:= -\phi(t) f(x, y) xy + \phi(t) f(x, 0) xy,\\ U_2 &:= -2\phi(t) f(x, y) y^2,\\ U_3 &:= -x \sum_{i=1}^{n} g_i(x) - x \sum_{i=1}^{n} \psi_i(t) h_i(x),\\ U_4 &:= (x + 2y) \sum_{i=1}^{n} \psi_i(t) \int_{t-\tau_i(t)}^{t} h_i'(x(s)) y(s) ds,\end{aligned}$$

and

$$U_5 := \dot{\phi}(t) \int_0^x f(\xi, 0)\xi d\xi + 2 \sum_{i=1}^{n} \dot{\psi}_i(t) \int_0^x h_i(\xi) d\xi.$$

Since $x f_y(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and applying the mean-value theorem, we get

$$U_1 := -\phi(t) \left[\frac{f(x, y) - f(x, 0)}{y} \right] xy^2 = -\phi(t) x f_y(x, y) y^2 \leq 0.$$

Additionally, from the conditions (H_1) and (H_2) such that $\phi(t) \geq \phi_0$ and $f(x, y) \geq 1$, for every $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$, so that

$$U_2 \leq -2\phi_0 y^2.$$

Moreover, $g_i(x) \geq \alpha_i x\ (x \neq 0)$, $h_i(x) \geq \beta_i x$ and $\psi_i(t) \geq \delta_i$, we obtain

$$U_3 \leq -\sum_{i=1}^{n} \alpha_i x^2 - \sum_{i=1}^{n} \delta_i \beta_i x^2 = -\sum_{i=1}^{n} (\alpha_i + \delta_i \beta_i) x^2.$$

Furthermore, from the condition (H_5) and (H_6) with the inequality $2|mn| \leq m^2 + n^2$, we get

$$\begin{aligned}U_4 &\leq \frac{1}{2} \sum_{i=1}^{n} \rho_i l_i \tau_i(t) x^2 + \sum_{i=1}^{n} \rho_i l_i \tau_i(t) y^2 + \frac{3}{2} \sum_{i=1}^{n} \rho_i l_i \int_{t-\tau_i(t)}^{t} y^2(s) ds\\ &\leq \frac{1}{2} \sum_{i=1}^{n} (\rho_i l_i) \gamma (x^2 + 2y^2) + \frac{3}{2} \sum_{i=1}^{n} \rho_i l_i \int_{t-\tau_i(t)}^{t} y^2(s) ds.\end{aligned}$$

Finally, since $\dot{\psi}_i(t) \leq \mu_i$, $h'(x) \leq \rho_i$, $\dot{\phi}(t) \leq \alpha$ and $f(x, y) \leq a_0$, for all $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$, we find that

$$U_5 \leq \frac{1}{2} \alpha a_0 x^2 + \sum_{i=1}^{n} \mu_i \rho_i x^2.$$

Using the above estimates of $U_j,\ j = 1, 2, 3, 4, 5$ and condition (H_6) such that $\dot{\tau}_i(t) \le \bar{\omega}$, we obtain

$$\begin{aligned}\mathcal{L}V_1(t, x_t, y_t) \le & -2\phi_0 y^2 + \frac{1}{2}\alpha a_0 x^2 - \sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i - \mu_i\rho_i)x^2 + y^2 \\ & + \frac{1}{2}\sum_{i=1}^{n}(\rho_i l_i)\gamma(x^2 + 2y^2) + \frac{3}{2}\sum_{i=1}^{n}\rho_i l_i \int_{t-\tau_i(t)}^{t} y^2(s)ds \\ & + \sum_{i=1}^{n}\lambda_i\gamma y^2 - \sum_{i=1}^{n}\lambda_i(1-\bar{\omega})\int_{t-\tau_i(t)}^{t} y^2(s)ds + \Delta^2 x^2.\end{aligned}$$

With some rearrangement of terms, we can easily get

$$\begin{aligned}\mathcal{L}V_1(t, x_t, y_t) \le & -\left\{\sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i - \mu_i\rho_i) - \frac{1}{2}\alpha a_0 - \Delta^2 - \frac{1}{2}\sum_{i=1}^{n}(\rho_i l_i)\gamma\right\}x^2 \\ & - \left\{2\phi_0 - 1 - \sum_{i=1}^{n}(\rho_i l_i + \lambda_i)\gamma\right\}y^2 \\ & + \sum_{i=1}^{n}\left\{\frac{3}{2}(\rho_i l_i) - \lambda_i(1-\bar{\omega})\right\}\int_{t-\tau_i(t)}^{t} y^2(s)ds.\end{aligned} \tag{2.3}$$

If we let

$$\lambda_i = \frac{3\rho_i l_i}{2(1-\bar{\omega})} \ge 0, \quad \text{for all } i = 1, 2, 3, ..., n.$$

It follows that

$$\begin{aligned}\mathcal{L}V_1(t, x_t, y_t) \le & -\left\{\sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i - \mu_i\rho_i) - \frac{1}{2}\alpha a_0 - \Delta^2 - \frac{1}{2}\sum_{i=1}^{n}(\rho_i l_i)\gamma\right\}x^2 \\ & - \left\{2\phi_0 - 1 - \sum_{i=1}^{n}\rho_i l_i\left(1 + \frac{3}{2(1-\bar{\omega})}\right)\gamma\right\}y^2.\end{aligned}$$

If, we now choose

$$M = \sum_{i=1}^{n}\rho_i l_i > 0, \quad A_0 = \sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i - \mu_i\rho_i) > 0 \ \text{ and } \ B_0 = 1 + \frac{3}{2(1-\bar{\omega})}.$$

Then we can observe

$$\mathcal{L}V_1(t, x_t, y_t) \le -\left\{A_0 - \frac{1}{2}\alpha a_0 - \Delta^2 - \frac{1}{2}\sum_{i=1}^{n}(\rho_i l_i)\gamma\right\}x^2 - \left\{2\phi_0 - 1 - B_0 M\gamma\right\}y^2.$$

If, we take

$$\gamma < \min\left\{\frac{A_0 - \frac{1}{2}\alpha a_0 - \Delta^2}{M}, \frac{2\phi_0 - 1}{B_0 M}\right\}, \quad \gamma > 0.$$

Then there exists a positive constant K_1 such that

$$\mathcal{L}V_1(t, x_t, y_t) \le -K_1(x^2 + y^2), \quad K_1 \in \mathbb{R}. \tag{2.4}$$

Thus, the inequality (2.4) establishes the condition

$$\mathcal{L}V(t, x) \le -r_3(|x|) \text{ for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Since $\int_{-\tau_i(t)}^{0}\int_{t+s}^{t} y^2(u)duds$ is non-negative, then we can write (2.1) as

$$V_1(t,x_t,y_t) \geq 2\sum_{i=1}^{n}\psi_i(t)\int_0^x h_i(\xi)d\xi + 2\sum_{i=1}^{n}\int_0^x g_i(\xi)d\xi + xy + y^2 + \phi(t)\int_0^x f(\xi,0)\xi d\xi.$$

From the assumptions $(H_1)-(H_5)$, we obtain

$$\begin{aligned} V_1(t,x_t,y_t) &\geq \sum_{i=1}^{n}(\delta_i\beta_i)x^2 + \sum_{i=1}^{n}\alpha_i x^2 + xy + y^2 + \frac{1}{2}\phi_0 x^2 \\ &\geq \sum_{i=1}^{n}(\delta_i\beta_i + \alpha_i + \frac{1}{2}\phi_0 - 1)x^2 + (x+\frac{1}{2}y)^2 + \frac{3}{4}y^2. \end{aligned}$$

Since $\sum_{i=1}^{n}(\delta_i\beta_i + \alpha_i + \frac{1}{2}\phi_0) > 1$, therefore there exists a positive constant K_2, small enough so that

$$V_1(t,x_t,y_t) \geq K_2(x^2+y^2), \quad K_2 > 0, \quad K_2 \in \mathbb{R}. \tag{2.5}$$

For the terms included in (2.1), using the assumptions $(H_1)-(H_5)$ of Theorem 2.1, and the estimate $2|mn| \leq m^2 + n^2$, we have the following inequalities

$$xy \leq \frac{1}{2}(x^2+y^2), \quad 2\sum_{i=1}^{n}\psi_i(t)\int_0^x h_i(\xi)d\xi \leq \sum_{i=1}^{n}(\rho_i l_i)x^2,$$

$$2\sum_{i=1}^{n}\int_0^x g_i(\xi)d\xi \leq b_i x^2, \quad \phi(t)\int_0^x f(\xi,0)\xi d\xi \leq \frac{1}{2}\phi_1 a_0 x^2,$$

$$\begin{aligned} \sum_{i=1}^{n}\lambda_i\int_{-\tau_i(t)}^{0}\int_{t+s}^{t} y^2(u)duds &\leq \sum_{i=1}^{n}\lambda_i\int_{t-\tau_i(t)}^{t}(u-t+\tau_i(t))y^2(u)du \\ &\leq \sum_{i=1}^{n}\lambda_i\|y\|^2\int_{t-\tau_i(t)}^{t}(u-t+\tau_i(t))du \\ &\leq \frac{1}{2}\sum_{i=1}^{n}\lambda_i\tau_i^2(t)\|y\|^2. \end{aligned}$$

It is clear from the above inequalities that

$$V_1(t,x_t,y_t) \leq \left\{\sum_{i=1}^{n}(\rho_i l_i + b_i) + \frac{1}{2}\phi_1 a_0 + \frac{1}{2}\right\}x^2 + \frac{3}{2}y^2 + \frac{1}{2}\|y\|^2\sum_{i=1}^{n}\lambda_i\tau_i^2(t).$$

Let $\chi = \frac{1}{2}\sum_{i=1}^{n}\lambda_i\tau_i^2(t)$, then we can write the last inequality as

$$V_1(t,x_t,y_t) \leq K_3(x^2+y^2) + \chi\|y\|^2, \quad K_3 > 0, \ \chi > 0, \text{ for all } K_3, \chi \in \mathbb{R}. \tag{2.6}$$

Therefore, by combining the inequalities (2.5) and (2.6), we get

$$K_2(x^2+y^2) \leq V_1(t,x_t,y_t) \leq K_3(x^2+y^2) + \chi\|y\|^2. \tag{2.7}$$

Therefore from (2.7), we note that the LF V_1 satisfies the inequalities

$$r_1(|x|) \leq V(t,x) \leq r_2(|x|).$$

Hence, by noting the discussion proceeded above, that is, by the inequalities (2.4) and (2.7), establish the hypotheses of theorems stability in [7, 20, 41].

Then, we can conclude that the zero solution of Liénard stochastic differential equation with multiple delays (1.1) is stochastically asymptotically stable.
The proof of Theorem 2.1 is finished.

3. STABILITY EXAMPLE

In this section, we display an example of how to illustrate the stability result for stochastic Liénard equation with the multiple variable delays, that we obtained in the previous section.
We consider the special case of Liénard stochastic equation (1.1) with variable delay $\tau_1(t)$ such that $n = 1$ and $e(t, x, y) \equiv 0$, as shown below

$$\begin{aligned}\ddot{x} &+ \left(4 + \frac{1}{10}e^{-\frac{10}{3}t}\right)\left(1 + \frac{1}{1+x^2}\right)\dot{x} + \left(\frac{x}{1+x^2} + x\right) \\ &+ \left(1 + \frac{1}{1+t}\right)\left(2x(t-\tau_1(t)) + \frac{x(t-\tau_1(t))}{1+|x(t-\tau_1(t))|}\right) + \Delta x(t)\dot{\vartheta}(t) = 0.\end{aligned} \tag{3.1}$$

Then we can express (3.1) as the equivalent system:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -\left(4 + \frac{1}{10}e^{-\frac{10}{3}t}\right)\left(1 + \frac{1}{1+x^2}\right)y - \left(\frac{x}{1+x^2} + x\right) \\ &\quad - \left(1 + \frac{1}{1+t}\right)\left(2x + \frac{x}{1+|x|}\right) - \Delta x(t)\dot{\vartheta}(t) \\ &\quad + \left(1 + \frac{1}{1+t}\right)\int_{t-\tau_1(t)}^{t}\left(2 + \frac{1}{(1+|x(s)|)^2}\right)y(s)ds.\end{aligned} \tag{3.2}$$

When we compare the systems (3.2) and (1.2), we see the following relationships:

$$\phi(t) = 4 + \frac{1}{10}e^{-\frac{10}{3}t}, \quad \phi_0 = 4 \le \phi(t) \le 5 = \phi_1,$$
$$\dot{\phi}(t) = |-\frac{1}{3}e^{-\frac{10}{3}t}| \le \frac{1}{3} = \alpha, \text{for all } t \ge 0.$$

Figure 1, depict the functions $\phi(t)$ and the derivative $\dot{\phi}(t)$ on the interval $t \in [0, 50]$.
Next, the function

$$f(x, y) = 1 + \frac{1}{1+x^2},$$

it follows that

$$1 \le f(x, y) \le 2 = a_0, \quad f_y(x, y) = 0, \quad xf_y(x, y) \ge 0, \quad \text{for all } x, y.$$

The function $f(x, y)$ is shown in Figure 2.
The function

$$g_1(x) = \frac{x}{1+x^2} + x, \quad \text{clearly } g_1(0) = 0, \quad \text{and that } \frac{g_1(x)}{x} = 1 + \frac{1}{1+x^2} \ge 1 = \alpha_1.$$

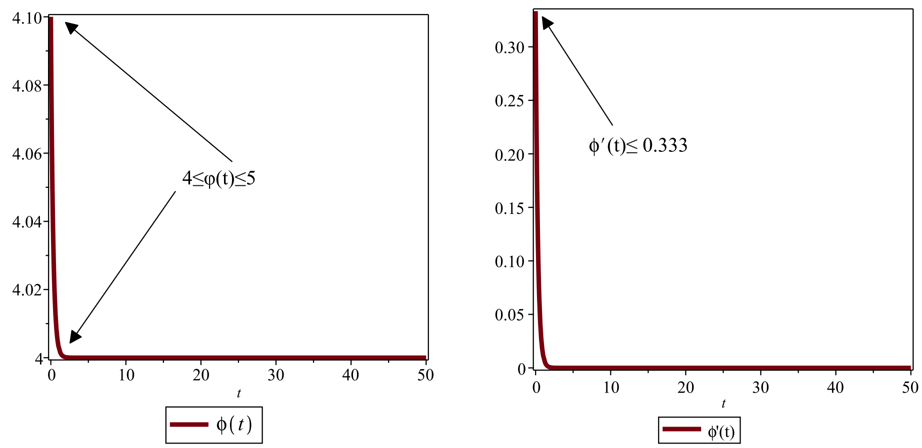

FIGURE 1. The paths of the functions $\phi(t)$ and $\dot{\phi}(t)$ for $t \in [0, 50]$.

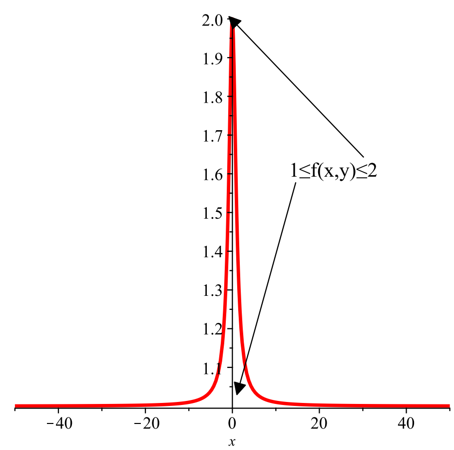

FIGURE 2. The path of the function $f(x, y)$ for $x \in [-40, 40]$.

Then, we have

$$0 < g_1'(x) = 1 + \frac{1 - x^2}{(1 + x^2)^2} \leq 2 = b_1.$$

The coinciding paths of $\frac{g_1(x)}{x}$ and $g_1'(x)$ are presented in Figure 3.
Moreover, the function

$$\psi_1(t) = \frac{1}{1+t}, \quad \text{since} \quad \delta_1 = 1 \leq \psi_1(t) \leq 2 = l_1,$$

furthermore, the derivative of the function $\psi_1(t)$ with respect to t is

$$\dot{\psi}_1(t) = \frac{-1}{(1+t)^2} \leq 0, \quad \text{for all} \quad t \geq 0,$$

See the bounds on the functions $\psi(t)$ and $\dot{\psi}(t)$ for $t \in [0, 50]$ in Figure 4.
We can choose $\mu_1 = \frac{1}{1000}$.
Also, we get

$$h_1(x) = 2x + \frac{x}{1 + |x|}, \; h_1(0) = 0, \text{ it tends to } \frac{h_1(x)}{x} = 2 + \frac{1}{1 + |x|} \geq 2 = \beta_1.$$

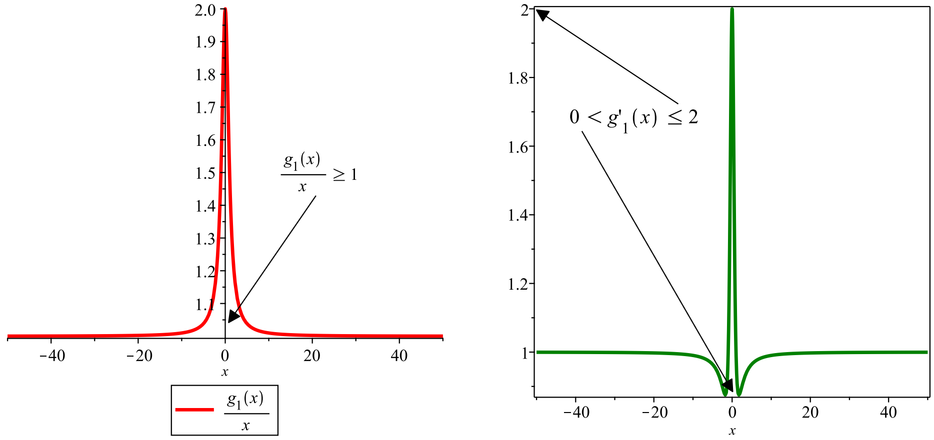

FIGURE 3. The behaviours of the functions $\frac{g_1(x)}{x}$ and $g_1'(x)$ for $x \in [-40, 40]$.

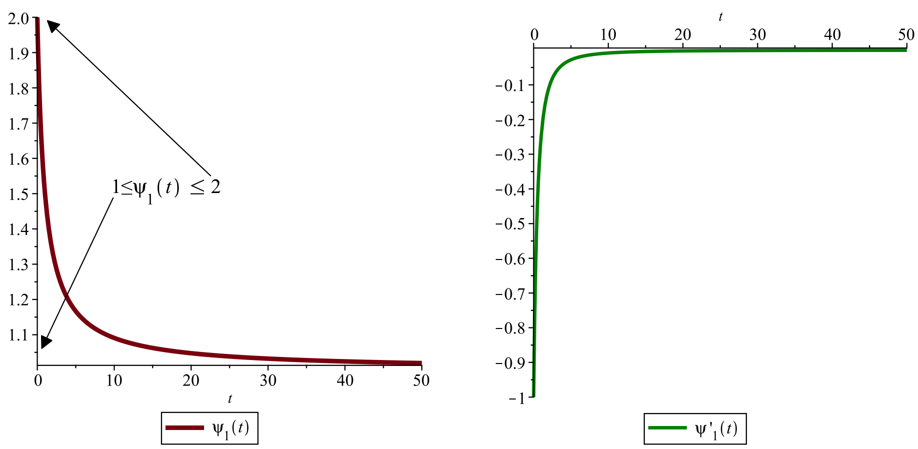

FIGURE 4. The trajectories of the functions $\psi(t)$ and $\dot{\psi}(t)$ for $t \in [0, 50]$.

Since $0 \leq \frac{1}{1+|x|} \leq 1$, for all x. So, the derivative of the function $h_1(x)$ with respect to x is

$$0 \leq h_1'(x) = 2 + \frac{1}{(1+|x|)^2} \leq 3 = \rho_1.$$

The behaviour of the functions $\frac{h_1(x)}{x}$ and $h_1'(x)$ for $x \in [-40, 40]$ are considered in Figure 5.

If we let the variable delay $\tau_1(t) = \frac{1}{16}\sin^2 t \leq \frac{1}{16} = \gamma$, with

$$\dot{\tau}_1(t) = \frac{1}{8}\sin t \cos t \leq \frac{1}{8} = \bar{\omega}.$$

Taking $\Delta = 1$, we can get the following estimates:

$$2\phi_0 - 1 = 7 > 0, \quad \delta_1\beta_1 + \alpha_1 + \frac{1}{2}\phi_0 = 5 > 1,$$

$$A_0 = \alpha_1 + \delta_1\beta_1 - \mu_1\rho_1 = 2.997 > 0,$$

$$M = \rho_1 l_1 = 6 > 0,$$

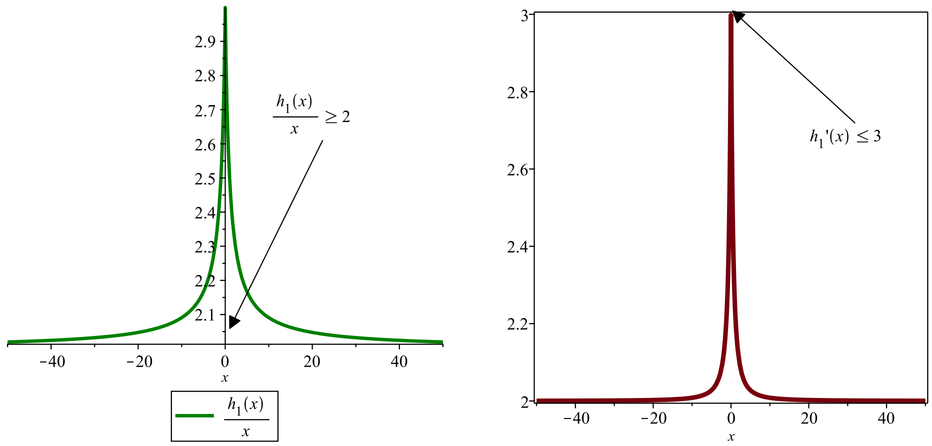

FIGURE 5. The paths of the functions $\frac{h_1(x)}{x}$ and $h_1'(x)$ for $x \in [-40, 40]$.

$$B_0 = \left\{1 + \frac{3}{2(1-\frac{1}{8})}\right\} \cong 2.7,$$

and

$$A_0 - \frac{1}{2}\alpha a_0 \cong 1.67 \geq 1 = \Delta^2.$$

Thus, the Liénard stochastic equation with variable delay (3.1) verifies all the hypotheses of Theorem 2.1.

Finally, if

$$\gamma = \frac{1}{16} < \min\{0.43, 0.11\} \cong 0.11.$$

Then the zero solution of (3.1) is stochastically asymptotically stable.

4. STOCHASTIC BOUNDEDNESS RESULT

Theorem 4.1. *In addition to the conditions* $(H_1)-(H_7)$ *in Theorem 2.1, we assume that* m *and* $\bar{\beta}$ *are positive constants such that:*

(H_8) $|e(t,x,y)| \leq m$.

(H_9) $2A_0 + 2\phi_0 \sum_{i=1}^n(\alpha_i + \delta_i\beta_i) - \bar{\beta}^2 - \alpha a_0 - (1+\bar{\beta})\sum_{i=1}^n b_i + \sum_{i=1}^n \mu_i\rho_i > (3+\bar{\beta})\Delta^2$.

(H_{10}) $2\phi_0(2+\bar{\beta}) - (2+\bar{\beta}^2) - (1+\bar{\beta})\sum_{i=1}^n b_i > 0$.

Then all the solutions of (1.1) *are uniformly stochastically bounded, provided that*

$$\gamma < \min\Bigg[\frac{2A_0 + 2\phi_0 \sum_{i=1}^n(\alpha_i + \delta_i\beta_i) - \bar{\beta}^2 - \alpha a_0 - (1+\bar{\beta})\sum_{i=1}^n b_i + \sum_{i=1}^n \mu_i\rho_i - (3+\bar{\beta})\Delta^2}{(1+\phi_0)M},$$
$$\frac{\{2\phi_0(2+\bar{\beta}) - (2+\bar{\beta}^2) - (1+\bar{\beta})\sum_{i=1}^n b_i\}(1-\bar{\omega})}{\{(3+\bar{\beta})(1-\bar{\omega}) + (4+\bar{\beta}+\phi_0)\}M}\Bigg].$$

Proof. It should be emphasised that in order to establish the uniform stochastic boundedness of solutions to the equation (1.1), we employ a methodology similar to that employed by [14, 17, 23].

Consider the boundedness of all (1.1) solutions. Assume $e(t,x,y)$ is bounded by a bound m and all conditions of Theorem 2.1 are met.
Consider the LF as

$$V(t,x_t,y_t) = V_1(t,x_t,y_t) + V_2(t,x_t,y_t), \tag{4.1}$$

where V_1 is defined as (2.1) and V_2 is defined as the following

$$\begin{aligned} V_2(t,x_t,y_t) =& \frac{1}{2}(\phi_0 x + y)^2 + \frac{1}{2}\bar{\beta}^2 x^2 + \frac{1}{2}\bar{\beta} y^2 + (1+\bar{\beta})\sum_{i=1}^{n} x g_i(x) \\ &+ (1+\bar{\beta})\sum_{i=1}^{n}\delta_i \int_0^x h_i(\eta)d\eta. \end{aligned} \tag{4.2}$$

From the conditions (H_3) and (H_5), we have

$$\sum_{i=1}^{n} x g_i(x) \geq \sum_{i=1}^{n}\alpha_i x^2 \text{ and } \sum_{i=1}^{n}\frac{h_i(x)}{x} \geq \sum_{i=1}^{n}\beta_i,$$

it follows that

$$V_2(t,x_t,y_t) \geq \frac{1}{2}(\phi_0 x + y)^2 + \frac{1}{2}\bar{\beta}^2 x^2 + \frac{1}{2}\bar{\beta} y^2 + (1+\bar{\beta})\sum_{i=1}^{n}\alpha_i x^2 + \frac{1}{2}(1+\bar{\beta})\sum_{i=1}^{n}(\delta_i\beta_i)x^2.$$

Hence, we get

$$V_2(t,x_t,y_t) \geq \frac{1}{2}(\phi_0 x + y)^2 + \frac{1}{2}\left\{\bar{\beta}^2 + (1+\bar{\beta})\sum_{i=1}^{n}(2\alpha_i + \delta_i\beta_i)\right\}x^2 + \frac{1}{2}\bar{\beta} y^2. \tag{4.3}$$

Then (2.6) and (4.3), show that there exists a positive constant D_1 such that

$$V(t,x_t,y_t) \geq D_1(x^2+y^2). \tag{4.4}$$

From the conditions (H_3) and (H_5), we are able to rewrite (4.2) as the following formula

$$V_2(t,x_t,y_t) \leq \frac{1}{2}(\phi_0 x + y)^2 + \frac{1}{2}\bar{\beta}^2 x^2 + \frac{1}{2}\bar{\beta} y^2 + (1+\bar{\beta})\sum_{i=1}^{n} b_i x^2 + \frac{1}{2}(1+\bar{\beta})\sum_{i=1}^{n}(\delta_i\rho_i)x^2.$$

Then, using the Cauchy-Schwarz inequality

$$V_2(t,x_t,y_t) \leq \frac{1}{2}\left\{\phi_0^2 + \bar{\beta}^2 + \phi_0 + (1+\bar{\beta})\sum_{i=1}^{n}(2b_i + \delta_i\rho_i)\right\}x^2 + \frac{1}{2}(\bar{\beta} + \phi_0 + 1)y^2. \tag{4.5}$$

So, from the inequalities (2.6) and (4.5) it is clear that there exists a positive constant D_2 such that

$$V(t,x_t,y_t) \leq D_2(x^2+y^2) + \chi\|y\|^2. \tag{4.6}$$

Going back to the above discussion, that is, from the inequalities (4.4) and (4.6), we can conclude that $V(t,x_t,y_t)$ satisfies the condition after

$$\|x\|^{q_1} \leq V(t,x) \leq \|x\|^{q_2},$$

where q_1 and q_2 are two positive constants, such that $q_1 \geq 1$.
Applying the Itô formula, the derivative of the functional $V_2(t, x_t, y_t)$ along the system (1.2), gives that

$$\begin{aligned}
\mathcal{L}V_2(t, x_t, y_t) =& (\phi_0 x + y)\Big\{\phi_0 y - \phi(t)f(x,y)y - \sum_{i=1}^n g_i(x) - \sum_{i=1}^n \psi_i(t)h_i(x) \\
&+ \sum_{i=1}^n \psi_i(t)\int_{t-\tau_i(t)}^t h_i'(x(s))y(s)ds + e(t,x,y)\Big\} + \bar{\beta}^2 xy \\
&+ \bar{\beta}y\Big\{ - \phi(t)f(x,y)y - \sum_{i=1}^n g_i(x) - \sum_{i=1}^n \psi_i(t)h_i(x) \\
&+ \sum_{i=1}^n \psi_i(t)\int_{t-\tau_i(t)}^t h_i'(x(s))y(s)ds + e(t,x,y)\Big\} \\
&+ (1+\bar{\beta})\Big\{\sum_{i=1}^n g_i(x)y + \sum_{i=1}^n g_i'(x)xy + \sum_{i=1}^n \delta_i h_i(x)y + \frac{1}{2}\Delta^2 x^2\Big\}.
\end{aligned} \tag{4.7}$$

Since $f(x,y) \geq 1$ and $\phi(t) \geq \phi_0$, it follows that

$$-\phi(t)f(x,y)y \leq -\phi_0 y.$$

Consequently, by the conditions $(H_1) - (H_5)$ and using the fact $2mn \leq m^2 + n^2$, we obtain the following inequalities

$$\begin{aligned}
&\phi_0 x \sum_{i=1}^n g_i(x) \geq \phi_0 \sum_{i=1}^n \alpha_i x^2, \quad -\sum_{i=1}^n \psi_i(t)h_i(x) \leq -\delta_i h_i(x), \\
&\phi_0 x \sum_{i=1}^n \psi_i(t)h_i(x) \geq \phi_0 \sum_{i=1}^n (\delta_i \beta_i)x^2, \quad \bar{\beta}^2 xy \leq \frac{1}{2}\bar{\beta}^2(x^2+y^2), \\
&(1+\bar{\beta})\sum_{i=1}^n g_i'(x)xy \leq (1+\bar{\beta})\sum_{i=1}^n b_i xy \leq \frac{1}{2}(1+\bar{\beta})\sum_{i=1}^n b_i(x^2+y^2).
\end{aligned}$$

Then we can write (4.7) as the following form

$$\begin{aligned}
\mathcal{L}V_2(t, x_t, y_t) \leq& -\frac{1}{2}\Big\{2\phi_0 \sum_{i=1}^n (\alpha_i + \delta_i\beta_i) - \bar{\beta}^2 - (1+\bar{\beta})\sum_{i=1}^n b_i - (1+\bar{\beta})\Delta^2\Big\}x^2 \\
&- \frac{1}{2}\Big\{2\bar{\beta}\phi_0 - \bar{\beta}^2 - (1+\bar{\beta})\sum_{i=1}^n b_i\Big\}y^2 \\
&+ (\phi_0 x + y + \bar{\beta}y)\Big\{\sum_{i=1}^n \psi_i(t)\int_{t-\tau_i(t)}^t h_i'(x(s))y(s)ds + e(t,x,y)\Big\}.
\end{aligned}$$

Since $|e(t,x,y)| \leq m$, $h_i'(x) \leq \rho_i$, $\tau_i(t) \leq \gamma$ and using the Cauchy-Schwarz inequality, we can write the above inequality as

$$
\begin{aligned}
\mathcal{L}V_2(t,x_t,y_t) \leq & -\frac{1}{2}\Big\{2\phi_0\sum_{i=1}^{n}(\alpha_i+\delta_i\beta_i)-\bar{\beta}^2-(1+\bar{\beta})\sum_{i=1}^{n}b_i-(1+\bar{\beta})\Delta^2\Big\}x^2 \\
& -\frac{1}{2}\Big\{2\bar{\beta}\phi_0-\bar{\beta}^2-(1+\bar{\beta})\sum_{i=1}^{n}b_i\Big\}y^2+m\phi_0|x|+(1+\bar{\beta})m|y| \\
& +\frac{1}{2}(1+\bar{\beta})\gamma\sum_{i=1}^{n}l_i\rho_i y^2+\frac{1}{2}(1+\bar{\beta})\sum_{i=1}^{n}(l_i\rho_i)\int_{t-\tau_i(t)}^{t}y^2(s)ds \\
& +\frac{1}{2}\phi_0\gamma\sum_{i=1}^{n}(l_i\rho_i)x^2+\frac{1}{2}\phi_0\sum_{i=1}^{n}l_i\rho_i\int_{t-\tau_i(t)}^{t}y^2(s)ds.
\end{aligned}
\tag{4.8}
$$

Now, we can deduce from the system (1.2), (2.3) and the condition (H_8) that

$$
\begin{aligned}
\mathcal{L}V_1(t,x_t,y_t) \leq & -\frac{1}{2}\Big\{\sum_{i=1}^{n}(2\alpha_i+2\delta_i\beta_i-\mu_i\rho_i)-\alpha a_0-\sum_{i=1}^{n}(\rho_i l_i)\gamma-2\Delta^2\Big\}x^2 \\
& -\frac{1}{2}\Big\{4\phi_0-2-2\sum_{i=1}^{n}(\rho_i l_i+\mu_i)\gamma\Big\}y^2+m|x|+2m|y| \\
& +\sum_{i=1}^{n}\Big\{\frac{3}{2}(\rho_i l_i)-\lambda_i(1-\bar{\omega})\Big\}\int_{t-\tau_i(t)}^{t}y^2(s)ds.
\end{aligned}
\tag{4.9}
$$

Hence, by combining two inequalities (4.8) and (4.9), we have

$$
\begin{aligned}
\mathcal{L}V \leq & -\frac{1}{2}\Big\{2(\phi_0+1)\sum_{i=1}^{n}(\alpha_i+\delta_i\beta_i)-\bar{\beta}^2-\alpha a_0-(1+\bar{\beta})\sum_{i=1}^{n}b_i-(\phi_0+1)\gamma\sum_{i=1}^{n}(\rho_i l_i) \\
& -\sum_{i=1}^{n}(\mu_i\rho_i)-(3+\bar{\beta})\Delta^2\Big\}x^2+(1+\phi_0)m|x|+(3+\bar{\beta})m|y| \\
& -\frac{1}{2}\Big\{2(2+\bar{\beta})\phi_0-(2+\bar{\beta}^2)-(1+\bar{\beta})\sum_{i=1}^{n}b_i-(3+\bar{\beta})\gamma\sum_{i=1}^{n}(\rho_i l_i)-2\sum_{i=1}^{n}\mu_i\gamma\Big\}y^2 \\
& +\sum_{i=1}^{n}\Big\{\frac{1}{2}(4+\bar{\beta}+\phi_0)l_i\rho_i-\lambda_i(1-\bar{\omega})\Big\}\int_{t-\tau_i(t)}^{t}y^2(s)ds.
\end{aligned}
$$

If we choose

$$
\lambda_i=\frac{(4+\bar{\beta}+\phi_0)l_i\rho_i}{2(1-\bar{\omega})}.
$$

Since $M = \sum_{i=1}^{n} \rho_i l_i > 0$ and $A_0 = \sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i - \mu_i\rho_i) > 0$, then we can rewrite the above inequality as

$$\begin{aligned}\mathcal{L}V \leq &- \frac{1}{2}\bigg\{2A_0 + 2\phi_0\sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i) - \bar{\beta}^2 - \alpha a_0 - (1+\bar{\beta})\sum_{i=1}^{n} b_i - (\phi_0+1)\gamma M \\ &\qquad + \sum_{i=1}^{n}(\mu_i\rho_i) - (3+\bar{\beta})\Delta^2\bigg\}x^2 + (1+\phi_0)m|x| + (3+\bar{\beta})m|y| \\ &- \frac{1}{2}\bigg\{2(2+\bar{\beta})\phi_0 - (2+\bar{\beta}^2) - (1+\bar{\beta})\sum_{i=1}^{n} b_i - (3+\bar{\beta})\gamma M - \frac{(4+\bar{\beta}+\phi_0)M}{2(1-\bar{\omega})}\gamma\bigg\}y^2.\end{aligned}$$

By taking

$$\begin{aligned}\gamma < \min\bigg[&\frac{2A_0 + 2\phi_0\sum_{i=1}^{n}(\alpha_i + \delta_i\beta_i) - \bar{\beta}^2 - \alpha a_0 - (1+\bar{\beta})\sum_{i=1}^{n} b_i + \sum_{i=1}^{n}\mu_i\rho_i - (3+\bar{\beta})\Delta^2}{(1+\phi_0)M}, \\ &\frac{\{2\phi_0(2+\bar{\beta}) - (2+\bar{\beta}^2) - (1+\bar{\beta})\sum_{i=1}^{n} b_i\}(1-\bar{\omega})}{\{(3+\bar{\beta})(1-\bar{\omega}) + (4+\bar{\beta}+\phi_0)\}M}\bigg].\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathcal{L}V(t,x_t,y_t) &\leq -D_3(x^2+y^2+z^2) + D_3\sigma(|x|+|y|) \\ &= -\frac{D_3}{2}(x^2+y^2) - \frac{D_3}{2}\bigg\{(|x|-\sigma)^2 + (|y|-\sigma)^2\bigg\} + D_3\sigma^2 \\ &\leq -\frac{D_3}{2}(x^2+y^2) + D_3\sigma^2, \text{ for some } D_3, \sigma > 0,\end{aligned}$$

where

$$\sigma = m\max\{1+\phi_0, 3+\bar{\beta}\}.$$

Taking $\nu(t) = \frac{D_3}{2}, \zeta(t) = D_3\sigma^2$ and $r = 2$, therefore the derivative of LF $V(t,x_t,y_t)$ is fulfilled the following condition

$$\mathcal{L}V(t,x) \leq -\nu(t)\|x\|^r + \zeta(t), \text{ for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

We can also see that the condition

$$V(t,x) - V^{r/q_2}(t,x) \leq \kappa,$$

is satisfied with $q_1 = q_2 = r = 2$ and $\kappa = 0$.
Then, according to [14, 17, 23], all hypotheses of Theorem 2.2 hold. As a result of using $\nu(t) = \frac{D_3}{2}, \zeta(t) = D_3\sigma^2$ and $\kappa = 0$, we find that

$$\begin{aligned}\int_{t_0}^{t}\{\kappa\nu(\theta) + \zeta(\theta)\}e^{-\int_\theta^t \nu(\phi)d\phi}d\theta &= D_3\sigma^2\int_{t_0}^{t} e^{-\frac{D_3}{2}\int_\theta^t d\phi}d\theta \\ &\leq 2\sigma^2, \text{ for all } t \geq t_0 \geq 0,\end{aligned} \tag{4.10}$$

We now have the following relationships

$$g^T = (0 \qquad -\Delta x),$$

$$V_x = (V_1)_x + (V_2)_x = 2\sum_{i=1}^{n} \psi_i(t)h_i(x) + 2\sum_{i=1}^{n} g_i(x) + \phi(t)f(x,0) + y + (\phi_0 x + y)\phi_0$$
$$+ \bar{\beta}^2 x + (1+\bar{\beta})\sum_{i=1}^{n} g_i(x) + (1+\bar{\beta})\sum_{i=1}^{n} x g_i'(x) + (1+\bar{\beta})\sum_{i=1}^{n} \delta_i h_i(x),$$

$$V_y = (V_1)_y + (V_2)_y = (1+\phi_0)x + (3+\bar{\beta})y.$$

Then, we have

$$\left|V_{x_i}(t,x_t)g_{ik}(t,x(t))\right| \leq \Delta\left\{\frac{1}{2}\left(5+2\phi_0+\bar{\beta}\right)x^2 + \frac{1}{2}\left(3+\bar{\beta}\right)y^2\right\} := \mathcal{N}(t).$$

Therefore, all solutions to equation (1.1) are uniformly asymptotically bounded and satisfied according to Lemma 2.4 [14, 17]

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \{\mathcal{C}x_0^2 + 2\sigma^2\}^{1/2}, \text{ for all } t \geq t_0 \geq 0 \text{ and } \mathcal{C} \text{ is a constant.}$$

This concludes the proof of Theorem 4.1.

Next

$$\int_{t_0}^{t} \{\kappa\nu(s) + \zeta(s)\} e^{-\int_{t_0}^{s}\nu(u)du} ds = D_3\sigma^2 \int_{t_0}^{t} e^{\frac{D_3}{2}\int_{t_0}^{s} du} ds$$
$$= 2\sigma^2 (e^{\frac{D_3}{2}(t-t_0)} - 1) \leq \mathcal{M}, \text{ for all } t \geq t_0 \geq 0,$$

where $\mathcal{M}$ is a positive constant.

Hence by Lemma 2.5 [14, 17, 23], the zero solution of (1.1) is α-uniformly exponentially asymptotically stable in probability with $N = \frac{1}{q_1} = \frac{1}{2}$.

5. BOUNDEDNESS EXAMPLE

In this section, we will show an example of how to apply the boundedness result obtained in the previous section.

Let $\bar{\beta} = 2$ and $m = 0.02$, therefore from the example in section 3, we obtain the following relations

$$2A_0 + 2\phi_0(\alpha_1 + \delta_1\beta_1) - \bar{\beta}^2 - \alpha a_0 - (1+\bar{\beta})b_1 + \mu_1\rho_1 \cong 19.330 > (3+\bar{\beta})\Delta^2 = 5,$$

$$2\phi_0(2+\bar{\beta}) - (2+\bar{\beta}^2) - (1+\bar{\beta})b_1 = 20 > 0.$$

So, we get

$$\gamma = \frac{1}{16} < \min\{0.478, 0.203\} \cong 0.203,$$

also, it follows that

$$\sigma = 0.02\max\{5,5\} = 0.1,$$

$$\mathcal{L}V(t,x_t,y_t) \leq -4.12x^2 - 6.92y^2 + 0.1|x| + 0.1|y|.$$

If we take

$$D_3 = 2.7085, \ \sigma = 0.1, \ \nu(t) = 2.06, \ \zeta(t) = 0.412 \text{ and } r = 2.$$

Thus all conditions of Theorem 2.2 [14, 17, 23] are satisfied with $q_1 = q_2 = r = 2$ and $\kappa = 0$. It is clear that

$$|V_{x_i}(t, x_t) g_{ik}(t, x(t))| \leq \frac{15}{2} x^2 + \frac{5}{2} y^2 := \mathcal{N}(t).$$

$$\int_{t_0}^{t} \{\kappa\nu(\theta) + \zeta(\theta)\} e^{-\int_\theta^t \nu(\phi) d\phi} d\theta \leq 0.02, \text{ for all } t \geq t_0 \geq 0.$$

As a result, according to Lemma 2.4 [14, 17, 23], all solutions of (3.1) are uniformly asymptotically bounded and satisfied

$$E^{x_0} \|x(t, t_0, x_0)\| \leq \{x_0^2 + 0.02\}^{1/2}, \text{ for all } t \geq t_0 \geq 0, \text{ and } \mathcal{C} \text{ is a constant.}$$

Next

$$\int_{t_0}^{t} \{\kappa\nu(s) + \zeta(s)\} e^{-\int_{t_0}^s \nu(u) du} ds = 0.02(e^{2.06(t-t_0)} - 1) \leq \mathcal{M}, \text{ for all } t \geq t_0 \geq 0,$$

where $\mathcal{M}$ is a positive constant.

As a result of Lemma 2.5 [14, 17, 23], the zero solution of (3.1) is α-uniformly exponentially asymptotically stable in probability with $N = \frac{1}{q_1} = \frac{1}{2}$.

6. NUMERICAL SIMULATIONS

In this section, we investigate the behaviour of the solution for equation (3.1) using an EM-based numerical method that allows us to obtain approximate numerical solutions for the considered system. We demonstrate the stability of the solutions for various values of the numerical method's step size h, and we can choose the initial solution of $(x(t), y(t))$ to be $(x(0) = 1, y(0) = 1)$.

Figure 6 depicts the behaviour of the solutions with $h = 0.1$, $\Delta = 0.5$ and the error value $\varepsilon = 0.0073$, indicating that we have a stable system.

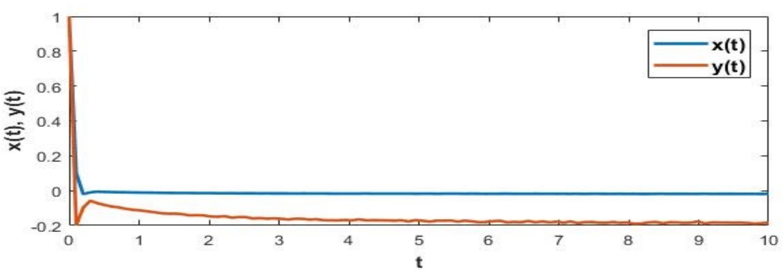

FIGURE 6. Trajectory of the solution for (3.1) with $\varepsilon = 0.0073, h = 0.1, \Delta = 0.5$.

Figures 7, 8, 9 are obtained by varying the value of h as $h = 0.2, h = 0.3, h = 0.5$ with $\varepsilon = 0.0804, \varepsilon = 0.0198, \varepsilon = 0.3473$ and $\Delta = 0.5$.

As can be seen, all of the solutions are stable. Figures 10 and 11 show the behaviour of solutions with $\Delta = 10$ and two different values of h as $h = 0.1, h = 0.3$ and $\varepsilon = 0.0084, \varepsilon = 0.6683$, respectively.

It is possible to see that the stochastic increases as the noise level increases and the error decreases as the noise level decreases.

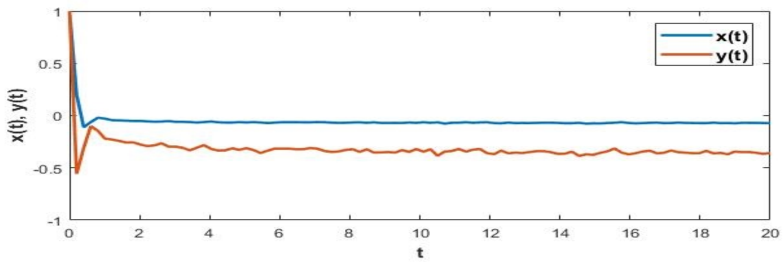

FIGURE 7. The solution's trajectory for (3.1) with $\varepsilon = 0.0804, h = 0.2, \Delta = 0.5$.

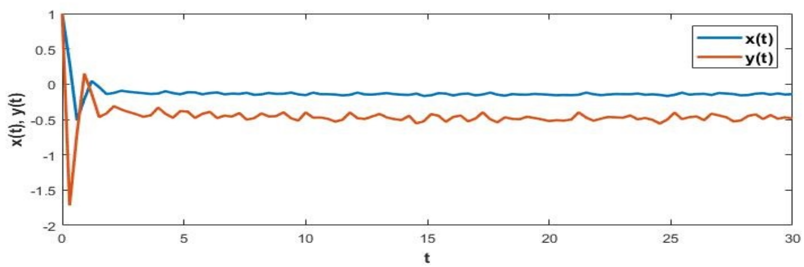

FIGURE 8. Trajectories of equation (3.1) with $\varepsilon = 0.0198, h = 0.3, \Delta = 0.5$.

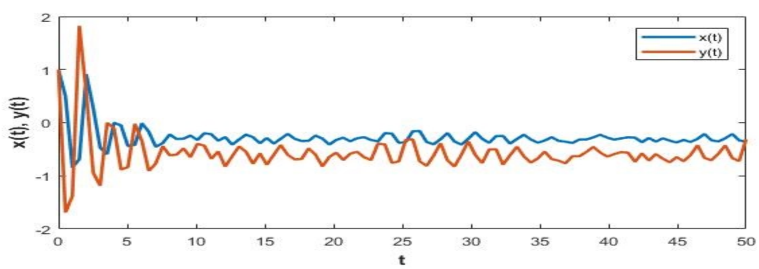

FIGURE 9. The path of the solution for (3.1) with $\epsilon = 0.3473, h = 50, \Delta = 0.5$.

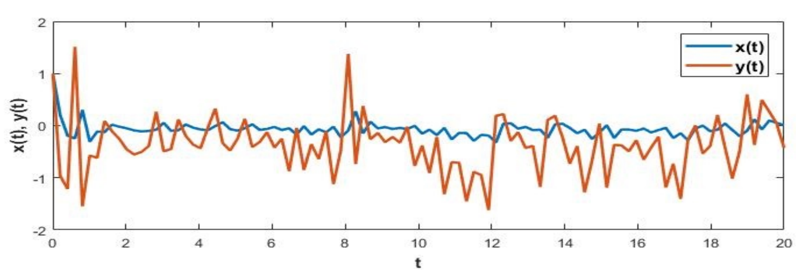

FIGURE 10. The behaviour of the solution for (3.1) with $\varepsilon = 0.0084, h = 0.2, \Delta = 10$.

Figure 12, on the other hand, depicts the behaviour of the solutions when $h = 1.5$ and $\Delta = 0.5$. We can see that as the value of h increases, so does the value of ε, and we no longer have a stable system.

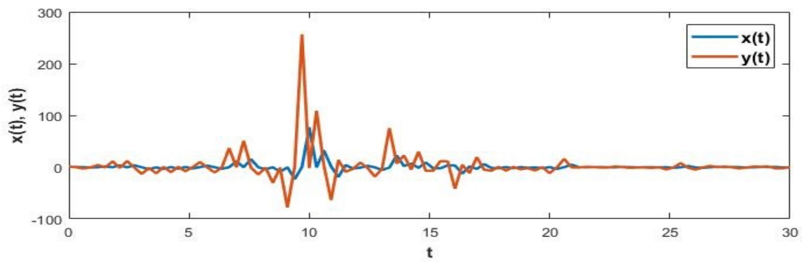

FIGURE 11. Stability diagram for (3.1) with $\varepsilon = 0.6683, h = 0.3, \Delta = 10$.

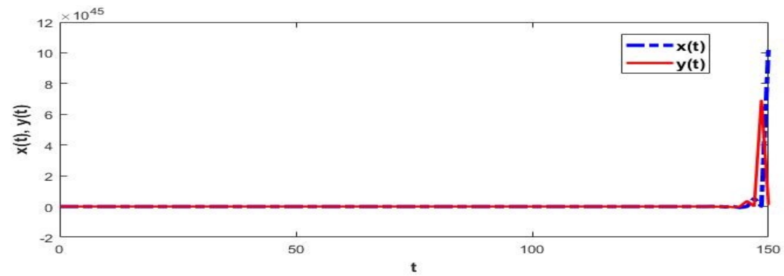

FIGURE 12. Trajectory of the solution for (3.1) with $\varepsilon = Inf., h = 1.5, \Delta = 0.5$.

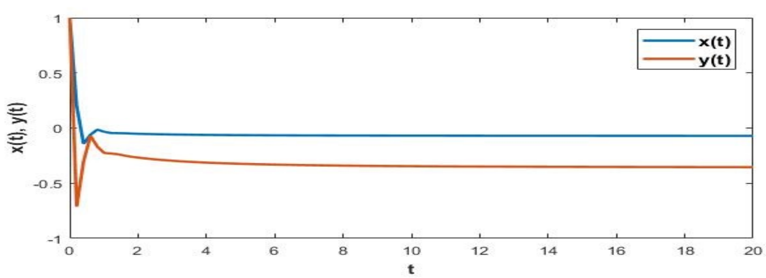

FIGURE 13. The diagram of stability for (3.1) with $\varepsilon = 0, \Delta = 0$.

Finally, Figure 13 gives the behaviour of the solutions for (3.1) when $\Delta = 0$, noting that in this case $\varepsilon = 0$ and the system is stable.

7. **CONCLUSIONS**

In this study, we analyse the uniformly stochastically boundedness and stochastically asymptotically stability of the underlying equation and the suggested numerical approach for the non-autonomous multi-delay stochastic Liénard equation. Here, some new sufficient conditions for the stability and boundedness in probability of solutions are discovered by defining appropriate LFs and using the EM-method with the sufficiently small step size to simulate the stability of solutions for stochastic Liénard equation with multiple delays. Our findings add to numerous excellent earlier discoveries in the literature. We will keep researching other qualitative characteristics of stochastic delay differential equation solutions in the future.

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