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EXISTENCE OF SOLUTIONS FOR THE ONE-DIMENSIONAL FOURTH-ORDER *P*-LAPLACIAN IMPULSIVE DIFFERENTIAL EQUATION INVOLVING NONLINEAR STIELTJES INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we present extended and improved results on the existence of solutions for the one-dimensional p-Laplacian impulsive differential equation with nonlinear Stieltjes integral boundary conditions, where the nonlinearity is a a. e. continuous function involving first order and second order as well as third order derivative of the unknown abstract function. We also provide examples to show the valid of our results. In particular, our results unify many known results.

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Key Words and Phrases. Stieltjes integral boundary conditions; Existence; impulsive differential equation; p-Laplacian.

1. INTRODUCTION

In this paper, we investigate the existence of solutions for the following one-dimensional singular p-Laplacian with nonlinear Stieltjes integral boundary conditions

$$
\begin{cases}
(\varphi_p(y'''))' = b(t)g(t,\, y,\, y',\, y'',\, y'''),\ t \in J' = J \setminus \{t_0, \cdots, t_{m+1}\}, \\
y(t_k^+) = y(t_k^-) + I_k(y(t_k)), k = 1, 2, \ldots, m, \\
y'(t_k^+) = y'(t_k^-) + N_k(y'(t_k)), k = 1, 2, \ldots, m, \\
y''(t_k^+) = y''(t_k^-) + L_k(y''(t_k)), k = 1, 2 \ldots, m, \\
y'''(t_k^+) = y'''(t_k^-) + R_k(y'''(t_k)), k = 1, 2, \ldots, m, \\
\eta y(0) - \lambda_1 y'(0) = \int_0^1 a_1(s)y(s)d\nu(s), \\
\eta y(1) + \lambda_2 y'(1) = \int_0^1 a_2(s)y(s)d\nu(s), \\
\eta y''(0) - \lambda_3 y'''(0) = \int_0^1 a_3(s)y''(s)d\nu(s), \\
\eta y''(1) + \lambda_4 y'''(1) = \int_0^1 a_4(s)y''(s)d\nu(s),
\end{cases}
\tag{1.1}
$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta > 0$, $\lambda_i > 0$ for $i = 1, 2, 3, 4$, $J = [0, 1]$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, where m is a fixed positive integer, ν, I_k, N_k, L_k and R_k are continuous and nondecreasing functions for $k = 1, \cdots, m$,

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as well as $y(t_k^+)$ with $y(t_k^-)$ represent the right-hand limit and left-hand limit of $y(t)$ at $t = t_k$, $b \in C(0,1)$, $b(t)$ may be singular at $t = 0$ and/or $t = 1$, together with $g > 0$ is a. e. continuous on $[0,1] \times (0,+\infty) \times (-\infty,+\infty)^3$.

Fourth-order p-Laplacian equations with nonlinear Stieltjes integral boundary conditions play an important role in both theory and applications. They have been attracted many people's attention over the years, see ([1]−[32]) and the references therein. They are often used to model various phenomena in physics, chemistry, biology, and infections diseases in the positive energy problems. However, in various situations, including the cases just mentioned above, based on the method of upper solution and lower solution, the existence of solution are easily established, one refers the reader to see ([2]−[33]) for some references along this line. However, the existence of solutions for p-Laplacian equations boundary value problems has been investigated by a lot of authors applying various nice methods such as topological degree, the Leray- Schauder continuation theorem and coincidence degree theory, maximum principle and so on, see ([9], [13], [22], [24]−[33]).

In [9](2004), He considered the existence of double positive solutions for the following three-point boundary value problems

$$\begin{cases} (\varphi_p(z'))' + \widehat{a}(t)\widehat{f}(z(t)) = 0, \quad 0 < t < 1, \\ z(0) - \mathfrak{B}_0(z'(\xi)) = 0, \quad z(1) - \mathfrak{B}_1(z'(1)) = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} (\varphi_p(z'))' + \widehat{a}(t)\widehat{f}(z(t)) = 0, \quad 0 < t < 1, \\ z(0) - \mathfrak{B}_0(z'(0)) = 0, \quad z(1) - \mathfrak{B}_1(z'(\xi)) = 0. \end{cases} \tag{1.3}$$

The author employed a fixed point theorem due to Avery and Henderson.

In [10](2004), He and Ge were concerned with the following two-point boundary value problems

$$\begin{cases} (\phi_p(z'))' + \widehat{q}(t)\widehat{f}(t, z(t)) = 0, \quad 0 \le t \le 1, \\ z(0) = g_1(z'(0)), \quad z(1) + g_2(z'(1)) = 0. \end{cases} \tag{1.4}$$

The main tool in the paper is the fixed point theorem in cones due to Krasnoselskii.

V*á*zquez [27](2022) obtained the existence, uniqueness together with quantitative estimates of solutions for a class of the fractional nonlinear diffusion equation

$$\partial_t z + \Upsilon_{s,p}(z) = 0, \tag{1.5}$$

where $\Upsilon_{s,p} = (-\Delta)_p^s$ is the standard fractional p-Laplacian operator, $0 < s < 1$ and $1 < p < 2$.

In [26] (2006), by making use of the weighted a priori estimate, v*á*zquez (2006) studied extinction in finite time of fast diffusion equations (1.5). In [22] (2017), Rynne pondered the following boundary value problem

$$(1.6)\qquad \begin{cases} -\left(t^{N-1}\phi_p(y'(t))\right)' - \widetilde{\lambda} t^{N-1}\widehat{f}(t,y(t)) = 0, \quad 0 < t < 1, \\ \mathfrak{BC}_N(y) = (0,0), \end{cases}$$

where $N \geq 1$ is an integer, $\phi_p(s) := |s|^{p-1}\text{sign}s$, $s \in \mathbb{R}$, $p \in \mathbb{R}$ satisfies $p > 1$ and $p \neq 2$, $\lambda \geq 0$, with

$$(1.7)\qquad \mathfrak{BC}_N(y) = \begin{cases} (y(0), y(1)), & \text{if } N = 1, \\ (y'(0), y(1)), & \text{if } N > 1, \end{cases}$$

The author presented the results on simple bifurcation and existence of a curve of positive solutions removing certain restriction.

In [16] (2012), by applying the analytic approaches such as comparison principle, vector calculus on networks and maximum principle, etc., Lee and Chung established the long time behaviors of nontrivial solutions for the p-Laplacian evolution $z_t = \Delta_p z$, with $p > 1$ and showed that the solution remains strictly positive for $p \geq 2$ and became extinct for $1 < p < 2$.

In [31](2022), Wettstein investigated the fractional harmonic gradient flow on S^1 getting value in $S^{n-1} \subset R^n$ for all $n \geq 2$, in particular establishing uniqueness and regularity of solutions in the so-called class through small enough energy for the weak fractional harmonic gradient flow: $z_t + (-\Delta)^{\frac{1}{2}} z = z|d_{\frac{1}{2}} z|^2$, satisfying $z(0,\cdot) = z_0$ in the sense $z(t,\cdot) \to z_0$ in L^2 as $t \to 0$, putting the existence of solutions. The author generalized and extended many known results (see [20]$-$[23]). Further, he contemplated convergence properties for solutions to the fractional gradient flow as $t \to \infty$.

Motivated by the results mentioned above, in the paper we study the existence of positive solutions for the problem (1.1). Usually, the problem (1.1) can be used to consider the numerical solutions. In this paper, however, we apply the analytic approaches, such as upper and lower solutions, comparison principle and uniqueness of solution, instead of numerical ones. As far as we know, a lot of nice of works of the problem (1.1) are concerned with the numerical approach, but few works are constructed by the analytic method and fixed point theory. We should also assert here that our results are new and generalize together with improve the results in ([2]$-$[10], [16]$-$[31]).

The rest of the paper is organized as follows. In Section 2, we first introduce several lemmas and definitions with notations frequently exploited through the paper. In Section 3, we foremost give a lemma and offer some key conditions. And then, we derive the interesting properties of solutions of the problem (1.1). We also present the main results as well as some their proofs. Finally, in Section 4, we supply some examples to show the valid of the main results.

2. PRELIMINARIES

Definition 2.1. [8] Let $\mathfrak{X}$ be a real Banach space. A nonempty closed convex set $\mathfrak{P} \subset \mathfrak{X}$ is said to be a cone provided that

(i) $y \in \mathfrak{P}$, $\tau \geq 0$ implies $\tau y \in \mathfrak{P}$;

(ii) $y \in \mathfrak{P}$, $-y \in \mathfrak{P}$ implies $y = 0$.

Definition 2.2. [8] Let $\mathfrak{X}$ be a real Banach space and $\mathfrak{P}$ be a cone in $\mathfrak{X}$. A mapping α is called to be the nonnegative continuous concave functional on $\mathfrak{P}$ if $\alpha : \mathfrak{P} \longrightarrow [0, +\infty)$ is continuous and

$$\alpha(\tau t + (1-\tau)s) \geq \tau\alpha(t) + (1-\tau)\alpha(s), \quad s,\, t \in \mathfrak{P},\ \tau \in [0,1].$$

Let $\mathfrak{X} = C[0,1]$ be a Banach space with the norm $\| \ y \ \| = \sup\limits_{0\leq t\leq 1} \mid y(t) \mid$, and let $\overline{K} = \{y \in \mathfrak{X} : y(t) \geq 0, \quad 0 \leq t \leq 1\}$. Then $\overline{K}$ is a positive cone in $\mathfrak{X}$.

Throughout the paper, the partial ordering is always given by $\overline{K}$. For the concepts and properties of Krein-Kutmann theorems and fixed point index theory, one refers the reader to see [8]. For $\theta \in (0, \frac{1}{2})$, let

$$\mathfrak{P} = \{y \in K \mid \min_{t\in[\theta,1-\theta]} y(t) \geq \theta\|y\|,\ y(\tau t+(1-\tau)s) \geq \tau y(t)+(1-\tau)y(s),\ s,\, t \in [0,1]\}.$$

Denote

$$PC[J,\mathbb{R}] = \left\{ \begin{array}{l} y \mid y \text{ is a map from } J \text{ onto } \mathbb{R} \text{ such that } y(t) \text{ is continuous at } t \neq t_k, \\ \text{left continuous at } t = t_k, \text{ and its right limit exists at } t = t_k \\ \text{(denoted by) } y(t_k^+), \text{ for } k = 1, \cdots, m. \end{array} \right\}$$

Evidently, $PC[J,\mathbb{R}]$ is a Banach space with norm $\|y\|_{PC(J,\mathbb{R})} = \sup\limits_{t\in J} \|y(t)\|$.

$$PC^1[J,\mathbb{R}] = \left\{ \begin{array}{l} y \mid y \text{ is a map from } J \text{ onto } \mathbb{R} \text{ such that } y'(t) \text{ is continuous at } t \neq t_k, \\ \text{left continuous at } t = t_k, \text{ and } y(t_k^-),\ y(t_k^+),\ y'(t_k^-),\ y'(t_k^+), \\ y(t_k^-) = y(t_k^+) = y(t_k), \text{ exist for } k = 1, \cdots, m. \end{array} \right\}$$

Obviously, $PC^1[J,\mathbb{R}]$ is a Banach space with norm

$$\|y\|_{PC^1(J,\mathbb{R})} = \sup_{t\in J}\{\|y\|_{PC(J,\mathbb{R})},\ \|y'\|_{PC(J,\mathbb{R})}\}.$$

It is noticed that $\mathfrak{P} \subset \overline{K} \subset \mathfrak{X}$. Denote $P_r = \{y \in \mathfrak{P} : \|y\| < r\}$, $\partial P_r = \{y \in \mathfrak{P} : \|y\| = r\}$, $\overline{P}_{r,R} = \{y \in \mathfrak{P} : r \leq \|y\| \leq R\}$, for any positive constants $0 < r < R < +\infty$. Let $y' = v$, $y(0) = 0$, $y(s) = \int_0^s v(t)dt + y(0) = \int_0^s v(t)dt$.

Now we study the following problem

$$
\left\{\begin{array}{l}
y''+x(t)=0, t\in J',\\
y\left(t_k^+\right)=y\left(t_k^-\right)+I_k\left(y\left(t_k\right)\right),\ k=1,\ \dots,m,\\
y'\left(t_k^+\right)=y'\left(t_k^-\right)+N_k\left(y'\left(t_k\right)\right),\ k=1,\ \dots,m,\\
y(0)-\frac{\lambda_1}{\eta}y'(0)=\frac{1}{\eta}\int_0^1 a_1(s)y(s)d\nu(s),\\
y(1)+\frac{\lambda_2}{\eta}y'(1)=\frac{1}{\eta}\int_0^1 a_2(s)y(s)d\nu(s).
\end{array}\right. \tag{2.1}
$$

We easily obtain the following results (2.1).

Lemma 2.3. *Let $x\in C[0,\,1]$ be positive on $[0,\,1]$. Then the problem (2.1) admits a unique solution y which is given by*

$$
\begin{aligned}
y(t)=\frac{1}{\eta}\Big\{&\int_0^1 a_1(s)x(s)d\nu(s)+\delta(\lambda_1+\eta t)\int_0^1(a_1(s)-a_2(s))x(s)d\nu(s)\\
&+\int_0^1 G(t,s)x(s)d\nu(s)+\sum_{0<t_k<t}[(t-t_k)N_k(y'(t_k))+I_k(y(t_k))]\\
&-\delta(\lambda_1+\eta t)\sum_{k=1}^{m}[\eta I_k(y(t_k))+(\eta-\eta t_k+\lambda_2)N_k(y'(t_k))]\Big\}
\end{aligned} \tag{2.2}
$$

where $\delta=\frac{1}{\eta+\lambda_1+\lambda_2}$, and

$$
G(t,s)=\left\{\begin{array}{ll}
\delta(\lambda_1+\eta s)(\eta t-\eta-\lambda_2) & \text{if}\ \ 0\le s<t\le 1,\\
\delta(\lambda_1+\eta t)(\eta s-\eta-\lambda_2) & \text{if}\ \ 0\le t\le s\le 1.
\end{array}\right. \tag{2.3}
$$

Proof. It is well known that the problem (2.1) is equivalent to the integral equation (2.2). □

Let

$$
\begin{aligned}
Ay(t)=\frac{1}{\eta}\Big\{&\int_0^1 a_1(s)x(s)d\nu(s)+\delta(\lambda_1+\eta t)\int_0^1(a_1(s)-a_2(s))x(s)d\nu(s)\\
&+\int_0^1 G(t,s)x(s)d\nu(s)+\sum_{0<t_k<t}[(t-t_k)N_k(y'(t_k))+I_k(y(t_k))]\\
&-\delta(\lambda_1+\eta t)\sum_{k=1}^{m}[\eta I_k(y(t_k))+(\eta-\eta t_k+\lambda_2)N_k(y'(t_k))]\Big\}
\end{aligned} \tag{2.4}
$$

where $G(t,s)$ is defined by (2.3). Obviously $A:PC[0,1]\longrightarrow PC[0,1]$ is completely continuous. We conclude that A has a unique nontrivial fixed point $y(t)$ in $PC[0,1]$. Therefore, the problem has a unique solution.

Lemma 2.4. *Let $y\in PC^1(J,\mathbb{R})\cap C^2(J,\mathbb{R})$ and*

$$
\left\{\begin{array}{l}
y''\le 0,\ t\in J',\\
y\left(t_k^+\right)=y\left(t_k^-\right)+I_k\left(y\left(t_k\right)\right),\ k=1,\dots,m,\\
y'\left(t_k^+\right)\le y'\left(t_k^-\right)+N_k\left(y'\left(t_k\right)\right),\ k=1,\dots,m,\\
y(0)-\frac{\lambda_1}{\eta}y'(0)\ge 0,\\
y(1)+\frac{\lambda_2}{\eta}y'(1)\ge 0.
\end{array}\right. \tag{2.5}
$$

Then $y(t)\ge 0$, for all $t\in J$.

Proof. By simple computation, we can easily obtain the result. Noticing that the graph of $y(t)$ on $[0,1]$ is concave. The proof is omitted. □

Throughout of the paper, we suppose that the following conditions hold:

$(\mathbf{A}_0)$ $g \in C(J\times(0,+\infty)\times(-\infty,+\infty)^3,\ [0,+\infty))$ for $t\neq t_k$, $k=1,\cdots,m$ with

$$\lim_{(s,x,\varrho,y,z)\to(s,x_0,\varrho_0,y_0,z_0)} g(s,x,\varrho,y,z), \quad \text{exists for} \quad t=t_k;$$

$(\mathbf{A}_1)$ $b\in L^1((0,1),\ [0,+\infty))$, $b(t)$ may be singular at $t=1$ and/or $t=0$, and

$$0<\int_0^1 b(s)d\nu(s)<+\infty. \tag{2.6}$$

$(\mathbf{A}_2)$ $g\,(t,\,y,\,y',\,y'',\,y''')\leq h(t,y)$, and $h(t,y)\in C([0,1]\times(0,+\infty),\ [0,+\infty))$, $h(t,y)$ may be singular at $y=0$ and for any $0<r<R<+\infty$, we have

$$\lim_{j\to+\infty}\sup_{y\in\overline{P}_{r,R}}\int_{\vartheta(j)} b(s)h(s,y(s))d\nu(s)=0,$$

where $\vartheta(j)=[0,\frac{1}{j}]\cup[\frac{j-1}{j},1]$, and $j>1$ is a certain natural number.

Remark 2.5. It is easy to know that $\varphi_q(s)=|s|^{q-2}s$. In fact, from $\frac{1}{p}+\frac{1}{q}=1$, we can get $(\varphi_q\varphi_p)(s)=|s|^{pq-2(p+q)+4}|s|^{p+q-4}s=|s|^{pq-(p+q)}s=s$. Thus $\varphi_p(s)=\varphi_q^{-1}(s)$.

Remark 2.6. By $(\mathbf{A}_1)$, there exists $t_0\in(0,1)$ such that $b(t_0)>0$. Obviously, if $h(t,y)$ is nonsingular at $y=0$, that is, $h\in C([0,1]\times[0,+\infty),\ [0,+\infty))$, then $(\mathbf{A}_2)$ is satisfied.

Now we investigate the existence of solutions for the following one-dimensional singular p-Laplacian equation with nonlinear boundary conditions

$$\begin{cases} (\varphi_p\,(y'''))'=b(t)g\,(t,\,y,\,y',\,y'',\,y''')\,,\ t\in J'=J\setminus\{t_0,1,\cdots,t_{m+1}\}, \\ y\left(t_k^+\right)=y\left(t_k^-\right)+I_k\left(y\left(t_k\right)\right),\ k=1,\,\ldots,m, \\ y'\left(t_k^+\right)=y'\left(t_k^-\right)+N_k\left(y'\left(t_k\right)\right),\ k=1,\,\ldots,m, \\ y''\left(t_k^+\right)=y''\left(t_k^-\right)+L_k\left(y''\left(t_k\right)\right),\ k=1,\,\ldots,m, \\ y'''\left(t_k^+\right)=y'''\left(t_k^-\right)+R_k\left(y'''\left(t_k\right)\right),\ k=1,\,\ldots,m, \\ \eta y(0)-\lambda_1y'(0)=\int_0^1 a_1(s)y(s)d\nu(s), \\ \eta y(1)+\lambda_2y'(1)=\int_0^1 a_2(s)y(s)d\nu(s), \\ \eta y''(0)-\lambda_3y'''(0)=\int_0^1 a_3(s)y''(s)d\nu(s), \\ \eta y''(1)+\lambda_4y'''(1)=\int_0^1 a_4(s)y''(s)d\nu(s), \end{cases} \tag{2.7}$$

where $\varphi_p(s)=|s|^{p-2}s$, $p>1$, $\varphi_q=(\varphi_p)^{-1}$, $\frac{1}{p}+\frac{1}{q}=1$, $\eta>0$, $\lambda_j>0$ for $j=1,2,3,4$, $J=[0,1]$, $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=1$, where m is a fixed positive integer, ν, I_k, N_k, L_k and R_k are continuous and nondecreasing functions for $k=1,\cdots,m$, as well as $y(t_k^+)$ with $y(t_k^-)$ represent the right-hand limit and left-hand limit of $y(t)$ at $t=t_k$, $b\in C(0,1)$, $b(t)$ may be singular at $t=0$ and/or $t=1$, $g\in C([0,1]\times(0,+\infty)\times(-\infty,+\infty)^3,\ (-\infty,+\infty))$.

Now we consider the following impulsive boundary value problem

$$(2.8)\qquad \begin{cases} (\varphi_p(y'''))' - Cy'' = b(t)H(t,\, y''),\ t \in J' = J \setminus \{t_0, \cdots, t_{m+1}\}, \\ y\left(t_k^+\right) = y\left(t_k^-\right) + I_k\left(y\left(t_k\right)\right),\ k = 1,\, \ldots, m, \\ y'\left(t_k^+\right) = y'\left(t_k^-\right) + N_k\left(y'\left(t_k\right)\right),\ k = 1,\, \ldots, m, \\ y''\left(t_k^+\right) = y''\left(t_k^-\right) + L_k\left(y''\left(t_k\right)\right),\ k = 1,\, \ldots, m, \\ y'''\left(t_k^+\right) = y'''\left(t_k^-\right) + R_k\left(y'''\left(t_k\right)\right),\ k = 1,\, \ldots, m, \\ \eta y(0) - \lambda_1 y'(0) = \int_0^1 a_1(s)y(s)d\nu(s) \triangleq \bar{a}_1, \\ \eta y(1) + \lambda_2 y'(1) = \int_0^1 a_2(s)y(s)d\nu(s) \triangleq \bar{a}_2, \\ \eta y''(0) - \lambda_3 y'''(0) = \int_0^1 a_3(s)y''(s)d\nu(s) \triangleq \bar{a}_3, \\ \eta y''(1) + \lambda_4 y'''(1) = \int_0^1 a_4(s)y''(s)d\nu(s) \triangleq \bar{a}_4, \end{cases}$$

where $b(t) \in L^1(J)$, $H(t,v)$ is measurable function with respect to $t \in J$ for a. e. $v \in \mathbb{R}$, and is Lebesgue integrable function with respect to $v \in \mathbb{R}$ for all $t \in J$; as well as $\bar{a}_1, \bar{a}_2, \bar{a}_3$ and $\bar{a}_4$ are real numbers with $C > 0$, $J' = J \setminus \{t_0, \cdots, t_{m+1}\}$.

We adopt the following assumptions for H, I_k, N_k, L_k and R_k, $k = 1, \cdots, m$:

($\mathbf{H}_1$) $H(t,v)$ is continuous with respect to $t \in J$ for a. e. $v \in \mathbb{R}$, and is decreasing with respect to $v \in \mathbb{R}$ for all $t \in J$;

($\mathbf{H}_2$) I_k, N_k, L_k and $R_k : \mathbb{R} \longrightarrow \mathbb{R}$ are nondecreasing for all $k = 1, \cdots, m$.

Lemma 2.7. *Suppose that the conditions* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *hold. If there exist y_1 and y_2 satisfies $y_i \in PC^1(J,\, \mathbb{R}) \cap C^4(J,\, \mathbb{R})$, $(\varphi_p(y_i'''))' \in PC^1(J,\, \mathbb{R})$ for $i = 1,\, 2$, and*

$$(2.9)\qquad \begin{cases} (\varphi_p(y_1'''))' - Cy_1'' - b(t)H(t,\, y_1'') \\ \le (\varphi_p(y_2'''))' - Cy_2'' - b(t)H(t,\, y_2''),\ t \in J', \\ y_1\left(t_k^+\right) - y_1\left(t_k^-\right) - I_k\left(y_1\left(t_k\right)\right) \\ = y_2\left(t_k^+\right) - y_2\left(t_k^-\right) - I_k\left(y_2\left(t_k\right)\right),\ k = 1, \ldots, m, \\ y_1'(t_k^+) - y_1'(t_k^-) - N_k(y_1'(t_k)) \\ \ge y_2'(t_k^+) - y_2'(t_k^-) - N_k(y_2'(t_k)),\ k = 1, \ldots, m, \\ y_1''(t_k^+) - y_1''(t_k^-) - L_k(y_1''(t_k)) \\ = y_2''(t_k^+) - y_2''(t_k^-) - L_k(y_2''(t_k)),\ k = 1, \ldots, m, \\ y_1'''(t_k^+) - y_1'''(t_k^-) - R_k(y_1'''(t_k)) \\ \ge y_2'''(t_k^+) - y_2'''(t_k^-) - R_k(y_2'''(t_k)),\ k = 1, \ldots, m, \\ \eta y_1(0) - \lambda_1 y_1'(0) \le \eta y_2(0) - \lambda_1 y_2'(0), \\ \eta y_1(1) + \lambda_2 y_1'(1) \le \eta y_2(1) + \lambda_2 y_2'(1), \\ \eta y_1''(0) - \lambda_3 y_1'''(0) \ge \eta y_2''(0) - \lambda_3 y_2'''(0), \\ \eta y_1''(1) + \lambda_4 y_1'''(1) \ge \eta y_2''(1) + \lambda_4 y_2'''(1). \end{cases}$$

Then $y_1(t) \le y_2(t)$ and $y_1''(t) \ge y_2''(t)$ for all $t \in J$.

Proof. Let $x_i = y_i''$ for $i = 1, 2$, and $k = 1, \cdots, m$, then we have

$$\text{(2.10)} \quad \begin{cases} (\varphi_p(x_1'))' - Cx_1 - b(t)H(t, x_1) \\ \le (\varphi_p(x_2'))' - Cx_2 - b(t)H(t, x_2),\ t \in J', \\ x_1(t_k^+) - x_1(t_k^-) - I_k(x_1(t_k)) = x_2(t_k^+) - x_2(t_k^-) - I_k(x_2(t_k)), \\ x_1'(t_k^+) - x_1'(t_k^-) - N_k(x_1'(t_k)) \ge x_2'(t_k^+) - x_2'(t_k^-) - N_k(x_2'(t_k)), \\ \eta x_1(0) - \lambda_3 x_1'(0) \ge \eta x_2(0) - \lambda_3 x_2'(0), \\ \eta x_1(1) + \lambda_4 x_1'(1) \ge \eta x_2(1) + \lambda_4 x_2'(1). \end{cases}$$

Thus, we easily get that $x_1(t) \ge x_2(t)$ for all $t \in J$, which implies that $y_1''(t) \ge y_2''(t)$. Let $\xi(t) = y_2(t) - y_1(t)$ for all $t \in J$. Then we have

$$\text{(2.11)} \quad \begin{cases} \xi''(t) \le 0,\ t \in J, \\ \xi\left(t_k^+\right) - \xi\left(t_k^-\right) - I_k\left(\xi\left(t_k\right)\right) = 0,\ k = 1, \ldots, m, \\ \xi'\left(t_k^+\right) - \xi'\left(t_k^-\right) - N_k\left(\xi'\left(t_k\right)\right) \le 0,\ k = 1, \ldots, m, \\ \eta\xi(0) - \lambda_3\xi'(0) \ge 0, \\ \eta\xi(1) + \lambda_4\xi'(1) \ge 0. \end{cases}$$

Thus, it follows from Lemma 2.2 that $\xi(t) \ge 0$ for all $t \in J$. Consequently, we obtain $y_1(t) \le y_2(t)$ for all $t \in J$. □

The following definitions and lemmas can be found in ([12]−[21]).

Definition 2.8. A function y is called a solution of the problem (2.8) if $y \in PC^1(J, \mathbb{R}) \cap C^4(J, \mathbb{R})$, as well as $(\varphi_p(y'''))' \in PC^1(J, \mathbb{R})$ and y satisfies (2.2).

Definition 2.9. A function y_* is called a lower solution of the problem (2.8) if

(i) $y_* \in PC^1(J, \mathbb{R}) \cap C^4(J, \mathbb{R})$ and $(\varphi_p(y_*'''))' \in PC^1(J, \mathbb{R})$;

(ii)

$$\text{(2.12)} \quad \begin{cases} (\varphi_p(y_*'''))' - Cy_*'' \le b(t)H\left(t, y_*''\right),\ t \in J' = J \setminus \{t_0, t_1, \cdots, t_{m+1}\}, \\ y_*\left(t_k^+\right) = y_*\left(t_k^-\right) + I_k\left(y_*\left(t_k\right)\right),\ k = 1, \ldots, m, \\ y_*'\left(t_k^+\right) \ge y_*'\left(t_k^-\right) + N_k\left(y_*'\left(t_k\right)\right),\ k = 1, \ldots, m, \\ y_*''\left(t_k^+\right) = y_*''\left(t_k^-\right) + L_k\left(y_*''\left(t_k\right)\right),\ k = 1, \ldots, m, \\ y_*'''\left(t_k^+\right) \ge y_*'''\left(t_k^-\right) + R_k\left(y_*'''\left(t_k\right)\right),\ k = 1, \ldots, m, \\ \eta y_*(0) - \lambda_1 y_*'(0) \le \int_0^1 a_1(s) y_*(s) d\nu(s) \triangleq \bar{a}_1, \\ \eta y_*(1) + \lambda_2 y_*'(1) \le \int_0^1 a_2(s) y_*(s) d\nu(s) \triangleq \bar{a}_2, \\ \eta y_*''(0) - \lambda_3 y_*'''(0) \ge \int_0^1 a_3(s) y_*''(s) d\nu(s) \triangleq \bar{a}_3, \\ \eta y_*''(1) + \lambda_4 y_*'''(1) \ge \int_0^1 a_4(s) y_*''(s) d\nu(s) \triangleq \bar{a}_4. \end{cases}$$

Definition 2.10. A function y^* is called a upper solution of the problem (2.8) if

(i) $y^* \in PC^1(J, \mathbb{R}) \cap C^4(J, \mathbb{R})$ and $(\varphi_p(y^{*'''}))' \in PC^1(J, \mathbb{R})$;

(ii)

$$\begin{cases} (\varphi_p(y^{*'''}))' - Cy^{*''} \geq b(t)H(t, y^{*''}),\ t \in J' = J \setminus \{t_0, t_1, \cdots, t_{m+1}\}, \\ y^*(t_k^+) = y^*(t_k^-) + I_k(y^*(t_k)),\ k = 1, \ldots, m, \\ y^{*\prime}(t_k^+) \leq y^{*\prime}(t_k^-) + N_k(y^{*\prime}(t_k)),\ k = 1, \ldots, m, \\ y^{*''}(t_k^+) = y^{*''}(t_k^-) + L_k(y^{*''}(t_k)),\ k = 1, \ldots, m, \\ y^{*'''}(t_k^+) \leq y^{*'''}(t_k^-) + R_k(y^{*'''}(t_k)),\ k = 1, \ldots, m, \\ \eta y^*(0) - \lambda_1 y^{*\prime}(0) \geq \int_0^1 a_1(s)y^*(s)d\nu(s) \triangleq \bar{a}_1, \\ \eta y^*(1) + \lambda_2 y^{*\prime}(1) \geq \int_0^1 a_2(s)y^*(s)d\nu(s) \triangleq \bar{a}_2, \\ \eta y^{*''}(0) - \lambda_3 y^{*'''}(0) \leq \int_0^1 a_3(s)y^{*''}(s)d\nu(s) \triangleq \bar{a}_3, \\ \eta y^{*''}(1) + \lambda_4 y^{*'''}(1) \leq \int_0^1 a_4(s)y^{*''}(s)d\nu(s) \triangleq \bar{a}_4. \end{cases} \tag{2.13}$$

Definition 2.11. We call that the function $g : J \times (0, +\infty) \times \mathbb{R}^3 \longrightarrow (0, +\infty)$ satisfies Nagumo-Wintner conditions corresponding to the couple of a lower solution y_* and a upper solution y^*, if there exist invertible functions φ_q and $\varphi_q^{-1} \in C([0, +\infty), (0, +\infty))$ and functions $b(t) \in L^1([0, 1], (0, +\infty))$, $K_1(t)$, $K_2(t) \in L^1([0, 1], (0, +\infty))$ such that

$$| g(t, \alpha, \sigma, \beta, \gamma) | \leq \varphi_q^{-1}(|\gamma|)(K_1(t) + K_2(t))|b(t)|^{-1}|\gamma|^{\frac{1}{q}}, \text{for } (t, \alpha, \sigma, \beta, \gamma) \in D, \tag{2.14}$$

where

$$D = \{(t, \alpha, \sigma, \beta, \gamma) \in J \times (0, +\infty) \times \mathbb{R}^3 \mid y_*(t) \leq y(t) \leq y^*(t),\ y^{*''}(t) \leq y''(t) \leq y_*''(t)\}$$

and

$$\int_0^{+\infty} \varphi_q(|s|^{q-1})d\nu(s) = +\infty. \tag{2.15}$$

Lemma 2.12. *Assume that the conditions* (H_0) *and* (H_1) *hold. Let* $b \in L^1(J, (0, +\infty))$ *and* $g : [0, 1] \times (0, +\infty) \times \mathbb{R}^3 \longrightarrow [0, +\infty)$ *satisfy Nagumo-Wintner conditions (2.14) and (2.15) in D. Then there exists a constant* $M > 0$ *such that every solution of problem (1.1) confirming* $y_*(t) \leq y(t) \leq y^*(t)$ *and* $y^{*''}(t) \leq y''(t) \leq y_*''(t)$ *for all* $t \in J$*, satisfies* $\|y'''\|_{PC(J,\mathbb{R})} \leq M$.

Proof. Suppose that there exists $s \in J$ such that $\|y'''(s)\|_{PC(J,\mathbb{R})} > M$. Then we have the following two cases:

Case A: There exists $k_0 \in \{0, 1, \cdots, m\}$ such that $s \in (t_{k_0}, t_{k_0+1}]$.

Case B: There exists $k_0 \in \{0, 1, \cdots, m\}$ such that $s = t_{k_0^+}$.

We only consider Case B. A similar argument holds for Case A. Since $y'''(t) \in C^3(J)$ and $y^{*''}(t) \leq y''(t) \leq y_*''(t)$, thus we have

$$\sup_{t \in [t_{k_0}^+, t_{k_0+1}^-]} | y'''(t) | \triangleq l_{k_0}.$$

Let $M^* > \max\left\{l_{k_0}, \|y_*'''\|_{PC(J,\mathbb{R})}, \|y^{*'''}\|_{PC(J,\mathbb{R})}\right\}$ such that

$$\int_{\varphi_p(l_{k_0})}^{\varphi_p(M^*)} \varphi_q(|s|^{q-1})d\nu(s) > \|K_1\|_{L^1} + \|K_2\|_{L^p}\omega^{\frac{1}{q}}, \tag{2.16}$$

with $\omega := \max\{y''(t_2) - y''(t_1) \mid t_1, t_2 \in [t_{k_0}^+, t_{k_0+1}^-]\}$. By the continuity of $y'''(t)$, we can find a constants such that $s_1, s_2 \in [t_{k_0}^+, t_{k_0+1}^-]$ such that $\|y'''(s_1)\|_{PC(J,\mathbb{R})} = l_{k_0}, \|y'''(s_2)\|_{PC(J,\mathbb{R})} = M^*$. Then we have one of the following situations

(i) $y'''(s_1) = l_{k_0}, y'''(s_2) = M^*$ and $l_{k_0} \le y'''(t) \le M^*$ for all $t \in (s_1, s_2)$.

(ii) $y'''(s_1) = l_{k_0}, y'''(s_2) = M^*$ and $l_{k_0} \le y'''(t) \le M^*$ for all $t \in (s_2, s_1)$.

(iii) $y'''(s_1) = -l_{k_0}, y'''(s_2) = -M^*$ and $-M^* \le y'''(t) \le -l_{k_0}$ for all $t \in (s_1, s_2)$.

(iv) $y'''(s_1) = -l_{k_0}, y'''(s_2) = -M^*$ and $-M^* \le y'''(t) \le -l_{k_0}$ for all $t \in (s_2, s_1)$.

Assume that the case (i) holds. The other can be handed in similar way. Since y is a solution of the problem (1.1) and by Nagumo Wininer conditions (2.14), thus we have

$$(\varphi_p(y'''))'(t) \le \varphi_q(y''')\left(K_1(t) + K_2(t) \mid y'''(t) \mid^{\frac{1}{q}}\right) \text{ for all } t \in (s_1, s_2). \tag{2.17}$$

If we put $s = \varphi_p(y'''(t))$, thus we have

$$\int_{\varphi_p(l_{k_0})}^{\varphi_p(M^*)} \varphi_q(|s|^{q-1}) d\nu(s) = \int_{s_1}^{s_2} (\varphi_p(y'''(t)))' d\nu(t).$$

Then by (2.17), we have

$$\begin{aligned}
&\int_{\varphi_p(l_{k_0})}^{\varphi_p(M^*)} \varphi_q\left(|s|^{q-1}\right) d\nu(s) \le \int_{s_1}^{s_2} (\varphi_p(y'''(t)))' \varphi_q(y'''(t))\, d\nu(t) \\
&\le \int_{s_1}^{s_2} b(t) g(t, y, y', y'', y''') \varphi_q(y'''(t))\, d\nu(t) \\
&\le \int_{s_1}^{s_2} \varphi_q^{-1}(y'''(t))\, \varphi_q(y'''(t)) \left(K_1(t) + K_2(t) \mid y'''(t) \mid^{\frac{1}{q}}\right) d\nu(t) \\
&\le \int_{s_1}^{s_2} \left[K_1(t) + K_2(t) \mid y'''(t) \mid^{\frac{1}{q}}\right] d\nu(t) \\
&\le \int_{s_1}^{s_2} K_1(t) d\nu(t) + \int_{s_1}^{s_2} K_2(t) \mid y'''(t) \mid^{\frac{1}{q}} d\nu(t) \\
&\le \|K_1\|_{L^1} + \left(\int_{s_1}^{s_2} (K_2(t))^p d\nu(t)\right)^{\frac{1}{p}} \left(\int_{s_1}^{s_2} \left((y'''(t))^{\frac{1}{q}}\right)^q d\nu(t)\right)^{\frac{1}{q}} \\
&\le \|K_1\|_{L^1} + \|K_2\|_{L^p} \left(\int_{s_1}^{s_2} y'''(t) d\nu(t)\right)^{\frac{1}{q}} \le \|K_1\|_{L^1} + \|K_2\|_{L^p} (y''(s_2) - y''(s_1))^{\frac{1}{q}} \\
&\le \|K_1\|_{L^1} + \|K_2\|_{L^p} \omega^{\frac{1}{q}},
\end{aligned}$$

which is a contradiction with (2.16), where $\|K_1\|_{L^1} = \int_0^1 K_1(t) d\nu(t)$. □

Lemma 2.13. *Suppose that condition* **(A**$_1$**)** *holds. Then there exists a constant* $\theta \in (0, \frac{1}{2})$ *satisfies*

$$0 < \int_\theta^{1-\theta} b(s) d\nu(s) < +\infty.$$

Proof. It follows from (A_1) and (2.6) that

$$0 < \int_{\theta}^{1-\theta} b(s)d\nu(s) < \int_0^1 b(s)d\nu(s) < +\infty.$$

The proof is completed. □

Lemma 2.14. *Suppose that conditions* $(\boldsymbol{A}_0)$ *as well as* $(\boldsymbol{A}_1)$ *and* $(\boldsymbol{A}_2)$ *hold. Then* $T : \overline{P}_{r,R} \to \mathfrak{P}$ *is completely continuous.*

Proof. It is easily to show that $T : \overline{P}_{r,R} \to \mathfrak{P}$. Next, for any positive constants $0 < r < R < +\infty$, we will show

$$\sup_{y \in \partial \overline{P}_{r,R}} \int_{[0,1]} b(s)h(s, y(s))d\nu(s) < +\infty, \tag{2.18}$$

which implies that $T : \mathfrak{P}\backslash\{0\} \to \mathfrak{P}$ is well defined.

By $(\mathbf{A}_2)$, for any $0 < r < R < +\infty$, there exists a natural number j such that

$$\sup_{y \in \partial \overline{P}_{r,R}} \int_{\vartheta(j)} b(s)h(s, y(s))d\nu(s) < 1. \tag{2.19}$$

For any $y \in \partial P_r$, let $y(t_0) = \max_{t\in[0,1]} |y(t)| = r, t_0 \in [0, 1]$. Denote

$$\chi_{\vartheta[a,\,b]}(t) = \begin{cases} 1, & t \in [a,\, b], \\ 0, & t \notin [a,\, b] \end{cases}$$

is the eigenvalue function of the set $\vartheta[a,\, b] = \{t \mid a \le t \le b\}$. Denote

$$\Theta^* = \max\left\{h(t,\, y) \mid (t,\, y) \in ([0, 1] \setminus \vartheta(j)) \times \left[\frac{r}{j},\, R\right],\, j \in \mathbb{Z}_+\right\}. \tag{2.20}$$

It follows from $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ with (2.19) –(2.20) that

$$\begin{aligned} \sup_{y \in \partial \overline{P}_{r,R}} \int_{[0,1]} b(s)h(s, y(s))d\nu(s) &\le \sup_{y \in \partial \overline{P}_{r,R}} \int_{\vartheta(j)} b(s)h(s, y(s))d\nu(s) \\ + \sup_{y \in \partial \overline{P}_{r,R}} \int_{[0,1]\setminus\vartheta(j)} b(s)h(s, y(s))d\nu(s) &\le 1 + \Theta^* \int_0^1 b(s)d\nu(s) < +\infty \end{aligned} \tag{2.21}$$

i. e., (2.18) holds. This also implies $T : \overline{P}_{r,R} \to \mathfrak{P}$ is well defined and $T(Q)$ is uniformly bounded for any bounded set $Q \subset \overline{P}_{r,R}$.

By simple computing and deducing, we can see that $T(\overline{P}_{r,R})$ is equicontinuous. Thus, by the Ascoli-Arzela theorem, we know that $T : \overline{P}_{r,R} \to \mathfrak{P}$ is compact.

Finally we known that $T : \overline{P}_{r,R} \to \mathfrak{P}$ is continuous. In fact, for any $y_n, y_0 \in \overline{P}_{r,R}$ and $\|y_n - y_0\| \to 0$ $(n \to \infty)$. Then $\|Ty_n - Ty_0\| \to 0$ $(n \to \infty)$. This completes the proof. □

3. MAIN RESULTS

In this section, we present and prove our main results.

We will assume that the existence of an ordered pair of lower and upper solutions y_* and y^* satisfying $y_*(t) \le y^*(t)$ and $y^{*\prime\prime}(t) \le y_*''(t)$, for all $t \in J$, and on the nonlinearity g, we shall impose the following additional conditions.

(A$_3$) $b(t)(g(t, y_1, \sigma, \beta, \gamma) - g(t, y_2, 1, \beta, \gamma)) \le 0$ for all $t \in J$,

$$y_*(t) \le y_1(t) \le y_2(t) \le y^*(t), \quad y^{*\prime\prime}(t) \le \beta(t) \le y_*''(t) \quad \text{and} \quad \sigma, \gamma \in \mathbb{R};$$

(A$_4$) There exists a real number $C > 0$ such that the function $\beta \mapsto b(t)g(t, y, \sigma, \beta, \gamma) - C\beta$ is decreasing for all $t \in J$,

$$y_*(t) \le y(t) \le y^*(t), \quad y^{*\prime\prime}(t) \le \beta(t) \le y_*''(t) \quad \text{and} \quad \sigma, \gamma \in \mathbb{R}.$$

Let $\gamma_*(t), \gamma^*(t) \in PC^1(J, \mathbb{R}) \cap C^4(J, \mathbb{R})$ be fixed such that

(i) $\varphi_p(\gamma_*'''), \varphi_p(\gamma^{*\prime\prime\prime}) \in PC^1(J, \mathbb{R})$.

(ii) $y_* \le \gamma_* \le \gamma^* \le y^*$ in J.

(iii) $y^{*\prime\prime} \le \gamma^{*\prime\prime} \le \gamma_*'' \le y_*''$ in J.

Denote

$$M_0 > \max\left\{M^*, \ \|y_*'''\|_{PC(J,\mathbb{R})}, \ \|y^{*\prime\prime\prime}\|_{PC(J,\mathbb{R})}\right\}.$$

We consider the following problems

$$\tag{3.1} \begin{cases} (\varphi_p(y'''))' - Cy'' = b(t)g(t, \gamma^*, 1, \gamma^{*\prime\prime}, M_0) - C\gamma^{*\prime\prime}, \ t \in J', \\ y(t_k^+) = y(t_k^-) + I_k(y(t_k)), \ k = 1, \ldots, m, \\ y'(t_k^+) = y'(t_k^-) + N_k(y'(t_k)), \ k = 1, \ldots, m, \\ y''(t_k^+) = y''(t_k^-) + L_k(y''(t_k)), \ k = 1, \ldots, m, \\ y'''(t_k^+) = y'''(t_k^-) + R_k(y'''(t_k)), \ k = 1, \ldots, m, \\ \eta y(0) - \lambda_1 y'(0) = \int_0^1 a_1(s)\gamma^*(s)d\nu(s), \\ \eta y(1) + \lambda_2 y'(1) = \int_0^1 a_2(s)\gamma^*(s)d\nu(s), \\ \eta y''(0) - \lambda_3 y'''(0) = \int_0^1 a_3(s)\gamma^{*\prime\prime}(s)d\nu(s), \\ \eta y''(1) + \lambda_4 y'''(1) = \int_0^1 a_4(s)\gamma^{*\prime\prime}(s)d\nu(s) \end{cases}$$

and

$$(3.2)\qquad \begin{cases} (\varphi_p(y'''))' - Cy'' = b(t)g(t,\gamma_*,1,\gamma_*'',M_0) - C\gamma_*'',\ t\in J',\\ y(t_k^+) = y(t_k^-) + I_k(y(t_k)),\ k=1,\dots,m,\\ y'(t_k^+) = y'(t_k^-) + N_k(y'(t_k)),\ k=1,\dots,m,\\ y''(t_k^+) = y''(t_k^-) + L_k(y''(t_k)),\ k=1,\dots,m,\\ y'''(t_k^+) = y'''(t_k^-) + R_k(y'''(t_k)),\ k=1,\dots,m,\\ \eta y(0) - \lambda_1 y'(0) = \int_0^1 a_1(s)\gamma_*(s)d\nu(s),\\ \eta y(1) + \lambda_2 y'(1) = \int_0^1 a_2(s)\gamma_*(s)d\nu(s),\\ \eta y''(0) - \lambda_3 y'''(0) = \int_0^1 a_3(s)\gamma_*''(s)d\nu(s),\\ \eta y''(1) + \lambda_4 y'''(1) = \int_0^1 a_4(s)\gamma_*''(s)d\nu(s) \end{cases}$$

The following preliminary Lemma will play a key role to prove our main results.

Lemma 3.1. *Let γ_* and γ^* be a lower and upper solutions respectively of problem (1.1) such that $\gamma_*\le\gamma^*$ and $\gamma_*\le 1,\gamma^{*\prime\prime}\le\gamma_*''$ in J. Assume that the hypothesis* (A_i) *for $i=0,1,2,3,4$ and* (H_1) *with* (H_2) *hold, as well as the Nagumo Wintner conditions (2.14) with (2.15) relative to a lower solution y_* and upper solution y^* respectively of problem (1.1) are satisfied. Then there exists a unique solutions $y^\curlywedge$ and $y_\curlyvee$ respectively for the problems (3.1) and (3.2) such that*

$$(3.3)\qquad y_*\le\gamma_*\le y_\curlyvee\le y^\curlywedge\le\gamma^*\le y^*\quad in\quad J,$$

and

$$(3.4)\qquad y^{*\prime\prime}\le\gamma^{*\prime\prime}\le y^{\curlywedge\prime\prime}\le y_\curlyvee''\le\gamma_*''\le y_*''\quad in\quad J.$$

Proof. The proof will be given in two steps.

Step I. γ_* is a lower solution of the problem (3.1).

If $t\in J'$, then by using $(\mathbf{A}_3)$ and $(\mathbf{A}_4)$, we have

$$(3.5)\qquad \begin{aligned} &(\varphi_p(\gamma_*'''))' - C\gamma_*'' \le b(t)g(t,\gamma_*,\gamma_*',\gamma_*'',\gamma_*''') - C\gamma_*''\\ &\le b(t)g(t,\gamma_*,\gamma_*',\gamma^{*\prime\prime},\gamma_*''') - C\gamma^{*\prime\prime} \le b(t)g(t,\gamma^*,\gamma_*',\gamma^{*\prime\prime},\gamma_*''') - C\gamma^{*\prime\prime}. \end{aligned}$$

That is

$$(3.6)\qquad (\varphi_p(\gamma_*'''))' - C\gamma_*'' \le b(t)g(t,\gamma^*,\gamma_*',\gamma^{*\prime\prime},\gamma_*''') - C\gamma^{*\prime\prime}.$$

Now since γ_* is a lower solution of the problem (1.1) and $y_*\le\gamma_*\le y^*$ in J, then by using a similar proof to that of Lemma 2.12, we have $\|\gamma_*'''\|_{PC(J,\mathbb{R})}\le M^*$. Then by (3.6) and $(\mathbf{A}_3)$ together with $(\mathbf{A}_4)$, we get

$$(3.7)\qquad (\varphi_p(\gamma_*'''))' - C\gamma_*'' \le b(t)g(t,\gamma^*,1,\gamma^{*\prime\prime},\|\gamma_*'''\|_{PC(J,\mathbb{R})}) - C\gamma^{*\prime\prime},\quad \forall\ t\in J.$$

In addition, we have

$$\begin{cases} \gamma_*\left(t_k^+\right) = \gamma_*\left(t_k^-\right) + I_k\left(\gamma_*\left(t_k\right)\right), \text{ if } t = t_k,\ k = 1,\dots,m, \\ \gamma_*'\left(t_k^+\right) \geq \gamma_*'\left(t_k^-\right) + N_k\left(\gamma_*'\left(t_k\right)\right), \text{ if } t = t_k,\ k = 1,\dots,m, \\ \gamma_*''\left(t_k^+\right) = \gamma_*''\left(t_k^-\right) + L_k\left(\gamma_*''\left(t_k\right)\right), \text{ if } t = t_k,\ k = 1,\dots,m, \\ \gamma_*'''\left(t_k^+\right) \geq \gamma_*'''\left(t_k^-\right) + R_k\left(\gamma_*'''\left(t_k\right)\right), \text{ if } t = t_k,\ k = 1,\dots,m, \\ \eta\gamma_*(0) - \lambda_1\gamma_*'(0) = \int_0^1 a_1(s)\gamma_*(s)d\nu(s) \leq \int_0^1 a_1(s)\gamma^*(s)d\nu(s), \\ \eta\gamma_*(1) + \lambda_2\gamma_*'(1) = \int_0^1 a_2(s)\gamma_*(s)d\nu(s) \leq \int_0^1 a_2(s)\gamma^*(s)d\nu(s), \\ \eta\gamma_*''(0) - \lambda_3\gamma_*'''(0) = \int_0^1 a_3(s)\gamma_*''(s)d\nu(s), \\ \eta\gamma_*''(1) + \lambda_4\gamma_*'''(1) = \int_0^1 a_4(s)\gamma_*''(s)d\nu(s) \end{cases} \tag{3.8}$$

Then, it follows (3.7) and (3.8) that γ_* is a lower solution of the problem (3.1).

Step II. γ^* is a upper solution of the problem (3.1). The proof is similar to that of **Step I.**, so it is omitted.

By **Step I.** and **Step II.** since $b(t) \in L^1(J), g(t, \gamma^*, 1, \gamma^{*''}, \|\gamma^{*'''}\|_{PC(J,\mathbb{R})})$ and $\|y'''\|_{PC(J,\mathbb{R})}$ are bounded, then by Lemma 2.7, there exists a unique solution $y^{\curlywedge}$ of the problem (3.1) such that $\gamma_* \leq y^{\curlywedge} \leq \gamma^*$ and $\gamma^{*''} \leq y^{\curlywedge''} \leq \gamma_*''$.

Similarly, we can prove that the problem (3.2) admits unique solution $y_{\curlyvee}$ such that $\gamma_* \leq y_{\curlyvee} \leq \gamma^*$ and $\gamma^{*''} \leq y_{\curlyvee}'' \leq \gamma_*''$.

Finally by using a proof similar to that of Lemma 2.7, we obtain $y_{\curlyvee} \leq y^{\curlywedge}$ and $y^{\curlywedge''} \leq y_{\curlyvee}''$ in J. The proof of Lemma 3.1 is complete. □

The main result of this work is as the following:

Theorem 3.2. *Let $y_*(t)$ and $y^*(t)$ be a lower and upper solution respectively for problem (1.1) such that $y_*(t) \leq y^*(t)$ and $y_*''(t) \geq y^{*''}(t)$ in J. Assume that the conditions **(A_i)** for $i = 0, 1, 2, 3, 4$ and (H_1) with (H_2) hold, and the Nagumo Wintner conditions (2.14) with (2.15) relative to a lower solution y_* and upper solution y^* respectively of problem (1.1) are satisfied. Then the problem (1.1) has maximal solution $y_\sharp$ and minimal solution $y^\sharp$ such that for every solution y of (1.1) with $y_*(t) \leq y(t) \leq y^*(t)$ in J, satisfying*

$$y_*(t) \leq y^\sharp(t) \leq y(t) \leq y_\sharp(t) \leq y^*(t), \qquad t \in J \tag{3.9}$$

and

$$y^{*''}(t) \leq y_\sharp''(t) \leq y''(t) \leq y^{\sharp''}(t) \leq y_*''(t), \quad t \in J. \tag{3.10}$$

Proof. There are three steps. We take $z_*,\ z^* \in PC^1(J,\mathbb{R})$ fixed such that

(i) $\left(\varphi_p\left(z_*'''\right)\right)', \left(\varphi_p\left(z^{*'''}\right)\right)' \in PC^1(J,\mathbb{R})$.

(ii) $y_* \leq z_* \leq z^* \leq y^*$ and $y^{*''} \leq z^{*''} \leq z_*'' \leq y_*''$ in J.

We define the sequences $\{y_{*_n}\}_{n\in\mathbb{N}}$ and $\{y^*_n\}_{n\in\mathbb{N}}$ by

$$(3.11)\quad \begin{cases} y_*^{\{0\}} = y_*, \\ \left(\varphi_p\left(y'''_{*_{n+1}}\right)\right)'(t) - Cy''_{*_{n+1}}(t) = b(t)g_n^{\curlywedge}(t),\ t\in J', \\ y_{*_{n+1}}\left(t_k^+\right) = y_{*_{n+1}}\left(t_k^-\right) + I_k\left(y_{*_{n+1}}\left(t_k\right)\right),\ k=1,\dots,m, \\ y'_{*_{n+1}}\left(t_k^+\right) = y'_{*_{n+1}}\left(t_k^-\right) + N_k\left(y'_{*_{n+1}}\left(t_k\right)\right),\ k=1,\dots,m, \\ y''_{*_{n+1}}\left(t_k^+\right) = y''_{*_{n+1}}\left(t_k^-\right) + L_k\left(y''_{*_{n+1}}\left(t_k\right)\right),\ k=1,\dots,m, \\ y'''_{*_{n+1}}\left(t_k^+\right) = y'''_{*_{n+1}}\left(t_k^-\right) + R_k\left(y'''_{*_{n+1}}\left(t_k\right)\right),\ k=1,\dots,m, \\ \eta y_{*_{n+1}}(0) - \lambda_1 y'_{*_{n+1}}(0) = \int_0^1 a_1(s)y_{*_n}(s)d\nu(s), \\ \eta y_{*_{n+1}}(1) + \lambda_2 y'_{*_{n+1}}(1) = \int_0^1 a_2(s)y_{*_n}(s)d\nu(s), \\ \eta y''_{*_{n+1}}(0) - \lambda_3 y'''_{*_{n+1}}(0) = \int_0^1 a_3(s)y''_{*_n}(s)d\nu(s), \\ \eta y''_{*_{n+1}}(1) + \lambda_4 y'''_{*_{n+1}}(1) = \int_0^1 a_4(s)y''_{*_n}(s)d\nu(s) \end{cases}$$

and

$$(3.12)\quad \begin{cases} y^{*\{0\}} = y^*, \\ \left(\varphi_p\left(y^{*'''}_{n+1}\right)\right)'(t) - Cy^{*''}_{n+1}(t) = b(t)g_n^{\curlyvee}(t),\ t\in J', \\ y^*_{n+1}\left(t_k^+\right) = y^*_{n+1}\left(t_k^-\right) + I_k\left(y^*_{n+1}\left(t_k\right)\right),\ k=1,\dots,m, \\ y^{*'}_{n+1}\left(t_k^+\right) = y^{*'}_{n+1}\left(t_k^-\right) + N_k\left(y^{*'}_{n+1}\left(t_k\right)\right),\ k=1,\dots,m, \\ y^{*''}_{n+1}\left(t_k^+\right) = y^{*''}_{n+1}\left(t_k^-\right) + L_k\left(y^{*''}_{n+1}\left(t_k\right)\right),\ k=1,\dots,m, \\ y^{*'''}_{n+1}\left(t_k^+\right) = y^{*'''}_{n+1}\left(t_k^-\right) + R_k\left(y^{*'''}_{n+1}\left(t_k\right)\right),\ k=1,\dots,m, \\ \eta y^*_{n+1}(0) - \lambda_1 y^{*'}_{n+1}(0) = \int_0^1 a_1(s)y^*_n(s)d\nu(s), \\ \eta y^*_{n+1}(1) + \lambda_2 y^{*'}_{n+1}(1) = \int_0^1 a_2(s)y^*_n(s)d\nu(s), \\ \eta y^{*''}_{n+1}(0) - \lambda_3 y^{*'''}_{n+1}(0) = \int_0^1 a_3(s)y^{*''}_n(s)d\nu(s), \\ \eta y^{*''}_{n+1}(1) + \lambda_4 y^{*'''}_{n+1}(1) = \int_0^1 a_4(s)y^{*''}_n(s)d\nu(s), \end{cases}$$

where

$$g_n^{\curlywedge}(t) = g\left(t, y_{*_n}, y'_{*_{n+1}}, y''_{*_n}, \|y'''_{*_{n+1}}\|_{PC(J,\mathbb{R})}\right) - Cy''_{*_n}(t)$$

and

$$g_n^{\curlyvee}(t) = g\left(t, y^*_n, y^{*'}_{n+1}, y^{*''}_n, \|y^{*'''}_{n+1}\|_{PC(J,\mathbb{R})}\right) - Cy^{*''}_n(t).$$

Noticing that by lemma 2.7., the sequences $\{y_{*_n}\}_{n\in\mathbb{N}}$ and $\{y^*_n\}_{n\in\mathbb{N}}$ are well defined.

Step I*. For all $n\in\mathbb{N}$, we have

$$y_*(t) \le y_{*_1}(t) \le \cdots \le y_{*_n}(t) \le y_{*_{n+1}}(t) \le y^*_{n+1}(t) \le y^*_n(t) \le \cdots \le y^*_1(t) \le y^*(t),$$
for all $t\in J$

and

$$y^{*''}(t) \le y_1^{*''}(t) \le \cdots \le y_n^{*''}(t) \le y_{n+1}^{*''}(t) \le y''_{*_{n+1}}(t) \le y''_{*_n}(t) \le \cdots \le y''_{*_1}(t) \le y''_*(t),$$
for all $t\in J$.

For $n=0$, we have

$$(3.13)\qquad \begin{cases} \left(\varphi_p\left(y'''_{*_1}\right)\right)'(t) - Cy''_{*_1}(t) \\ = b(t)g\left(t, y_*, y'_*, y''_*, \|y'''_{*_1}\|_{PC^1(J,\mathbb{R})}\right) - Cy''_*(t), t\in J' \\ y_{*_1}\left(t_k^+\right) = y_{*_1}\left(t_k^-\right) + I_k\left(y_{*_1}\left(t_k\right)\right),\ k=1,\ldots,m \\ y'_{*_1}\left(t_k^+\right) = y'_{*_1}\left(t_k^-\right) + N_k\left(y'_{*_1}\left(t_k\right)\right),\ k=1,\ldots,m \\ y''_{*_1}\left(t_k^+\right) = y''_{*_1}\left(t_k^-\right) + L_k\left(y''_{*_1}\left(t_k\right)\right),\ k=1,\ldots,m \\ y'''_{*_1}\left(t_k^+\right) = y'''_{*_1}\left(t_k^-\right) + R_k\left(y'''_{*_1}\left(t_k\right)\right),\ k=1,\ldots,m \\ \eta y_{*_1}(0) - \lambda_1 y'_{*_1}(0) = \int_0^1 a_1(s)y_*(s)d\nu(s), \\ \eta y_{*_1}(1) + \lambda_2 y'_{*_1}(1) = \int_0^1 a_2(s)y_*(s)d\nu(s), \\ \eta y''_{*_1}(0) - \lambda_3 y'''_{*_1}(0) = \int_0^1 a_3(s)y''_*(s)d\nu(s), \\ \eta y''_{*_1}(1) + \lambda_4 y'''_{*_1}(1) = \int_0^1 a_4(s)y''_*(s)d\nu(s) \end{cases}$$

and

$$(3.14)\qquad \begin{cases} \left(\varphi_p\left(y_1^{*\prime\prime\prime}\right)\right)'(t) - Cy_1^{*\prime\prime}(t) \\ = b(t)g\left(t, y^*, y^{*\prime}, y^{*\prime\prime}, \|y_1^{*\prime\prime\prime}\|_{PC^1(J,\mathbb{R})}\right) - Cy^{*\prime\prime},\quad t\in J' \\ y_1^*\left(t_k^+\right) = y_1^*\left(t_k^-\right) + I_k\left(y_1^*\left(t_k\right)\right),\ k=1,\ldots,m \\ y_1^{*\prime}\left(t_k^+\right) = y_1^{*\prime}\left(t_k^-\right) + N_k\left(y_1^{*\prime}\left(t_k\right)\right),\ k=1,\ldots,m \\ y_1^{*\prime\prime}\left(t_k^+\right) = y_1^{*\prime\prime}\left(t_k^-\right) + L_k\left(y_1^{*\prime\prime}\left(t_k\right)\right),\ k=1,\ldots,m \\ y_1^{*\prime\prime\prime}\left(t_k^+\right) = y_1^{*\prime\prime\prime}\left(t_k^-\right) + R_k\left(y_1^{*\prime\prime\prime}\left(t_k\right)\right),\ k=1,\ldots,m \\ \eta y_1^*(0) - \lambda_1 y_1^{*\prime}(0) = \int_0^1 a_1(s)y^*(s)d\nu(s), \\ \eta y_1^*(1) + \lambda_2 y_1^{*\prime}(1) = \int_0^1 a_2(s)y^*(s)d\nu(s), \\ \eta y_1^{*\prime\prime}(0) - \lambda_3 y_1^{*\prime\prime\prime}(0) = \int_0^1 a_3(s)y^{*\prime\prime}(s)d\nu(s), \\ \eta y_1^{*\prime\prime}(1) + \lambda_4 y_1^{*\prime\prime\prime}(1) = \int_0^1 a_4(s)y^{*\prime\prime}(s)d\nu(s). \end{cases}$$

Since y_* and y^* are respectively lower and upper solutions of the problem (1.1), then by the Lemma 3.1, we have

$$y_*(t) \le y_{*_1}(t) \le y_1^*(t) \le y^*(t), \quad \text{for all} \quad t\in J$$

and

$$y^{*\prime\prime}(t) \le y_1^{*\prime\prime}(t) \le y''_{*_1}(t) \le y''_*(t), \quad \text{for all} \quad t\in J.$$

Assume that for a fixed $n>1$, we have

$$y_*(t) \le y_{*_{n-1}}(t) \le y_{*_n}(t) \le y_n^*(t) \le y_{n-1}^*(t) \le y^*(t), \quad \text{for all} \quad t\in J$$

and

$$y^{*\prime\prime}(t) \le y_{n-1}^{*\prime\prime}(t) \le y_n^{*\prime\prime}(t) \le y''_{*_n}(t) \le y''_{*_{n-1}}(t) \le y''_*(t), \quad \text{for all} \quad t\in J.$$

Thus we prove that

$$y_*(t) \le y_{*_n}(t) \le y_{*_{n+1}}(t) \le y_{n+1}^*(t) \le y_n^*(t) \le y^*(t), \quad \text{for all} \quad t\in J$$

and

$$y^{*\prime\prime}(t) \le y_n^{*\prime\prime}(t) \le y_{n+1}^{*\prime\prime}(t) \le y''_{*_{n+1}}(t) \le y''_{*_n}(t) \le y''_*(t), \quad \text{for all} \quad t\in J.$$

Then we show that

$$y_*(t) \leq y_{*_n}(t) \leq y_{*_{n+1}}(t) \leq y^*_{n+1}(t) \leq y^*_n(t) \leq y^*(t), \quad \text{for all} \quad t \in J$$

and

$$y^{*\prime\prime}(t) \leq y^{*\prime\prime}_n(t) \leq y^{*\prime\prime}_{n+1}(t) \leq y''_{*_{n+1}}(t) \leq y''_{*_n}(t) \leq y''_*(t), \quad \text{for all} \quad t \in J.$$

If $t \in J'$, we have

$$\left(\varphi_p\left(y^{*\prime\prime\prime}_n\right)\right)'(t) - Cy^{*\prime\prime}_n(t) = b(t)g(t, y^*_{n-1}, y^{*\prime}_n, y^{*\prime\prime}_{n-1}, \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}) - Cy^{*\prime\prime}_{n-1}(t), \quad t \in J'.$$

Since $y^*_n(t) \leq y^*_{n-1}(t)$ and $y^{*\prime\prime}_{n-1} \leq y^{*\prime\prime}_n$ and by using the hypothesis $(\mathbf{A}_3)$ and $(\mathbf{A}_4)$, we obtain

$$\begin{aligned} &b(t)g\left(t, y^*_{n-1}, y^{*\prime}_n, y^{*\prime\prime}_{n-1}, \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}\right) - Cy^{*\prime\prime}_{n-1} \\ &\geq b(t)g\left(t, y^*_n, y^{*\prime}_n, y^{*\prime\prime}_{n-1}, \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}\right) - Cy^{*\prime\prime}_{n-1} \\ &\geq b(t)g\left(t, y^*_n, y^{*\prime}_n, y^{*\prime\prime}_n, \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}\right) - Cy^{*\prime\prime}_n \end{aligned}$$

which implies that

$$\left(\varphi_p\left(y^{*\prime\prime\prime}_n\right)\right)'(t) \geq b(t)g\left(t, y^*_n, y^{*\prime}_n, y^{*\prime\prime}_n, \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}\right). \tag{3.15}$$

Now by using a proof similar to that of Lemma 2.12., we have $\|y'''_n\|_{PC(J,\mathbb{R})} \leq M^*$, then by making use of (3.15), it follows that

$$\left(\varphi_p\left(y^{*\prime\prime\prime}_n\right)\right)'(t) \geq b(t)g\left(t, y^*_n, y^{*\prime}_n, y^{*\prime\prime}_n, \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}\right), \quad t \in J'. \tag{3.16}$$

On the other hand, for $k = 1, \cdots, m$, we have

$$\begin{cases} y^*_n\left(t^+_k\right) = y^*_n\left(t^-_k\right) + I_k\left(y^*_n\left(t_k\right)\right), \ k = 1, \ldots, m, \\ y^{*\prime}_n\left(t^+_k\right) = y^{*\prime}_n\left(t^-_k\right) + N_k\left(y^{*\prime}_n\left(t_k\right)\right), \ k = 1, \ldots, m, \\ y^{*\prime\prime}_n\left(t^+_k\right) = y^{*\prime\prime}_n\left(t^-_k\right) + L_k\left(y^{*\prime\prime}_n\left(t_k\right)\right), \ k = 1, \ldots, m, \\ y^{*\prime\prime\prime}_n\left(t^+_k\right) = y^{*\prime\prime\prime}_n\left(t^-_k\right) + R_k\left(y^{*\prime\prime\prime}_n\left(t_k\right)\right), \ k = 1, \ \ldots, m \end{cases} \tag{3.17}$$

and

$$\begin{cases} \eta y_n(0) - \lambda_1 y^{*\prime}_n(0) = \int_0^1 a_1(s) y^*_{n-1}(s) d\nu(s) \geq \int_0^1 a_1(s) y^*_n(s) d\nu(s), \\ \eta y_n(1) + \lambda_2 y^{*\prime}_n(1) = \int_0^1 a_2(s) y^*_{n-1}(s) d\nu(s) \geq \int_0^1 a_2(s) y^*_n(s) d\nu(s), \\ \eta y^{*\prime\prime}_n(0) - \lambda_3 y^{*\prime\prime\prime}_n(0) = \int_0^1 a_3(s) y^{*\prime\prime}_{n-1}(s) d\nu(s) \leq \int_0^1 a_3(s) y^{*\prime\prime}_n(s) d\nu(s), \\ \eta y^{*\prime\prime}_n(1) + \lambda_4 y^{*\prime\prime\prime}_n(1) = \int_0^1 a_4(s) y^{*\prime\prime}_{n-1}(s) d\nu(s) \leq \int_0^1 a_4(s) y^{*\prime\prime}_n(s) d\nu(s) \end{cases} \tag{3.18}$$

Then by (3.16) with (3.17) and (3.18), it follows that y^*_n is a upper solution of the problem the problem (1.1).

Similarly, we show that y_{*_n} is a lower solution of the problem (1.1) and consequently by Lemma 3.1, there exist a lower solution $y_{*_{n+1}}$ and a upper solution y^*_{n+1} of the problem (3.11) and (3.12) respectively such that

$$y_*(t) \leq y_{*_n}(t) \leq y_{*_{n+1}}(t) \leq y^*_{n+1}(t) \leq y^*_n(t) \leq y^*(t), \quad \text{for all} \quad t \in J$$

and

$$y^{*\prime\prime}(t) \leq y^{*\prime\prime}_n(t) \leq y^{*\prime\prime}_{n+1}(t) \leq y''_{*_{n+1}}(t) \leq y''_{*_n}(t) \leq y''_*(t), \quad \text{for all} \quad t \in J$$

which implies that for all $n \in \mathbb{N}$, we have

$$y_*(t) \leq y_{*_n}(t) \leq y_{*_{n+1}}(t) \leq y^*_{n+1}(t) \leq y^*_n(t) \leq y^*(t), \quad \text{for all} \quad t \in J$$

and

$$y^{*\prime\prime}(t) \leq y^{*\prime\prime}_n(t) \leq y^{*\prime\prime}_{n+1}(t) \leq y''_{*_{n+1}}(t) \leq y''_{*_n}(t) \leq y''_*(t), \quad \text{for all} \quad t \in J.$$

Step II*. The sequences $\{y^*_n\}_{n\geq 1}$ converges to a maximal solution of the problem (1.1).

By **Step I*.,** and since $\|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})} \leq M^*$, for all $n \in \mathbb{N}$, it is clear that the sequence $\{y^{*\prime\prime\prime}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $PC^1(J,\mathbb{R})$. We put $J_1 = [0,t_1], J_2 = (t_1,t_2], \cdots, J_m = (t_{m-1},t_m], J_{m+1} = (t_m,1]$, then $J = \bigcup_{k=1}^{m+1} J_k$.

Let $\varepsilon > 0$ and $t,s \in J_1$, such that $t < s$, then for all $n \in \mathbb{N}$, and by $(\mathbf{A_4})$ we have

$$\begin{aligned}
&\left(\varphi_p\left(y^{*\prime\prime\prime}_{n+1}(s)\right)\right)' - \left(\varphi_p\left(y^{*\prime\prime\prime}_{n+1}(t)\right)\right)' \\
&\leq \left|\int_t^s \left[b(\tau)g\left(\tau, y^*_n(\tau), y^{*\prime}_n(\tau), y^{*\prime\prime}_n(\tau), \|y^{*\prime\prime\prime}_n\|_{PC(J,\mathbb{R})}\right) - C\left(y^{*\prime\prime}_n(\tau) - y^{*\prime\prime}_{n+1}\right)\right] d\nu(\tau)\right| \\
&\leq (C_1(g) + 2CC_2)|s-t|,
\end{aligned}$$

where

$$C_1(g) := \max\{|b(t)g(t,\alpha,\sigma,\beta,\gamma)| \mid t \in J, y_* \leq y \leq y^*, y^{*\prime\prime}(t) \leq \beta \leq y''_*(t), |\gamma| \leq M_0\}$$

and

$$C_2 = \max\{y''(t),\ y_* \leq y \leq y^*,\ y^{*\prime\prime}(t) \leq y''(t) \leq y''_*(t)\}.$$

If we put $M_1 = (C_1(g) + 2CC_2)$, one has

$$\left|\left(\varphi_p\left(y^{*\prime\prime\prime}_{n+1}(s)\right)\right)' - \left(\varphi_p\left(y^{*\prime\prime\prime}_{n+1}(t)\right)\right)'\right| \leq M_1|s-t|.$$

Then if we take $|s-t| \leq \frac{\varepsilon}{M_1+1}$, we get

$$\left|\left(\varphi_p\left(y^{*\prime\prime\prime}_{n+1}(s)\right)\right)' - \left(\varphi_p\left(y^{*\prime\prime\prime}_{n+1}(t)\right)\right)'\right| < \varepsilon.$$

Therefore the sequence $\left(\varphi_p\left(y^{*\prime\prime\prime}_n(t)\right)\right)'_{n\in\mathbb{N}}$ is equi-continuous on J_1.

Now since φ_p^{-1} is an increasing homecomorphism from $\mathbb{R}$ to $\mathbb{R}$, we infer from

$$\left|y^{*\prime\prime\prime}_{n+1}(s) - y^{*\prime\prime\prime}_{n+1}(t)\right| = \left|\varphi_p^{-1}\left(\varphi_p(y^{*\prime\prime\prime}_{n+1}(s))\right) - \varphi_p^{-1}\left(\varphi_p(y^{*\prime\prime\prime}_{n+1}(t))\right)\right| < \varepsilon$$

that the sequence $\{y^{*\prime\prime\prime}_n\}_{n\in\mathbb{N}}$ is equicontinuous on J_1 and it is not difficult to prove that $\{y^*_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^3(J_1)$.

Hence by Ascoli-Arzela's theorem there exists a subsequence $\left\{y^{*\{a_1\}}_n\right\}_{n\in\mathbb{N}}$ of $\{y^*_n\}_{n\in\mathbb{N}}$ which converges in $C^3(J_1)$.

Consider the subsequence $\left\{y^{*\{a_1\}}_n\right\}_{n\in\mathbb{N}}$ on the interval J_2. On this interval the subsequence $\left\{y^{*\{a_1\}}_n\right\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous. So, it has a subsequence $\left\{y^{*\{a_2\}}_n\right\}_{n\in\mathbb{N}}$ will converge uniformly on the interval $(t_1,t_2]$.

Continuing this process for the intervals $(t_2,t_3],\cdots,(t_m,t_{m+1}]$, we see that the sequence $\{y_n^*\}_{n\in\mathbb{N}}$ has a subsequence $\left\{y_n^{*\{a_{m+1}\}}\right\}_{n\in N}$ which will converge uniformly on the interval J. Let $y^{a_{m+1}}=\lim_{n\to\infty} y_n^{*\{a_{m+1}\}}$. Then $(y^{a_{m+1}})^{(i)}=\lim_{n\to\infty}\left(y_n^{*\{a_{m+1}\}}\right)^{(i)}$ for $i=1,2,3$.

But by **Step I*.** the sequence $\{y_n^*\}_{n\in N}$ is decreasing and bounded from below, then the pointwise limit of this sequence exists and it is denoted by $y_\sharp$. Hence, we have $y^{a_{m+1}}=y_\sharp$.

Let $k\in\{0,1,\cdots,m\}$ be fixed, and $t,s\in(t_k,t_{k+1})$, we obtain

$$\varphi_p\left(y_{n+1}^{*\prime\prime\prime}(s)\right)=\varphi_p\left(y_{n+1}^{*\prime\prime\prime}(t)\right)+\int_t^s G_n(\tau)d\nu(\tau),$$

where

$$G_n(t)=b(t)g\left(\tau,y_n^*(\tau),y_n^{*\prime}(\tau),y_n^{*\prime\prime}(\tau),\|y_{n+1}^{*\prime\prime\prime}(\tau)\|_{PC(J,\mathbb{R})}\right)-C(y_n^{*\prime\prime}(\tau)-y_{n+1}^{*\prime\prime}(\tau)).$$

Now, as $n\longrightarrow+\infty$, we obtain

$$G_n(t)\longrightarrow b(t)g\left(\tau,y_\sharp(\tau),y_\sharp'(\tau),y_\sharp''(\tau),\|y\|_\sharp'''(\tau)\|_{PC(J,\mathbb{R})}\right).$$

Also, there exists a positive number $L_4>0$ such that for $n\in\mathbb{N}$ and $\tau\in J$, we have

$$\|G_n(t)\|\leq L_4.$$

Hence, the dominated convergence theorem of Lebesgue implies that

$$\varphi_p\left(y_\sharp(t)'''\right)=\varphi_p\left(y_\sharp'''(s)\right)+\int_s^t b(\tau)g\left(\tau,y_\sharp(\tau),y_\sharp'(\tau),y_\sharp''(\tau),\|y_\sharp'''(\tau)\|_{PC(J,\mathbb{R})}\right)d\nu(\tau).$$

Thus, for $k=0,\ 1,\ \cdots,\ m$, we get

$$\left(\varphi_p\left(y_\sharp'''\right)\right)'=b(t)g\left(t,y_\sharp(t),y_\sharp'(t),y_\sharp''(t),\|y_\sharp'''(t)\|_{PC(J,\mathbb{R})}\right),\quad t\in(t_k,t_{k+1}). \tag{3.19}$$

That is

$$\left(\varphi_p\left(y_\sharp'''\right)\right)'=b(t)g(t,y_\sharp(t),y_\sharp'(t),y_\sharp''(t),\|y_\sharp'''(t)\|_{PC(J,\mathbb{R})}),\quad t\in J'. \tag{3.20}$$

On the other hand, since the functions $a_j\ (j=1,2,3,4)$ are continuous, we have

$$\begin{cases}\eta y_\sharp(0)-\lambda_1 y_\sharp'(0)=\int_0^1 a_1(s)y_\sharp(s)d\nu(s)\\ \eta y_\sharp(1)+\lambda_2 y_\sharp'(1)=\int_0^1 a_2(s)y_\sharp(s)d\nu(s)\\ \eta y_\sharp''(0)-\lambda_3 y_\sharp'''(0)=\int_0^1 a_3(s)y_\sharp''(s)d\nu(s)\\ \eta y_\sharp''(1)+\lambda_4 y_\sharp'''(1)=\int_0^1 a_4(s)y_\sharp''(s)d\nu(s)\end{cases} \tag{3.21}$$

Similarly since the functions I_k,N_k,L_k and R_k are continuous for $k=1,\cdots,m$. Thus we have

$$\begin{cases}y_\sharp\left(t_k^+\right)=y_\sharp\left(t_k^-\right)+I_k\left(y_\sharp\left(t_k\right)\right),\ k=1,\ldots,m\\ y_\sharp'\left(t_k^+\right)=y_\sharp'\left(t_k^-\right)+N_k\left(y_\sharp'\left(t_k\right)\right),\ k=1,\ldots,m\\ y_\sharp''\left(t_k^+\right)=y_\sharp''\left(t_k^-\right)+L_k\left(y_\sharp''\left(t_k\right)\right),\ k=1,\ldots,m\\ y_\sharp'''\left(t_k^+\right)=y_\sharp'''\left(t_k^-\right)+R_k\left(y_\sharp'''\left(t_k\right)\right),\ k=1,\ldots,m\end{cases} \tag{3.22}$$

Now using a proof similar to that of Lemma 2.12., we prove that $\|y_\sharp'''\|_{PC(J,\mathbb{R})} \leq M^*$. Hence, $y_\sharp$ is a solution of the problem (1.1).

Now, we prove that if y_Υ is another lower solution of problem (1.1) such that $y_* \leq y_\Upsilon \leq y^*$ and $y^{*\prime\prime} \leq y_\Upsilon'' \leq y_*''$ in J, then $y_\Upsilon \leq y_\sharp$ and $y_\sharp'' \leq y_\Upsilon''$ in J. Since y_Υ is a lower solution of problem (1.1), then by **Step I⋆.** we obtain $y_\Upsilon \leq y_n^*$ and $y_n^{*\prime\prime} \leq y_\Upsilon''$, $\forall n \in \mathbb{Z}_+$. Letting $n \longrightarrow +\infty$, we obtain $y_\Upsilon \leq \lim_{n\to+\infty} y_n^*$ and $\lim_{n\to+\infty} y_n^{*\prime\prime} \leq y_\Upsilon''$, which means that $y_\sharp$ is a maximal solution of the problem (1.1).

Step III⋆. The sequence $\{y_{*n}\}_{n\in\mathbb{N}}$ converges to minimal solution $y^\sharp(t)$ of problem (1.1).

Similar to the proof of the part of **Step II⋆.**, we make a little change to get a conclusion. What needs special explanation is that we have to construct subsequence $\{y_{*n\{a_{m+1}\}}\}_{n\in\mathbb{N}}$ of the sequence $\{y_{*n}\}_{n\in\mathbb{N}}$ which is increasing and bounded, then the pointwise limit of this sequence exists and it is denoted by $y^\sharp$. Let $y_{a_{m+1}} = \lim_{n\to\infty} y_{*n\{a_{m+1}\}}$. Hence, we have $y_{a_{m+1}} = y^\sharp$. The remaining parts of description is omitted. Consequently, the proof of our main result is complete. □

Remark 3.3. It follows from Theorem 3.1 we know that the maximal solution $y_\sharp$ and minimal solution $y^\sharp$ for the problem (1.1) have been obtained by constructing the iterative sequence $\{y_n^*\}_{n\in\mathbb{N}}$ and $\{y_{*n}\}_{n\in\mathbb{N}}$ with y^* and y_* respectively as the initial value.

4. EXAMPLES AND DISCUSSIONS

In this section, we would like to use the previous result to present the following examples. We would also give some discussions.

Example 4.1. We consider the following boundary value problem

$$\begin{cases} (\varphi_p(y'''))' = b(t)g(t, y, y', y'', y'''), \ t \in J' = [0,1] \setminus \{\frac{1}{3}\}, \\ y\left(\frac{1}{3}^+\right) = y\left(\frac{1}{3}^-\right) + 4, \ \ y'\left(\frac{1}{3}^+\right) = y'\left(\frac{1}{3}^-\right), \\ y''\left(\frac{1}{3}^+\right) = y''\left(\frac{1}{3}^-\right), \quad y'''\left(\frac{1}{3}^+\right) = y'''\left(\frac{1}{3}^-\right), \\ \eta y(0) - \lambda_1 y'(0) = \frac{4}{3}\int_0^1 \frac{1}{s} y(s) d\nu(s), \\ \eta y(1) + \lambda_2 y'(1) = \frac{1}{2}\int_0^1 s^2 y(s) d\nu(s), \\ \eta y''(0) - \lambda_3 y'''(0) = -\frac{1}{2}\int_0^1 s^3 y''(s) d\nu(s), \\ \eta y''(1) + \lambda_4 y'''(1) = \frac{1}{6}\int_0^1 s^4 y''(s) d\nu(s), \end{cases} \tag{4.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\nu(s) = s^2$, η, λ_1, λ_2, λ_3 and λ_4 are positive real numbers such that $\eta = 6$, $0 < \lambda_1 \leq \frac{28}{27}$, $\frac{2273}{1215} \leq \lambda_2 \leq 2$, $0 < \lambda_3 = \frac{73}{84}$, $0 < \lambda_4 \leq \frac{1}{90}$ and $f : J \longrightarrow \mathbb{R}$ is a function defined by

$$f(t) = \begin{cases} 2t + \frac{1}{3}, & \text{if } \ t \in [0, \frac{1}{3}], \\ \frac{1-\cos 2(t-\frac{1}{3})}{(t-\frac{1}{3})^2} + 2, & \text{if } \ t \in (\frac{1}{3}, 1]. \end{cases}$$

The function $f : J \longrightarrow \mathbb{R}$ is continuous for $t \neq \frac{1}{3}$, $f\left(\frac{1}{3}^-\right) = 1$ and $f\left(\frac{1}{3}^+\right) = 4$. We put by definition

$$b(t)g\left(t, y, y', y'', y'''\right) = f(t)\left(3t + 2y - ty' - ty'' + 8y'''\right) \tag{4.2}$$

Let

$$y_*(t) = \begin{cases} -3t, & \text{if } t \in [0, \frac{1}{3}], \\ -3t + 4, & \text{if } t \in (\frac{1}{3}, 1] \end{cases}$$

and

$$y^*(t) = \begin{cases} -t^5 - t^4 - t^3 + t + 16, & \text{if } t \in [0, \frac{1}{3}], \\ -t^5 - t^4 - t^3 + t + 20, & \text{if } t \in (\frac{1}{3}, 1]. \end{cases}$$

We have $y_*(t) \leq y^*(t)$ and $y^{*\prime\prime}(t) \leq y_*''(t)$ for all $t \in J$. It is easy to check that $\left(\varphi_p\left(y'''(t)\right)\right)' = 0$ for all $t \in J'$, and

$$b(t)g\left(t, y_*, y_*', y_*'', y_*'''\right) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{3}], \\ 8f(t), & \text{if } t \in (\frac{1}{3}, 1]. \end{cases}$$

Then we have

$$\left(\varphi_p\left(y_*'''(t)\right)\right)' \leq b(t)g\left(t, y_*, y_*', y_*'', y_*'''\right), \quad \text{for all } t \in J' \tag{4.3}$$

Also we have

$$\begin{cases} y_*\left(\frac{1}{3}^+\right) = y_*\left(\frac{1}{3}^-\right) + 4, \quad y_*'\left(\frac{1}{3}^+\right) = y_*'\left(\frac{1}{3}^-\right), \\ y_*''\left(\frac{1}{3}^+\right) = y_*''\left(\frac{1}{3}^-\right), \qquad y_*'''\left(\frac{1}{3}^+\right) = y_*'''\left(\frac{1}{3}^-\right), \\ \eta y_*(0) - \lambda_1 y_*'(0) = 3\lambda_1 \leq \frac{4}{3}\int_0^1 \frac{1}{s} y_*(s) d\nu(s) = \frac{28}{9}, \\ \eta y_*(1) + \lambda_2 y_*'(1) = 6 - 3\lambda_2 \leq \frac{1}{2}\int_0^1 s^2 y_*(s) d\nu(s) = \frac{157}{405}, \\ \eta y_*''(0) - \lambda_3 y_*'''(0) = 0 \geq -\frac{1}{2}\int_0^1 s^3 y_*''(s) d\nu(s) = 0, \\ \eta y_*''(1) + \lambda_4 y_*'''(1) = 0 \geq \frac{1}{6}\int_0^1 s^4 y_*''(s) d\nu(s) = 0. \end{cases} \tag{4.4}$$

Then by (4.3) and (4.4), it follows that y_* is a lower solution of problem (4.1).

Similarly, we have $\left(\varphi_p\left(y^{*\prime\prime\prime}(t)\right)\right)' = 0$ for all $t \in J'$, and

$$b(t)g\left(t, y^*, y^{*\prime}, y^{*\prime\prime}, y^{*\prime\prime\prime}\right) = \frac{1}{t}\begin{cases} (-t^4 + 3t^3 + 6t^2 - 10t)f(t), & \text{if } t \in [0, \frac{1}{3}], \\ (-t^4 + 2t^3 + 3t^2 - 11t)f(t), & \text{if } t \in (\frac{1}{3}, 1]. \end{cases}$$

Thus, we obtain

$$\left(\varphi_p\left(y^{*\prime\prime\prime}(t)\right)\right)' \geq b(t)g\left(t, y^*, y^{*\prime}, y^{*\prime\prime}, y^{*\prime\prime\prime}\right), \quad \text{for all } t \in J' \tag{4.5}$$

Also, we have

$$
(4.6)\quad \begin{cases} y^*\left(\frac{1}{3}^+\right) = y^*\left(\frac{1}{3}^-\right) + 4, \quad y^{*\prime}\left(\frac{1}{3}^+\right) = y^{*\prime}\left(\frac{1}{3}^-\right), \\ y^{*\prime\prime}\left(\frac{1}{3}^+\right) = y^{*\prime\prime}\left(\frac{1}{3}^-\right), \quad y^{*\prime\prime\prime}\left(\frac{1}{3}^+\right) = y^{*\prime\prime\prime}\left(\frac{1}{3}^-\right), \\ \eta y^*(0) - \lambda_1 y^{*\prime}(0) = 96 - \lambda_1 \geq \frac{4}{3}\int_0^1 \frac{1}{s} y^*(s)d\nu(s) = \frac{742}{15}, \\ \eta y^*(1) + \lambda_2 y^{*\prime}(1) = 108 - 11\lambda_2 \geq \frac{1}{2}\int_0^1 s^2 y^*(s)d\nu(s) \geq \frac{109061}{22680}, \\ \eta y^{*\prime\prime}(0) - \lambda_3 y^{*\prime\prime\prime}(0) = 6\lambda_3 \leq -\frac{1}{2}\int_0^1 s^3 y^{*\prime\prime}(s)d\nu(s) = \frac{73}{14}, \\ \eta y^{*\prime\prime}(1) + \lambda_4 y^{*\prime\prime\prime}(1) = -288 - 90\lambda_4 \leq \frac{1}{6}\int_0^1 s^4 y^{*\prime\prime}(s)d\nu(s) = -\frac{233}{63}, \end{cases}
$$

Then, by making use of (4.5) and (4.6), it follows that y^* is a upper solution of problem (4.1).

On the other hand, it is easy to show that the function g defined by (4.2) satisfies the hypothesis of Theorem 3.1 and therefore, it follows that the problem (4.1) has a minimal and a maximal solution. Consequently, the problem (4.1) has at least two solutions. □

Example 4.2. We study the following boundary value problem

$$
(4.7)\quad \begin{cases} (\varphi_p(y'''))' = b(t)g(t, y, y', y'', y'''), \ t \in J' = [0,1] \setminus \{\frac{3}{4}\}, \\ y\left(\frac{3}{4}^+\right) = y\left(\frac{3}{4}^-\right) + 1, \quad y'\left(\frac{3}{4}^+\right) = y'\left(\frac{3}{4}^-\right), \\ y''\left(\frac{3}{4}^+\right) = y''\left(\frac{3}{4}^-\right), \quad y'''\left(\frac{3}{4}^+\right) = y'''\left(\frac{3}{4}^-\right), \\ \eta y(0) - \lambda_1 y'(0) = -\frac{1}{6}\int_0^1 s y(s)d\nu(s), \\ \eta y(1) + \lambda_2 y'(1) = 2\int_0^1 s^2 y(s)d\nu(s), \\ \eta y''(0) - \lambda_3 y'''(0) = \frac{1}{7}\int_0^1 s^4 y''(s)d\nu(s), \\ \eta y''(1) + \lambda_4 y'''(1) = \frac{1}{9}\int_0^1 s^5 y''(s)d\nu(s), \end{cases}
$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\nu(s) = 2s + 1$, $\eta, \lambda_1, \lambda_2, \lambda_3$ and λ_4 are positive real numbers such that $\eta = 1$, $0 < \lambda_1 \leq \frac{25}{288}$, $\lambda_2 = \frac{29}{10}$, $\lambda_3 = \frac{89}{210}$, $\lambda_4 = \frac{11}{10003}$ and $f : J \longrightarrow \mathbb{R}$ is a function defined by

$$
f(t) = \begin{cases} t + \frac{5}{4}, & \text{if } t \in [0, \frac{3}{4}], \\ \frac{\tan(t-\frac{3}{4})}{t-\frac{3}{4}} + 2, & \text{if } t \in (\frac{3}{4}, 1]. \end{cases}
$$

The function $f : J \longrightarrow \mathbb{R}$ is continuous for $t \neq \frac{3}{4}$, $f\left(\frac{3}{4}^-\right) = 2$ and $f\left(\frac{3}{4}^+\right) = 3$. We put by definition

$$
(4.8)\quad b(t)g(t, y, y', y'', y''') = f(t)\left(3 + 3t + y + y' - t^2 y'' + 6ty'''\right)
$$

Let

$$
y_*(t) = \begin{cases} -3t, & \text{if } t \in [0, \frac{3}{4}], \\ -3t + 1, & \text{if } t \in (\frac{3}{4}, 1] \end{cases}
$$

and

$$y^*(t) = \begin{cases} -\frac{1}{3}t^3 - \frac{1}{2}t^2 + 3t + 8, & \text{if } t \in [0, \frac{3}{4}], \\ -\frac{1}{3}t^3 - \frac{1}{2}t^2 + 3t + 9, & \text{if } t \in (\frac{3}{4}, 1]. \end{cases}$$

We have $y_*(t) \leq y^*(t)$ and $y^{*\prime\prime}(t) \leq y_*''(t)$ for all $t \in J$. It is easy to check that $(\varphi_p(y'''(t)))' = 0$ for all $t \in J'$, and

$$b(t)g(t, y_*, y_*', y_*'', y_*''') = \begin{cases} 0, & \text{if } t \in [0, \frac{3}{4}], \\ f(t), & \text{if } t \in (\frac{3}{4}, 1]. \end{cases}$$

Then we have

$$(\varphi_p(y_*'''(t)))' \leq b(t)g(t, y_*, y_*', y_*'', y_*'''), \quad \text{for all } t \in J' \tag{4.9}$$

Also we have

$$\begin{cases} y_*\left(\frac{3}{4}^+\right) = y_*\left(\frac{3}{4}^-\right) + 1, \quad y_*'\left(\frac{3}{4}^+\right) = y_*'\left(\frac{3}{4}^-\right), \\ y_*''\left(\frac{3}{4}^+\right) = y_*''\left(\frac{3}{4}^-\right), \quad y_*'''\left(\frac{3}{4}^+\right) = y_*'''\left(\frac{3}{4}^-\right), \\ \eta y_*(0) - \lambda_1 y_*'(0) = 3\lambda_1 \leq -\frac{1}{6}\int_0^1 s y_*(s)d\nu(s) = \frac{25}{96}, \\ \eta y_*(1) + \lambda_2 y_*'(1) = -2 - 3\lambda_2 \leq 2\int_0^1 s^2 y_*(s)d\nu(s) = -\frac{107}{48}, \\ \eta y_*''(0) - \lambda_3 y_*'''(0) = 0 \geq \frac{1}{7}\int_0^1 s^4 y_*''(s)d\nu(s) = 0, \\ \eta y_*''(1) + \lambda_4 y_*'''(1) = 0 \geq \frac{1}{9}\int_0^1 s^5 y_*''(s)d\nu(s) = 0. \end{cases} \tag{4.10}$$

Then by (4.9) and (4.10), it follows that y_* is a lower solution of problem (4.7).

Similarly, we have $(\varphi_p(y^{*\prime\prime\prime}(t)))' = 0$ for all $t \in J'$, and

$$b(t)g(t, y^*, y^{*\prime}, y^{*\prime\prime}, y^{*\prime\prime\prime}) = \begin{cases} (-t^3 + 6t^2 + 5t - 12)f(t), & \text{if } t \in [0, \frac{3}{4}], \\ (-t^3 + 6t^2 + 5t - 13)f(t), & \text{if } t \in (\frac{3}{4}, 1]. \end{cases}$$

Thus, we obtain

$$(\varphi_p(y^{*\prime\prime\prime}(t)))' \geq b(t)g(t, y^*, y^{*\prime}, y^{*\prime\prime}, y^{*\prime\prime\prime}), \quad \text{for all } t \in J' \tag{4.11}$$

Also we have

$$\begin{cases} y^*\left(\frac{3}{4}^+\right) = y^*\left(\frac{3}{4}^-\right) + 1, \quad y^{*\prime}\left(\frac{3}{4}^+\right) = y^{*\prime}\left(\frac{3}{4}^-\right), \\ y^{*\prime\prime}\left(\frac{3}{4}^+\right) = y^{*\prime\prime}\left(\frac{3}{4}^-\right), \quad y^{*\prime\prime\prime}\left(\frac{3}{4}^+\right) = y^{*\prime\prime\prime}\left(\frac{3}{4}^-\right), \\ \eta y^*(0) - \lambda_1 y^{*\prime}(0) = 8 - 3\lambda_1 \geq -\frac{1}{6}\int_0^1 s y^*(s)d\nu(s) = -\frac{2413}{1440}, \\ \eta y^*(1) + \lambda_2 y^{*\prime}(1) = \frac{67}{6} + \lambda_2 \geq 2\int_0^1 s^2 y^*(s)d\nu(s) \geq \frac{9947}{720}, \\ \eta y^{*\prime\prime}(0) - \lambda_3 y^{*\prime\prime\prime}(0) = -1 + 2\lambda_3 \leq \frac{1}{7}\int_0^1 s^4 y^{*\prime\prime}(s)d\nu(s) = -\frac{16}{105}, \\ \eta y^{*\prime\prime}(1) + \lambda_4 y^{*\prime\prime\prime}(1) = -3 - 2\lambda_4 \leq \frac{1}{9}\int_0^1 s^5 y^{*\prime\prime}(s)d\nu(s) = -\frac{19}{189}. \end{cases} \tag{4.12}$$

Then by (4.11) and (4.12), it follows that y^* is a upper solution of the problem (4.7).

On the other hand, it is easy to show that the function $b(t)g(t, y^*, y^{*\prime}, y^{*\prime\prime}, y^{*\prime\prime\prime})$ defined by (4.8) satisfies the hypothesis of Theorem 3.1 and therefore, it follows that the problem (4.7) has a minimal and a maximal solution. Consequently, the problem (4.7) has at least two solutions. □

We discuss the conditions in the paper. It is easy to know that the functions satisfying the conditions of the theorem 3.1 are rather wide. For example, we can obtain the following corollary:

Corollary 4.3. *Let $y_*(t)$ and $y^*(t)$ be a lower and upper solution respectively for problem (1.1) such that $y_*(t) \leq y^*(t)$ in J. Assume that the conditions $(\boldsymbol{A}_i)$ for $i = 0, 1, 2, 3, 4$ and (H_1) with (H_2) hold, and the Nagumo-Wintner conditions relative to y_* and y^* are satisfied. Then the problem (1.1) has a solution $y^\clubsuit$ with $y_*(t) \leq y^\clubsuit(t) \leq y^*(t)$ in J.*

Remark 4.4. From above discussions, it is clear that our results unify, improve and extend the results in [16], [22] and [23] with [31].

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REFERENCES

[1] Z. Bai, B. Huang and W. Ge, The iterative solution for some fourth order p-Laplacian equation boundary value problems, *Appl. Math. Lett.,* 19(1): 8–14, 2006.

[2] S. Chen, The existence of multiple positive solutions for a class of third-order p-Laplacian operator singular boundary value problems, *Acta Math. Sci.,* 26(5): 794–800, 2006. (in Chinese)

[3] B. Ahmad, A. Alsaedi and D. Garout, Monotone iterative schemes for impulsive three-point nonlinear boundary value problems with quadratic convergence, *J. Korean Math. Soc.,* 45(5): 1275–1295, 2008.

[4] M. Derhab and H. Mekni, Existence of minimal and maximal solution for a second order differential equation with nonlocal boundary conditions, *Comm. Appl. Nonlinear Anal.,* 18(2): 1–19, 2011.

[5] M. Derhab and B. Messirdi, Existence of minimal and maximal solutions for a fourth order quasilinear impulsive differential equations with integral boundary conditions, *Comm. Appl. Nonlinear Anal.,* 22(1): 1–21, 2015.

[6] K. Deimling, *Nonlinear Functional Analysis*, New York, Springer-Verlag, 1985.

[7] D. Guo, Existence of solution of boundary value problems for nonlinear second order impulsive differential equations in Banach space, *J. Math. Anal. Appl.,* 181(2): 407–421, 1994.

[8] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[9] X. He, Double positive solutions of a three-point boundary value problems for the one dimensional p-Laplacian, *Appl. Math. Lett.,* 17: 867–873, 2004.

[10] X. He and W. Ge, Twin positive solutions for one dimensional p-Laplacian boundary value problems, *Nonlinear Anal.,* 56: 975–984, 2004.

[11] D. Ji and W. Ge, Multiple positive solutions for some p-Laplacian boundary value problems, *Appl. Math. Comput.,* 187: 1315–1325, 2007.

[12] N. Khattabi, M. Frigon and N. Ayyadi, Multiple solutions of boundary value problem with $\Phi-$Laplacian operators and under a Wintner-Nagumon growth condition, *Bound. Value Probl.,* 2013: 1–21, 2013.

[13] L. Kong and J. Wang, Multiple positive solutions for the one dimensional p-Laplacian, *Nonlinear Anal.,* 42: 1327–1333, 2000.

[14] D. Kong, L. Liu and Y. Wu, Triple positive solutions of a boundary value problems for nonlinear singular second-order differential equation of mixed type with p-Laplacian, *Comput. Math. Appl.,* 58: 1425–1432, 2009.

[15] K. Lan and J. Webb, Positive solution of semilinear differential equation with singularities, *J. Differential Equations,* 148: 407–421, 1998.

[16] Y. Lee and S. Chun, Extinction and positivity of solution of the p-Laplacian evolution equations on networks, *J. Math. Anal. Appl.,* 386: 581–592, 2012.

[17] X. Li, Multiple positive solutions for some four-point boundary value problems with p-Laplacian, *Appl. Math. Comput.,* 202: 413–426 2008.

[18] B. Liu, Positive solutions of three-point boundary value problem for the one-dimensional p-Laplacian with infinitely many singularities, *Appl. Math. Lett.,* 17: 655–661, 2004.

[19] D. Ma, Z. Du and W. Ge, Existence and iteration of monotone positive solutions for multipoint boundary value problems with p-Laplacian operator, *Comput. Math. Appl.,* 50: 729–739, 2005.

[20] K. Mazowiecka and A. Schikorra, Fractional div-curl guantities and appplications to nonlocal geometric equations, *J. Funct. Anal.,* 275: 1–44, 2018.

[21] V. D. Milman and A. D. Myshkis, On the Stability of motion in the presence of impulses, *Sib. Math. J.,* 1(2): 233–237, 1960. (in Russian)

[22] B. P. Rynne, Simple bifurcation and global curves of solution of p-Laplacian problems with radial symmetry, *J. Differential Equations,* 263: 3611–3626, 2017.

[23] M. Struwe, On the evolution of harmonic mappings of Riemannian surface, *Comment. Math. Helv.,* 60: 558–581, 1985.

[24] H. Su, Z. Wei, B. Wang, The existence of positive solution for a nonlinear four-point singular boundary value problem with a p-Laplacian operator, *Nonlinear Anal.,* 66: 2204–2217, 2007.

[25] Y. Sun, L. Liu, J. Zhang and R. P. Agarwal, Positive solution of singular three-point boundary value problems for second-order differential equations, *J. Comput. Appl. Math.,* 230(1): 738–750, 2009.

[26] J. L. Vázquez, Smoothing and decay estimates for nonlinear diffusion equation, *Oxford Lecture Series in Math. and its Appli.,* Oxford University Press, Oxford, No.33, 2006.

[27] J. L. Vázquez, Growing solutions of the fractional p-Laplacian equations in the fast diffusion range, *Nonlinear Anal.,* 214: 112575, 2022.

[28] Y. Wang, C. Hou, Existence of multiple positive solutions for one-dimensional p-Laplacian, *J. Math. Anal. Appl.,* 315: 144–153, 2006.

[29] Y. Wang and W. Ge, Multiple positive solutions for multipoint boundary value problem with one-dimensional p-Laplacian, *J. Math. Anal. Appl.,* 327: 1381–1395, 2007.

[30] Z. Wang and J. Zhang, Positive solutions for one-dimensional p-Laplacian boundary value problem with dependence on the first order derivative, *J. Math. Anal. Appl.,* 314: 618–630, 2006.

[31] J. Wettstein, Uniquess and regularity of the fractional harmonic gradient flow in S^{n-1}, *Nonlinear Anal.,* 214: 112592, 2022.

[32] C. Yang and J. Yan, Positive solution for third-order Sturm-Liouville boundary value problem with p-Laplacian, *Comput. Math. Appl.,* 59: 2059–2066, 2010.

[33] D. Zhao, H. Wang and W. Ge, Existence of triple positive solutions to a class of p-Laplacian boundary value problem, *J. Math. Anal. Appl.,* 328: 972–983, 2007.