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A MARTINGALE APPROACH TO ASYMPTOTIC STABILITY OF NONLINEAR STOCHASTIC DIFFERENCE EQUATIONS WITH BOUNDED NOISE IN $\mathbb{R}^1$

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ABSTRACT. Necessary and sufficient conditions for almost sure asymptotic stability of solutions of stochastic dynamical systems generated by linear and nonlinear, nonautonomous ordinary stochastic difference equations (SDE) in $\mathbb{R}^1$

$$X_{n+1} \quad = \quad X_n\Big(1-\alpha_n f(X_n)+\sigma_n g(X_n)\xi_{n+1}\Big)$$

driven by square-integrable independent random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ with uniformly bounded quantities $\sigma_n\xi_{n+1}$ are in the center of this presentation. All conditions are explicitly expressed in terms of the coefficients α_n, σ_n, f and g. Kolmogorov's variant of the strong law of large numbers as well as martingale convergence and martingale representation theorems are applied to prove related results.

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1. INTRODUCTION

The search for efficient necessary and sufficient conditions (abbreviated by iff-conditions here) for almost sure asymptotic stability of the solution of stochastic difference equations (SDE) is highly complex and important for applications, especially in mathematical finance (asset price evolutions in discrete (B, S)-markets) and mathematical biology (population dynamics). Mostly, authors have found sufficient conditions while using Lyapunov-Krasovskii functional's technique, see e.g. Kolmanovskii and Shaikhet [18], [19] Kolmanovskii V.B., Kosareva N.P. and Shaikhet L.E. [17], Kolmanovskii, Koroleva and Kosareva [16], Rodkina, Mao and Kolmanovskii [33], Rodkina [31], [32], Rodkina and Nosov [34], Rodkina and Schurz [35], [36], [37] and [38]. We are aiming at obtaining efficient criteria (i.e. iff-conditions) to guarantee almost sure asymptotic stability without applying Lyapunov-Krasovskii functional's method while using martingale convergence theorems. All conditions shall be expressed in terms of the coefficients of related stochastic equations.

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Our studies are devoted to the problem of almost sure asymptotic stability of the solutions of ordinary stochastic difference equations

$$X_{n+1} = X_n\Big(1 - \alpha_n f(X_n) + \sigma_n g(X_n)\xi_{n+1}\Big) \tag{1.1}$$

driven by the sequence of independent random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ with moments $\mathbb{E}[\xi_{n+1}] = 0$ and $\mathbb{E}[\xi_{n+1}]^2 < +\infty$ such that $m_{n+1} := \sum_{i=0}^{n} \xi_{i+1}$ defines a square-integrable martingale with respect to the naturally generated filtration

$$\mathcal{F}_{n+1} = \sigma(\{\xi_{i+1} : i = 0, 1, ..., n\}). \tag{1.2}$$

Such equations occur as natural discretizations of asset prices in mathematical finance or population dynamics in mathematical biology - phenomena which are modelled by stochastic differential equations (SDEs) driven by independent Wiener processes. For related theory and numerics, see Allen [1], Arnold [3], [4], Dynkin [6], Evans [7], Freidlin and Wentzell [8], Friedman [9], Gard [10], [12], Khas'minskij [13], Kloeden, Platen and Schurz [15], Krylov [21], Ladde and Sambandham [22], Mao [24], [25], Mil'shtein [26], Mohammed [27], Schurz [40], [41], [42], [43] and further applications in Kliemann and Sri [14].

We are going to prove that, for almost sure asymptotic stability of the trivial solution of (1.1), it is necessary and sufficient that at least one of the sums $\sum_{n=0}^{+\infty} \alpha_n$ or $\sum_{n=0}^{+\infty} \sigma_n^2 \eta_n$ with $\eta_n = \mathbb{E}[\xi_{n+1}]^2 < +\infty$ have to be divergent almost surely. Our generalizable proofs are not confined to identically distributed random variables Z_n such that we could refer to the case of dependent $Z_n = \sigma_n g(X_l : l \le n)\xi_{n+1}$ too. Moreover, it is worth to note that, to the best of our knowledge up to now, such results are hardly met in view of possible applications to stochastic numerical analysis, even not often for linear equations (discrete or continuous) where almost sure stability was proved only under the assumption $\sum_{n=0}^{+\infty} \sigma_n^2 \eta_n = +\infty$, except for the groundbreaking papers of Higham [11] and Schurz [42], where results concerning almost sure asymptotic stability were obtained for the trivial solution of equations

$$X_{n+1} = X_n\Big(1 + \sigma_0 \xi_{n+1}\Big) + |c_0 \xi_{n+1}|(X_n - X_{n+1})$$

with zero drift and real parameters $|c_0| \ge |\sigma_0| > 0$ (i.e., more precisely speaking, equations interpreted as linear-implicit discretizations of Girsanov's stochastic differential equations $dX(t) = \sigma_0 X(t) dW(t)$ through balanced implicit methods with unbounded martingale-type of noise $\xi_{n+1} = W(t_{n+1}) - W(t_n) \in \mathcal{N}(0, t_{n+1} - t_n)$ along partitions $0 = t_0 < t_1 < \cdots < t_N = T$ driven by an underlying Wiener process W in a more general context). For stability investigations of stochastic numerical methods using Lyapunov functions in any dimension, cf. also [40], [41], [42], [43]. For a.s. asymptotic stability of linear systems of drift-implicit stochastic Theta methods in $\mathbb{R}^d$, see [44].

For the general setting, we assume the following. Fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ (i.e. complete with respect to the $\mathbb{P}$-null sets of $\mathcal{F}$, see Kolmogorov (1933) [20]). The notation $X = (X_n(\omega))_{n\in\mathbb{N}}$ denotes a $(\mathcal{F}_n)$-adapted stochastic process with $X_n(\omega) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$. $\mathcal{B}(S)$ represents the σ-field of all Borel-sets of the set S. We also use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. Moreover, "SLLN" stands for "law of large numbers" and "MCT" for "martingale convergence theorems" (see Doob [5], Liptser and Shiryaev [23], Neveu [29], Protter [30] or Shiryaev [46] for more details concerning these concepts).

The paper is organized by 7 sections as follows. After this introduction Section 2 commences with a compilation of necessary preliminaries in order to prove our main results. Section 3 discusses a.s. asymptotic stability for linear equations without drift parts. In Section 4 we extend our investigations to the stability behavior of linear equations with drift parts. Section 5 reports on results with respect to nonlinear equations with trivial solution. Eventually, in Sections 6 and 7 we work on some relaxations of the previously presented conditions which guarantee a.s. asymptotic stability of related class of stochastic difference equations (1.1).

2. PRELIMINARIES

A series of preliminary results is needed for the proof of main results. For its statement, we borrow some results from Shiryaev [46], chapter IV, paragraph 3, p. 389 and derive some further simple conclusions.

Theorem 2.1 (Kolmogorov's SLLN). *Assume that $\xi_0, \xi_1, \xi_2, \ldots$ is a sequence of independent real-valued random variables with finite second moments and set $\theta_i^2 = Var(\xi_i)$ for $i \in \mathbb{N}$. Let $S_n = \xi_0 + \xi_1 + \cdots + \xi_n$ and $b_n > 0$ be real numbers such that $b_n \uparrow +\infty$ (i.e. monotonically increasing) as $n \to +\infty$ and*

$$\sum_{i=0}^{+\infty} \frac{\theta_i^2}{b_i^2} < +\infty \tag{2.1}$$

Then, we have $\mathbb{P}$-a.s.

$$\lim_{n\to+\infty} \frac{S_n - \mathbb{E}\, S_n}{b_n} = 0. \tag{2.2}$$

Corollary 2.2. *Assume that, in addition to the conditions of Theorem 2.1, the hypothesis*

$$\sum_{i=0}^{+\infty} \theta_i^2 = +\infty \tag{2.3}$$

holds. Then, for any $\delta > 0$*, we have* $\mathbb{P}$*-a.s.*

$$\lim_{n\to+\infty} \frac{S_n - \mathbb{E}\, S_n}{(\sum_{i=0}^{n} \theta_i^2)^{\frac{1}{2}+\delta}} \;=\; 0. \tag{2.4}$$

Proof. Let $\delta > 0$ be any real constant. Define

$$A_n = \sum_{i=0}^{n} \theta_i^2, \quad b_n = A_n^{\frac{1+\delta}{2}},$$

and find some $N = N(\omega) > 0$ such that, for $n > N$, we have

$$\frac{A_n}{1+A_n} = 1 - \frac{1}{1+A_n} > \frac{1}{2}.$$

Then

$$\begin{aligned}\sum_{i=N}^{n} \frac{\theta_i^2}{b_i^2} &= \sum_{i=N}^{n} \frac{\theta_i^2}{A_i^{1+\delta}} < 2^{1+\delta} \sum_{i=N}^{n} \frac{\theta_i^2}{(1+A_i)^{1+\delta}} \le 2^{1+\delta} \sum_{i=N}^{n} \int_{A_{i-1}}^{A_i} \frac{dt}{(1+t)^{1+\delta}} \\ &< 2^{1+\delta} \int_0^{+\infty} \frac{dt}{(1+t)^{1+\delta}} < +\infty.\end{aligned}$$

Now, it remains to apply Kolmogorov's SLLN (i.e. Theorem 2.1). □

Corollary 2.3. *Assume that under the conditions of Corollary 2.2 we have* $\mathbb{E}\,[\xi_n] = 0$ *for all* $n \in \mathbb{N}$*. Then, for any* $\delta, \varepsilon > 0$*, there exists a random integer* $N = N(\varepsilon, \omega) > 0$ *such that, for all* $n > N$*, we have* $\mathbb{P}$*-a.s.*

$$|S_n| \;\le\; \varepsilon \left(\sum_{i=0}^{n} \theta_i^2 \right)^{\frac{1}{2}+\delta}. \tag{2.5}$$

Lemma 2.4. *If* $u \in \mathbb{R}^1$ *and* $|u| \le k < 1$ *then*

$$0 \;<\; 1+u \;\le\; \exp\left(u - \frac{u^2}{2(1+k)^2} \right) \quad \text{and} \tag{2.6}$$

$$1+u \;\ge\; \exp\left(u - \frac{u^2}{2(1-k)^2} \right) \;>\; 0. \tag{2.7}$$

Proof. Simple manipulation using properties of ln function and its Taylor series. □

Lemma 2.5. *If* $u \in \mathbb{R}^1$ *and* $|u| \le k \le 1$ *then*

$$0 \;\le\; 1+u \;\le\; \exp\left(u - \frac{1}{2e^k} u^2 \right) \tag{2.8}$$

and, if additionally $k^2 \exp(k) < 2$ *then*

$$1+u \;\ge\; \exp\left(u - \frac{e^k}{2} u^2 \right) \;\ge\; \exp\left(u - l(k) u^2 \right) \;>\; 0 \tag{2.9}$$

where

$$l(k) \;=\; \frac{e^k}{2\,(1 - k^2 e^k/2)}. \tag{2.10}$$

Proof. An expansion of e^x into its 2nd order Taylor series yields that

$$\exp(u) = 1 + u + \frac{u^2}{2!}\exp(\theta) \tag{2.11}$$

where θ is an intermediate value satisfying $0 < |\theta| < |u|$. Then

$$1 + u = \exp(u) - \frac{u^2}{2!}\exp(\theta) = \exp(u)\left(1 - \frac{u^2\exp(\theta)}{2\exp(u)}\right). \tag{2.12}$$

Let $|u| \le k \le 1$. For $u > 0$, we have the elementary estimates

$$0 \le \theta \le u \le k, \quad 1 \le \exp(\theta) \le \exp(u) \le \exp(k),$$
$$\text{hence} \quad \frac{1}{\exp(k)} \le \frac{1}{\exp(u)} \le \frac{\exp(\theta)}{\exp(u)} = \exp(\theta - u) \le 1, \tag{2.13}$$

and, for $u < 0$, the estimates

$$0 \ge \theta \ge u \ge -k, \quad 1 \ge \exp(\theta) \ge \exp(u) \ge \exp(-k),$$
$$\text{hence} \quad 1 \le \frac{\exp(\theta)}{\exp(u)} = \exp(\theta - u) \le \exp(k) \tag{2.14}$$

hold. Thus, summarizing, for any $|u| \le k \le 1$, we may conclude that

$$\exp(k) \ge \frac{\exp(\theta)}{\exp(u)} \ge \frac{1}{\exp(k)}, \quad 0 \le 1 + u \le \exp\left(u - \frac{1}{2e^k}u^2\right). \tag{2.15}$$

Consequently, inequalities (2.8) are proven. For the proof of inequalities (2.9) we use the other sides of inequalities (2.13) and (2.14). Namely, we use the estimate

$$\frac{\exp(\theta)}{\exp(u)} \le \max\{1, e^k\} = \exp(k).$$

Thus, while assuming that $k^2\exp(k) < 2$ and $|u| \le k$, we arrive at

$$1 + u = \exp(u) \cdot \left(1 - \frac{u^2\exp(\theta)}{2\exp(u)}\right) \ge \exp(u) \cdot \left(1 - \frac{u^2\exp(k)}{2}\right) > 0. \tag{2.16}$$

Now, we make use of a Taylor expansion of $\ln(1 + u)$ up to the first order term, namely $\ln(1 + u) = u/(1 + \theta)$ with intermediate value $\theta \in (\min(0, u), \max(0, u))$ and $|u| \le k < 1$. Estimate the second factor in the last product in (2.16) as follows. Consider

$$0 < 1 - \frac{u^2\exp(k)}{2} = \exp\left(\ln\left(1 - \frac{u^2\exp(k)}{2}\right)\right) = \exp\left(-\frac{u^2\exp(k)}{2(1 + \zeta_i)}\right) \tag{2.17}$$

with intermediate value $\zeta_i \in (-u^2\exp(k)/2, 0)$. Recall that $k^2\exp(k) < 2$ holds. Note that

$$1 \le \frac{1}{1 + \zeta_i} \le \frac{1}{1 - u^2e^k/2} \le \frac{1}{1 - k^2e^k/2}. \tag{2.18}$$

Define l(k) as in (2.10). Combining (2.18) with (2.17) leads to

$$1 - \frac{u^2\exp(k)}{2} \ge \exp(-l(k)u^2) > 0.$$

Plugging this estimate into (2.16) brings us to the conclusion of Lemma 2.5. □

3. ALMOST SURE STABILITY FOR LINEAR EQUATIONS WITHOUT DRIFT

Consider linear stochastic difference equations

$$X_{n+1} \;=\; X_n\Big(1+\sigma_n\xi_{n+1}\Big) \tag{3.1}$$

started at some initial value X_0 which is independent of $\mathcal{F}_n$ for all $n \in \mathbb{N}$, where σ_n are nonrandom parameters and ξ_{n+1} are uniformly bounded random variables such that

$$\forall i \in \mathbb{N} : |\sigma_i\xi_{i+1}| \;\leq\; k \;<\; 1, \quad \text{and} \quad k^2\exp(k) \;<\; 2 \tag{3.2}$$

where $k > 0$ is a real constant. Assume that ξ_{n+1} are independent random variables which have mean $\mathbb{E}\,[\xi_{i+1}] = 0$ and finite 2nd moments $\mathbb{E}\,[\xi_{i+1}]^2 = \eta_i < +\infty$. In the statement of the theorem below, we shall also suppose that

$$\sum_{i=0}^{+\infty}\sigma_i^2\eta_i \;=\; +\infty. \tag{3.3}$$

Theorem 3.1. *Assume that condition (3.2) is satisfied. Then, condition (3.3) is fulfilled if and only if* $\lim_{n\to+\infty} X_n = 0$ *holds* $\mathbb{P}$*-a.s. for all solutions* $(X_n)_{n\in\mathbb{N}}$ *of equation (3.1).*

Proof. Suppose that $(X_n)_{n\in\mathbb{N}}$ solves (3.1). First, some preliminary considerations. Applying Lemma 2.5, more precisely from its proof-step (2.12), we get to the representation

$$X_{n+1} \;=\; X_0\prod_{i=0}^{n}(1+\sigma_i\xi_{i+1}) \;=\; X_0\prod_{i=0}^{n}\exp(\sigma_i\xi_{i+1})\left(1-\frac{(\sigma_i\xi_{i+1})^2\exp(\theta_i)}{2\exp(\sigma_i\xi_{i+1})}\right) \tag{3.4}$$

where $\theta_i \in [0, |\sigma_i\xi_{i+1}|] \subseteq [0,k]$. Let $\phi_i : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ be the moment generating function related to the random variable $-\xi^2_{i+1}$. Then, an expansion of e^x into its Taylor series leads to

$$\phi_i(t) \;=\; \mathbb{E}\,[\exp(-t\xi^2_{i+1})] \;=\; \mathbb{E}\,\left[1-t\xi^2_{i+1}\exp(\zeta_i)\right]$$

for $t \in \mathbb{R}^1_+$, where $\zeta_i \in [-t\xi^2_{i+1}, 0]$. For $t\xi^2_{i+1} \leq k^2$, one arrives at

$$-k^2 \leq -t\xi^2_{i+1} \leq \zeta_i \leq 0, \quad -\exp(-k^2) \geq -\exp(-t\xi^2_{i+1}) \geq -\exp(\zeta_i) \geq -1.$$

This implies that

$$\begin{aligned} \phi_i(t) \;&=\; 1-t^2\mathbb{E}\,[\xi^2_{i+1}\exp\{\zeta_i\}] \\ &\leq\; 1-t^2\exp(-k^2)\mathbb{E}\,[\xi^2_{i+1}] \;=\; 1-t^2\exp(-k^2)\eta_i \end{aligned} \tag{3.5}$$

and, in particular for $t = \sigma_i^2$, we have

$$0 \;<\; \phi_i(\sigma_i^2) \;\leq\; 1-\sigma_i^2\exp(-k^2)\eta_i. \tag{3.6}$$

We also note that

$$M_{n+1} \quad = \quad \prod_{i=0}^{n} \frac{\exp(-\sigma_i^2 \xi_{i+1}^2)}{\phi_i(\sigma_i^2)}$$

defines a nonnegative martingale $M = (M_n)_{n\in\mathbb{N}}$ started at initial value $M_0 = 1$, since the expectation of any independent factor in the above product is 1, i.e., more precisely, $\mathbb{E}\left[\exp(-\sigma_i^2\xi_{i+1}^2)/\phi_i(\sigma_i^2)\right] = 1$. Thanks to well-known martingale convergence theorems (MCT, see [5], [23], [29], [30], [46]), this martingale M converges ($\mathbb{P}$-a.s.) to a finite random variable $M_{+\infty}$. Therefore, it is uniformly bounded above by some $\mathbb{P}$-a.s. finite random variable $H_1 = H_1(\omega) > 0$, i.e.

$$1 \;\le\; \sup_{n\in\mathbb{N}} M_n(\omega) \;\le\; H_1(\omega) \;<\; +\infty \tag{3.7}$$

for $\omega \in \Omega$ ($\mathbb{P}$-a.s.). Furthermore, from the elementary estimate $1 + u \le e^u$ and (3.6), we know that

$$0 \;<\; \prod_{i=0}^{n} \phi_i(\sigma_i^2) \;\le\; \prod_{i=0}^{n} \left(1 - \exp(-k^2)\sigma_i^2\eta_i\right) \;\le\; \exp\left(-\exp(-k^2)\sum_{i=0}^{n}\sigma_i^2\eta_i\right). \tag{3.8}$$

Second, return to the estimation of (3.4). Suppose that condition (3.3) is fulfilled. For the estimation of $|X_{n+1}|$ for the solution X_{n+1} of (3.1) from above, we apply inequality $1 + u \le e^u$, Lemma 2.5 and simple estimation (2.15). Thus, by returning to representation (3.4), we obtain

$$|X_{n+1}| \;\;\le\;\; |X_0| \exp\left(\sum_{i=0}^{n}\sigma_i\xi_{i+1} - \frac{1}{2e^k}\sum_{i=0}^{n}\sigma_i^2\xi_{i+1}^2\right). \tag{3.9}$$

To estimate the term $\sum_{i=0}^{n}\sigma_i\xi_{i+1}$ we apply Corollary 2.3 from the preliminaries in section 2. Consequently, there exists some $\mathbb{P}$-a.s. finite random variable $H_2 = H_2(\omega) > 0$ (i.e. ε from Corollary 2.3) such that, for any $\delta \in (0, 1/2)$, we have $\mathbb{P}$-a.s.

$$\sum_{i=0}^{n}\sigma_i\xi_{i+1} \;\;\le\;\; \left|\sum_{i=0}^{n}\sigma_i\xi_{i+1}\right| \le H_2\left(\sum_{i=0}^{n}\sigma_i^2\eta_i\right)^{\frac{1}{2}+\delta} \tag{3.10}$$

for all $n \geq N(H_2, \omega)$. Combining (3.7), (3.8), (3.9) and (3.10) leads to (for all $n \geq N(H_2, \omega)$)

$$\begin{aligned} |X_{n+1}| &\leq |X_0| \cdot \exp\left(H_2 \left[\sum_{i=0}^{n} \sigma_i^2 \eta_i \right]^{\frac{1}{2}+\delta} \right) \cdot \exp\left(-\frac{1}{2e^k} \sum_{i=0}^{n} \sigma_i^2 \xi_{i+1}^2 \right) \\ &= |X_0| \cdot \exp\left(H_2 \left[\sum_{i=0}^{n} \sigma_i^2 \eta_i \right]^{\frac{1}{2}+\delta} \right) \cdot \left(\prod_{i=0}^{n} \frac{\exp(-\sigma_i^2 \xi_{i+1}^2)}{\phi_i(\sigma_i^2)} \right)^{1/(2e^k)} \cdot \left(\prod_{i=0}^{n} \phi_i(\sigma_i^2) \right)^{1/(2e^k)} \\ &\leq |X_0| \cdot \exp\left(H_2 \left[\sum_{i=0}^{n} \sigma_i^2 \eta_i \right]^{\frac{1}{2}+\delta} \right) \cdot H_1^{1/(2e^k)} \cdot \exp\left(-\frac{\exp(-k^2)}{2e^k} \sum_{i=0}^{n} \sigma_i^2 \eta_i \right) \end{aligned}$$

$$\leq |X_0| \cdot H_1^{1/(2e^k)} \cdot \exp\left(H_2 \left[\sum_{i=0}^{n} \sigma_i^2 \eta_i \right]^{\frac{1}{2}+\delta} - \frac{\exp(-k^2 - k)}{2} \sum_{i=0}^{n} \sigma_i^2 \eta_i \right). \tag{3.11}$$

Due to condition (3.3), for all real numbers $\varepsilon > 0$, there exists some random integer $N_1 = N_1(\varepsilon, \omega)$ such that, for all $n \geq N_1$ and all $\delta \in (0, 1/2)$, we can estimate

$$H_2 \left[\sum_{i=0}^{n} \sigma_i^2 \eta_i \right]^{\frac{1}{2}+\delta} \leq \varepsilon \sum_{i=0}^{n} \sigma_i^2 \eta_i.$$

Then, for fixed $\varepsilon > 0$ with $\varepsilon < \exp(-k^2 - k)/4$ and all $n \geq \max\{N_1(\varepsilon, \omega), N(H_2, \omega)\}$, we arrive at

$$|X_{n+1}| \leq |X_0| \cdot H_1^{1/(2e^k)} \cdot \exp\left(-\frac{\exp(-k^2 - k)}{4} \sum_{i=0}^{n} \sigma_i^2 \eta_i \right),$$

hence, a.s. stability of equation (3.1) can be established and the asymptotic property

$$\lim_{n \to +\infty} X_n = 0 \tag{3.12}$$

holds $\mathbb{P}$-a.s.

Third, consider the verification of the backwards conclusion in iff-part of Theorem 3.1. Suppose that $\lim_{n \to \infty} X_n = 0$ holds $\mathbb{P}$-a.s. Return to the estimation of representation (3.4). An estimation of $|X_{n+1}|$ from below using the second part of Lemma 2.5 (more precisely, its proof-step (2.12)) yields that

$$|X_{n+1}| \geq |X_0| \cdot \prod_{i=0}^{n} \exp(\sigma_i \xi_{i+1}) \left(1 - \frac{(\sigma_i \xi_{i+1})^2 \exp(\theta_i)}{2 \exp(\sigma_i \xi_{i+1})} \right)$$

$$\geq |X_0| \cdot \prod_{i=0}^{n} \exp(\sigma_i \xi_{i+1}) \left(1 - \frac{(\sigma_i \xi_{i+1})^2 \exp(k)}{2} \right) > 0. \tag{3.13}$$

To proceed with the estimation of $|X_{n+1}|$ from below, consider the expression $1 - ((\sigma_i \xi_{i+1})^2 \exp(k))/(2)$. Now, we make use of a Taylor expansion of $\ln(1 + u)$ up to the first order term, namely $\ln(1 + u) = u/(1 + \theta)$ with intermediate value $\theta \in$

$(\min(0,u),\max(0,u))$ and $|u| \le k < 1$. We estimate the factors of the product in (3.13) as in the proof of Lemma 2.5 as follows. Set $u = \sigma_i\xi_{i+1}$ and get to

$$\begin{aligned} 0 \;<\; 1-\frac{(\sigma_i\xi_{i+1})^2\exp(k)}{2} \;&=\; \exp\left(\ln\left(1-\frac{(\sigma_i\xi_{i+1})^2\exp(k)}{2}\right)\right) \\ &=\; \exp\left(-\frac{(\sigma_i\xi_{i+1})^2\exp(k)}{2(1+\zeta_i)}\right) \end{aligned} \tag{3.14}$$

with intermediate value $\zeta_i \in (-(\sigma_i\xi_{i+1})^2\exp(k)/2, 0)$. Recall that $k^2\exp(k) < 2$ holds by assumption (3.2). Note that the term $1/(1+\zeta_i)$ can be estimated from above as in (2.18). Define $l(k)$ as in (2.10). Applying estimation (2.18) to (3.14) implies that

$$\prod_{i=0}^{n}\exp(\sigma_i\xi_{i+1})\cdot\left(1-\frac{(\sigma_i\xi_{i+1})^2\exp(k)}{2}\right) \ge \prod_{i=0}^{n}\exp\left(\sigma_i\xi_{i+1}-l(k)(\sigma_i\xi_{i+1})^2\right).$$

Then, by exploiting the prior observations, estimation (3.13) renders to

$$|X_{n+1}| \;\ge\; |X_0|\cdot\exp\left(\sum_{i=0}^{n}\sigma_i\xi_{i+1}-l(k)\sum_{i=0}^{n}(\sigma_i\xi_{i+1})^2\right). \tag{3.15}$$

Next, for further estimation from below, consider $\phi_i^{(2)} : \mathbb{R}_+^1 \to \mathbb{R}_+^1$ as the moment generating function for the random variable ξ_i^2, defined by $\phi_i^{(2)}(t) = \mathbb{E}\left[\exp(t\xi_{i+1}^2)\right]$ for $0 \le t \le \sigma_i^2$. Then, from an expansion of e^u into its Taylor series $e^u = 1 + ue^\theta$ where $\theta \in (0,u)$, we conclude that

$$\phi_i^{(2)}(t) \;=\; \mathbb{E}\left[\exp(t\xi_{i+1}^2)\right] \;=\; \mathbb{E}\left[1+t\xi_{i+1}^2\exp(\zeta_i)\right],$$

where $0 \le |\zeta_i| \le k^2$ and $\exp(k^2) \;\ge\; \exp(\zeta_i) \;\ge\; \exp(-k^2)$. An estimation of $\phi^{(2)}(t)$ from above gives

$$\phi_i^{(2)}(t) \;=\; 1+t\mathbb{E}\left[\xi_{i+1}^2\exp(\zeta_i)\right] \;\le\; 1+t\exp(k^2)\mathbb{E}\left[\xi_{i+1}\right]^2 \;=\; 1+t\exp(k^2)\eta_i$$

for $0 \le t \le \sigma_i^2$. Hence, in particular, we have

$$\phi_i^{(2)}(\sigma_i^2) \;\le\; 1+\sigma_i^2\exp(k^2)\eta_i.$$

We note that $M^{(2)} = (M_n^{(2)})_{n\in\mathbb{N}}$ satisfying

$$M_{n+1}^{(2)} \;=\; \prod_{i=0}^{n}\frac{\exp(\sigma_i^2\xi_{i+1}^2)}{\phi_i^{(2)}(\sigma_i^2)}$$

started at $M_0^{(2)} = 1$ forms a nonnegative martingale with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and its expectation $\mathbb{E}\left[M_n^{(2)}\right] = 1$. Then, by MCT (see [5], [23], [29], [30], [46]), M converges $\mathbb{P}$-a.s. and is uniformly bounded by some $\mathbb{P}$-a.s. finite random variable $H_3 = H_3(\omega) \ge 1$. Similarly to the uniform boundedness (3.7), we find that

$$1 \;\le\; \sup_{n\in\mathbb{N}} M_n^{(2)} \;\le\; H_3 \;<\; +\infty, \tag{3.16}$$

hence

$$\frac{1}{M_n^{(2)}} \;\geq\; \frac{1}{\sup_{n\in\mathbb{N}} M_n^{(2)}} \;\geq\; \frac{1}{H_3} \;>\; 0 \tag{3.17}$$

for all $n \in \mathbb{N}$. Then, using the latter observations, we have

$$\prod_{i=0}^{n} \exp(-\sigma_i^2 \xi_{i+1}^2) \;=\; \prod_{i=0}^{n} \frac{\phi^{(2)}(\sigma_i^2)}{\exp(\sigma_i^2\xi_{i+1}^2)} \cdot \prod_{i=0}^{n} \frac{1}{\phi^{(2)}(\sigma_i^2)} \;=\; \frac{1}{M_{n+1}^{(2)}} \cdot \prod_{i=0}^{n} \frac{1}{\phi^{(2)}(\sigma_i^2)}$$

$$\geq\; \frac{1}{H_3 \prod_{i=0}^{n} \bar{\phi}_i^{(2)}(\sigma_i^2)} \;\geq\; \frac{1}{H_3 \prod_{i=0}^{n} (1+\sigma_i^2 \exp(k^2)\eta_i)}. \tag{3.18}$$

Exploiting the elementary relation $1+u \leq e^u$ provides us the estimate

$$\prod_{i=0}^{n} (1+\sigma_i^2 \exp(k^2)\eta_i) \;\leq\; \exp\left(\exp(k^2) \sum_{i=0}^{n} \sigma_i^2 \eta_i\right). \tag{3.19}$$

Combining (3.18) and (3.19) leads to

$$\prod_{i=0}^{n} \exp(-\sigma_i^2 \xi_{i+1}^2) \;\geq\; \frac{1}{H_3} \exp\left(-\exp(k^2) \sum_{i=0}^{n} \sigma_i^2 \eta_i\right) \tag{3.20}$$

which can be applied to estimate the 2nd part of exponential at the right hand side of (3.15). Next, we are going to estimate its 1st part. For this purpose, let $\phi_i^{(1)} : \mathbb{R}^1 \to +\mathbb{R}^1$ be the moment generating function for the random variable $-\xi_{i+1}$, i.e. $\phi_i^{(1)}(t) = \mathbb{E}[\exp(-t\xi_{i+1})]$ for $t \in \mathbb{R}^1$. Then, by the expansion of e^u into its 2nd order Taylor series $e^u = 1 + u + u^2 e^\theta/2$ where the intermediate value θ satisfies $0 \leq |\theta| < |u|$, we have

$$\phi_i^{(1)}(t) \;=\; \mathbb{E}[\exp(-t\xi_{i+1})] = \mathbb{E}\left[1 - t\xi_{i+1} + \frac{t^2\xi_{i+1}^2 \exp(\zeta_i)}{2}\right] = \mathbb{E}\left[1 + \frac{t^2\xi_{i+1}^2 \exp(\zeta_i)}{2}\right]$$

where $|\zeta_i| \leq k$ and $\exp(k) \;\geq\; \exp(\zeta_i) \;\geq\; \exp(-k)$. Using these observations we can estimate

$$\phi_i^{(1)}(t) \;=\; 1 + \frac{1}{2} t^2 \mathbb{E}[\xi_{i+1}^2 \exp(\zeta_i))] \leq 1 + \frac{1}{2} t^2 \exp(k) \mathbb{E}[\xi_{i+1}]^2 = 1 + \frac{1}{2} t^2 \exp(k) \eta_i$$

and, in particular for $t = \pm\sigma_i$, we have

$$1 \;\leq\; \phi_i^{(1)}(\pm\sigma_i) \;\leq\; 1 + \frac{1}{2}\sigma_i^2 \exp(k)\eta_i.$$

Similarly as before, $M^{(1)} = (M_n^{(1)})_{n\in\mathbb{N}}$ started at $M_0 = 1$ and satisfying

$$M_n^{(1)} \;=\; \prod_{i=0}^{n} \frac{\exp(\sigma_i \xi_{i+1})}{\phi_i^{(1)}(-\sigma_i)}$$

forms a nonnegative martingale with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Then, by MCT (see [5], [23], [29], [30], [46]), and the fact that $\phi_i^{(1)}(t) = \phi_i^{(1)}(-t) \geq 1$ for all $i \in \mathbb{N}$ and all $t \in \mathbb{R}^1$, the exponential martingale $M^{(1)}$ converges $\mathbb{P}$-a.s. to $M_{+\infty}^{(1)}$ with $\mathbb{E}[M_{+\infty}^{(1)}] = 1$.

Therefore, we can estimate $M^{(1)}$ from below as follows. There is some $\mathbb{P}$-a.s. finite random variable $H_4 = H_4(\omega) > 0$ such that, for all $n \in \mathbb{N}$, we have

$$\left(M_{n+1}^{(1)}\right)^{-1} = \prod_{i=0}^{n} \exp\left(-\sigma_i \xi_{i+1}\right) \phi_i^{(1)}(-\sigma_i)$$

$$= \prod_{i=0}^{n} \frac{\exp\left(-\sigma_i \xi_{i+1}\right)}{\phi_i^{(1)}(\sigma_i)} \cdot \prod_{l=0}^{n} \phi_l^{(1)}(\sigma_l)\phi_l^{(1)}(-\sigma_l) \le H_4 \cdot \exp\left(\exp(k) \sum_{i=0}^{n} \sigma_i^2 \eta_i\right) < +\infty.$$

This fact leads to

$$M_{n+1}^{(1)} \ \ge \ \frac{1}{H_4} \cdot \exp\left(-\exp(k) \sum_{i=0}^{n} \sigma_i^2 \eta_i\right) \ > \ 0. \tag{3.21}$$

Applying the latter estimate implies that

$$\prod_{i=0}^{n} \exp(\sigma_i \xi_{i+1}) \ = \ \prod_{i=0}^{n} \left(\frac{\exp(\sigma_i \xi_{i+1})}{\phi_i^{(1)}(-\sigma_i)} \phi_i^{(1)}(-\sigma_i) \right) \ = \ M_{n+1}^{(1)} \cdot \prod_{i=0}^{n} \phi_i^{(1)}(-\sigma_i)$$

$$\ge \ \frac{1}{H_4} \cdot \exp\left(-\exp(k) \sum_{i=0}^{n} \sigma_i^2 \eta_i\right) \ > \ 0 \tag{3.22}$$

for all $n \in \mathbb{N}$. Now, return to the estimation (3.15). Suppose that $|X_0| > 0$. Then, after plugging estimates (3.20) and (3.22) into (3.15), we arrive at

$$|X_{n+1}| \ \ge \ |X_0| \cdot \exp\left(\sum_{i=0}^{n} \sigma_i \xi_{i+1} - l(k) \sum_{i=0}^{n} (\sigma_i \xi_{i+1})^2\right)$$

$$\ge \ \frac{|X_0|}{H_4 H_3^{l(k)}} \cdot \exp\left(-(\exp(k) + l(k) \exp(k^2)) \sum_{i=0}^{n} \sigma_i^2 \eta_i\right) \ > \ 0 \tag{3.23}$$

for all $n \in \mathbb{N}$. Recall that we have supposed that $\lim_{n\to+\infty} X_n = 0$ for all X_0 ($\mathbb{P}$-a.s.). Note also that the nonrandom constant $\exp(k) + l(k)\exp(k^2) > 0$ is strictly positive. Therefore, from (3.23), we may conclude that the nonrandom sum $\sum_{i=0}^{n} \sigma_i^2 \eta_i$ has to tend to $+\infty$ as $n \to +\infty$. Consequently, Theorem 3.1 is proved. □

Remark 3.2. The proof of Theorem 3.1 can be simplified using the standard Central Limit and Monotone Convergence Theorems (see [46]) in conjunction with properties of the ln function, based on the independence of $(\xi_{n+1})_{n\in\mathbb{N}}$. However, for the sake of transparency of all succeeding proofs, we prefer to avoid a simplification in this way in view of an impossible extension of such a simplified proof-technique to the case of nonlinear equations with possibly dependent noise terms. Furthermore, some of the above proof-steps are needed for the simplification of proofs and some necessary references below. Moreover, under the hypothesis that all random variables ξ_{n+1} are independent of each other, either the limits $X_\infty = 0$ or $X_\infty > 0$ for $X_0 > 0$ ($\mathbb{P}$-a.s.) by the well-known Kolmogorov 0-1 Law (see [46]).

Remark 3.3. Because of the independence of random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ the martingale (3.1) is of the likelihood ratio type covered by Kakutani's theorem on singularity / equivalence of product measures (see, e.g. Neveu [29], Proposition III-2-6). Suppose that $X_0 \geq 0$ for simplicity. Then this theorem asserts that the limit of such a martingale X is either a.s. positive or a.s. zero, and is positive if and only if

$$\prod_{n=0}^{+\infty} \mathbb{E}\,\sqrt{1+\sigma_n\xi_{n+1}} \;\; > \;\; 0.$$

This would give another set of necessary and sufficient conditions to ensure asymptotic stability, from which the result of Theorem 3.1 could be deduced too.

Remark 3.4. Using estimates (3.11) and (3.23) it is possible to obtain very rough estimates for the exponential decay rate of solutions of equation (3.1). More precisely, for some nonrandom constants $H_1 = H_1(k)$, $H_2 = H_2(k)$, $K_1 = K_1(k)$ and $K_2 = K_2(k)$ satisfying $0 < H_1(k) \leq H_2(k)$ and $0 < K_1(k) \leq K_2(k)$, we find that

$$H_1|X_0|\exp\left(-K_2\sum_{i=0}^{n}\sigma_i^2\eta_i\right) \leq |X_{n+1}| \leq H_2|X_0|\exp\left(-K_1\sum_{i=0}^{n}\sigma_i^2\eta_i\right) \tag{3.24}$$

for all $n \in \mathbb{N}$. In particular, when ξ_i are independent and identically distributed discrete random variables taking on only 2 values 1 and -1 with equal probabilities 1/2, we have $\eta_i = 1$ for all $i \in \mathbb{N}$. Suppose that $\sigma_i = 1/\sqrt{2(i+1)}$, then conditions (3.2) and (3.3) hold with $k = 1/\sqrt{2}$. In this case, as a result of our analysis above, we can even find polynomial-type decay rates as follows. Estimates (3.24) take on the form

$$H_1|X_0|\exp\left(-\frac{K_2}{2}\sum_{i=0}^{n}\frac{1}{i+1}\right) \leq |X_{n+1}| \leq H_2|X_0|\exp\left(-\frac{K_1}{2}\sum_{i=0}^{n}\frac{1}{i+1}\right).$$

Now, apply the estimates $\sum_{i=0}^{n} 1/(i+1) \leq 1+\ln(n+1)$ and $\ln(n+2) < \sum_{i=0}^{n} 1/(i+1)$ which are derived from an application of monotonicity of Riemann sums to the integrals $\ln(n) = \int_1^n (1/x)dx$. Consequently, we get to

$$\frac{H_1|X_0|e^{-K_2/2}}{(n+1)^{K_2/2}} \leq |X_{n+1}| \leq \left(\frac{n+1}{n+2}\right)^{K_1/2} \cdot \frac{H_2|X_0|}{(n+1)^{K_1/2}} \leq \frac{H_2|X_0|}{(n+1)^{K_1/2}} \tag{3.25}$$

with strictly positive constants $K_1 \leq K_2$ and $H_2 \leq H_1$. Therefore, we can even establish polynomial-type decay rates for discrete stochastic difference equations using techniques from the proof of Theorem 3.1.

4. ALMOST SURE STABILITY FOR LINEAR EQUATIONS WITH DRIFT

Consider ordinary stochastic difference equations

$$X_{n+1} \;=\; X_n\Big(1-\alpha_n+\sigma_n\xi_{n+1}\Big) \tag{4.1}$$

started at some initial value X_0 which is independent of $\mathcal{F}_n$ for all $n \in \mathbb{N}$, where α_n and σ_n are not random, $(\xi_{n+1})_{n\in\mathbb{N}}$ are independent random variables with mean $\mathbb{E}[\xi_{n+1}] = 0$ and second moments $\mathbb{E}[\xi_{n+1}]^2 = \eta_n < +\infty$. Assume that there exists some nonrandom constant $k \in (0,1)$ such that, for all $n \in \mathbb{N}$, we have $\mathbb{P}$-a.s.

$$-k \le \sigma_n\xi_{n+1} \le k, \tag{4.2}$$

$$1 - \alpha_n + \sigma_n\xi_{n+1} > 0. \tag{4.3}$$

Furthermore, for the statement of Theorem 4.2 below, we introduce the following conditions.

$$\lim_{n\to+\infty}\left[\sum_{i=0}^{n}\alpha_i + \sum_{i=0}^{n}\sigma_i^2\eta_i\right] = +\infty, \tag{4.4}$$

$$\lim_{n\to+\infty}\sum_{i=0}^{n}\alpha_i = +\infty, \tag{4.5}$$

$$\forall n \in \mathbb{N}\ \alpha_n \ge 0. \tag{4.6}$$

Remark 4.1. Condition (4.4) is fulfilled if one of the conditions (4.5) or (3.3) is satisfied.

Theorem 4.2. *Assume that conditions (4.2), (4.3) and (4.6) are satisfied. Then, condition (4.4) is fulfilled if and only if* $\lim_{n\to+\infty} X_n = 0$ *holds* $\mathbb{P}$*-a.s. for all solutions* $(X_n)_{n\in\mathbb{N}}$ *of equation (4.1).*

Proof. First of all, we represent solutions of (4.1) in the form

$$X_n = X_0\prod_{i=0}^{n}(1-\alpha_i+\sigma_i\xi_{i+1}) = X_0\prod_{i=0}^{n}(1+\sigma_i\xi_{i+1})\prod_{i=0}^{n}\left(1-\frac{\alpha_i}{1+\sigma_i\xi_{i+1}}\right). \tag{4.7}$$

We note that, since condition (4.3) is fulfilled, the second product at the right side of (4.7) is positive. The positivity of the first product at the right side of (4.7) is due to condition (4.2). Moreover, due to condition (4.2) and inequality $1+u \le \exp(u)$, we can estimate the second product at the right side of (4.7) from above as follows.

$$\begin{aligned}\prod_{i=0}^{n}\left(1-\frac{\alpha_i}{1+\sigma_i\xi_{i+1}}\right) &\le \prod_{i=0}^{n}\exp\left(-\frac{\alpha_i}{1+\sigma_i\xi_{i+1}}\right) = \exp\left(-\sum_{i=0}^{n}\frac{\alpha_i}{1+\sigma_i\xi_{i+1}}\right)\\ &\le \exp\left(-\frac{1}{1+k}\sum_{i=0}^{n}\alpha_i\right).\end{aligned} \tag{4.8}$$

Next, we estimate it from below. For this purpose, for $u > -1$, we can expand $\ln(1+u)$ in its Taylor series up to the first order term, namely $\ln(1+u) = u/(1+\theta)$ where the intermediate value θ satisfies $\theta \in (\min(0,u), \max(0,u))$ and $|u| \le k < 1$.

We obtain

$$\begin{aligned}\prod_{i=0}^{n}\left(1-\frac{\alpha_i}{1+\sigma_i\xi_{i+1}}\right) &= \prod_{i=0}^{n}\exp\left(\ln\left(1-\frac{\alpha_i}{1+\sigma_i\xi_{i+1}}\right)\right)\\ &= \prod_{i=0}^{n}\exp\left(-\frac{\alpha_i}{(1+\sigma_i\xi_{i+1})(1+\zeta_i)}\right)\\ &\geq \exp\left(-\frac{1}{(1-k)^2}\sum_{i=0}^{n}\alpha_i\right), \qquad (4.9)\end{aligned}$$

since (4.3) holds and $1/(1+k) \leq 1/(1+\sigma_i\xi_{i+1}) \leq 1/(1-k)$, $|\zeta_i| \leq k$ and therefore $1/(1+k) \leq 1/(1+\zeta_i) \leq 1/(1-k)$.

Suppose that condition (4.4) is fulfilled. As noted in Remark 4.1, this condition can be only satisfied if either (4.5) or (3.3) is fulfilled. First, suppose that condition (4.5) is fulfilled. Using estimate (4.8), the trivial fact that

$$M_{n+1}^{(3)} = \prod_{i=0}^{n}(1+\sigma_i\xi_{i+1}), \quad M_0^{(3)} = 1 \qquad (4.10)$$

forms a nonnegative martingale $M = (M_n^{(3)})_{n\in\mathbb{N}}$ with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ because of $\mathbb{E}\,[1+\sigma_i\xi_{i+1}] = 1$ and standard MCT (see [5], [23], [29], [30], [46]), we find some $\mathbb{P}$-a.s. finite random variable $H_5 = H_5(\omega) > 0$ such that

$$|X_{n+1}| \leq |X_0|H_5\exp\left(-\frac{1}{1+k}\sum_{i=0}^{n}\alpha_i\right). \qquad (4.11)$$

Apparently, under (4.5), X_{n+1} must converge ($\mathbb{P}$-a.s.) to 0 as $n \to +\infty$.

Second, suppose that condition (3.3) is fulfilled. Then, by applying estimates (3.9), (3.10) and (3.11), it follows that, for all integers $n \geq N_1(\omega)$ (where N_1 is chosen in the same way as in Theorem 3.1), we have $\mathbb{P}$-a.s.

$$|X_{n+1}| \leq |X_0|H_1^{1/(2e^k)}\exp\left(-\frac{1}{1+k}\sum_{i=0}^{n}\alpha_i - \frac{\exp(-k^2-2e^k)}{4}\sum_{i=0}^{n}\sigma_i^2\eta_i\right), \qquad (4.12)$$

hence X_{n+1} converges ($\mathbb{P}$-a.s.) to 0 as $n \to +\infty$ too.

Now, the backwards conclusion. Suppose that $\lim_{n\to+\infty} X_n = 0$ ($\mathbb{P}$-a.s.). Acting as in the last part of the proof of Theorem 3.1, we can arrive at similar estimates from below as in (3.23). However, for this purpose, we need to estimate from below the nonnegative product martingale $M^{(3)} = (M_n^{(3)})_{n\in\mathbb{N}}$ defined as in (4.10). We may apply the 2nd order Taylor expansion $1/(1+u) = 1-u+u^2/(1+\theta)^3$ where θ is an imtermediate value satisfying $\theta \in [-k,k]$. Thus, thanks to MCT (see [5], [23], [29],

[30], [46]), we can estimate the nonnegative inverse of $M^{(3)}$ by

$$\left(M_{n+1}^{(3)}\right)^{-1} = \prod_{i=0}^{n} \frac{1}{1+\sigma_i\xi_{i+1}} = \prod_{i=0}^{n} \frac{\frac{1}{1+\sigma_i\xi_{i+1}}}{\mathbb{E}\left[\frac{1}{1+\sigma_i\xi_{i+1}}\right]} \cdot \prod_{i=0}^{n} \mathbb{E}\left[\frac{1}{1+\sigma_i\xi_{i+1}}\right]$$
$$\leq H_6 \cdot \exp\left(\frac{1}{(1-k)^3}\sum_{i=0}^{n}\sigma_i^2\eta_i\right)$$

where $H_6 = H_6(\omega) > 0$ is a $\mathbb{P}$-a.s. finite random variable on Ω. Consequently, we find that

$$M_{n+1}^{(3)} \geq \frac{1}{H_6} \cdot \exp\left(-\frac{1}{(1-k)^3}\sum_{i=0}^{n}\sigma_i^2\eta_i\right) \tag{4.13}$$

for all $n \in \mathbb{N}$. Finally, apply (4.9) and (4.13) to representation (4.7) in order to get to

$$|X_{n+1}| \geq |X_0|\frac{1}{H_6}\exp\left(-\frac{1}{(1-k)^2}\sum_{i=0}^{n}\alpha_i - \frac{1}{(1-k)^3}\sum_{i=0}^{n}\sigma_i^2\eta_i\right).$$

Recall that $0 < k < 1$. Thus, (4.14) means that at least one of the expressions $\sum_{i=0}^{n}\alpha_i$ or $\sum_{i=0}^{n}\sigma_i^2\eta_i$ has to tend to $+\infty$ as $n \to +\infty$ in order to have $X_n \to 0$ for all X_0 ($\mathbb{P}$-a.s.) as $n \to +\infty$. Hence, the proof of Theorem 4.2 is complete. □

Remark 4.3. It is not difficult to recognize that Theorem 4.2 remains valid if, instead of conditions (4.6), we just require that, for some nonrandom constant $K > 0$, we have

$$\forall n \in \mathbb{N} : \sum_{i=0}^{n}\alpha_i > -K > -\infty. \tag{4.14}$$

Remark 4.4. Instead of representation (4.7) we can also consider the splitting

$$X_{n+1} = X_0\prod_{i=0}^{n}\frac{1-\alpha_i+\sigma_i\xi_{i+1}}{1-\alpha_i}\cdot\prod_{i=0}^{n}(1-\alpha_i) = X_0\prod_{i=0}^{n}\left(1+\frac{\sigma_i\xi_{i+1}}{1-\alpha_i}\right)\cdot\prod_{i=0}^{n}(1-\alpha_i).$$

We note that

$$M_{n+1} = \prod_{i=0}^{n}\left(1+\frac{\sigma_i\xi_{i+1}}{1-\alpha_i}\right) \tag{4.15}$$

started at $M_0 = 1$ forms a martingale with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ since

$$\mathbb{E}\left[1+\frac{\sigma_i\xi_{i+1}}{1-\alpha_i}\right] = 1$$

for all $i \in \mathbb{N}$. Then, instead of conditions (4.2) and (4.3), we require that, for some nonrandom constant $k \in (0,1)$ with $k^2\exp(k) < 2$,

$$\forall i \in \mathbb{N} : -k \leq \frac{\sigma_i\xi_{i+1}}{1-\alpha_i} \leq k. \tag{4.16}$$

holds. Apparently, this condition guarantees the strict positivity of the product martingale $(M_n)_{n\in\mathbb{N}}$ following (4.15) and MCT can be applied as before, hence we achieve more freedom for the choice of parameters α_n.

5. ALMOST SURE STABILITY FOR NONLINEAR EQUATIONS WITH TRIVIAL SOLUTION

Consider ordinary nonlinear stochastic difference equations

$$X_{n+1} = X_n\Big(1-\alpha_n f(X_n)+\sigma_n g(X_n)\xi_{n+1}\Big) \tag{5.1}$$

started at some initial value X_0 which is independent of $\mathcal{F}_n$ for all $n\in\mathbb{N}$, where $(\xi_{n+1})_{n\in\mathbb{N}}$ are independent random variables with $\mathbb{E}\,[\xi_{i+1}]=0$ and $\mathbb{E}\,[\xi_{i+1}]^2=\eta_i$. Let $f:\mathbb{R}^1\to\mathbb{R}^1_+$ and $g:\mathbb{R}^1\to\mathbb{R}^1$ be two continuous functions such that

$$\forall\varepsilon>0\ \exists H(\varepsilon)>0\ :\quad \inf_{|x|>\varepsilon} f(x)\ge H(\varepsilon)\quad\text{and}\quad \inf_{|x|>\varepsilon}|g(x)|\ge H(\varepsilon), \tag{5.2}$$

$$\forall x\in\mathbb{R}^1\ :\quad 0\ \le\ f(x),|g(x)|\ \le\ 1. \tag{5.3}$$

Suppose also that

$$\forall x\in\mathbb{R}^1\ \forall i\in\mathbb{N}\ :\quad 1-\alpha_i f(x)+\sigma_i\xi_{i+1}g(x)\ >\ 0. \tag{5.4}$$

Remark 5.1. Assumption (5.4) is fulfilled if, in addition to conditions (5.3) and (4.2), the condition

$$\forall i\in\mathbb{N}\ :\quad 0\ \le\ \alpha_i\ \le\ 1-k$$

holds. Indeed, we have

$$\forall u\in\mathbb{R}^1\ \forall i\in\mathbb{N}\ :\quad \alpha_i f(u)\ \le\ \alpha_i\ \le\ 1-k\ \le\ 1+\sigma_i\xi_{i+1}\ \le\ 1+\sigma_i\xi_{i+1}g(u).$$

Theorem 5.2. *Assume that conditions (4.2), (4.3), (4.6), (5.2), (5.3) and (5.4) are satisfied. Then, condition (4.4) is fulfilled if and only if the limit* $\lim_{n\to+\infty}X_n=0$ *holds* $\mathbb{P}$*-a.s. for all solutions* $(X_n)_{n\in\mathbb{N}}$ *of equation (5.1).*

Proof. We have a representation for the solution X_n of equation (5.1)

$$\begin{aligned} X_{n+1} &= X_0\cdot\prod_{i=0}^{n}(1-\alpha_i f(X_i)+\sigma_i g(X_i)\xi_{i+1})\\ &= X_0\cdot\prod_{i=0}^{n}(1+\sigma_i g(X_i)\xi_{i+1})\cdot\prod_{i=0}^{n}\left(1-\frac{\alpha_i f(X_i)}{1+\sigma_i g(X_i)\xi_{i+1}}\right) \end{aligned} \tag{5.5}$$

From (5.5) we can easily conclude that the limit $\lim_{n\to\infty}X_n$ exists. Indeed, the first product $M_{n+1}=\prod_{i=0}^{n}(1+\sigma_i g(X_i)\xi_{i+1})$ forms a nonnegative martingale $M=(M_n)_{n\in\mathbb{N}}$ started at $M_0=1$ with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ since $\mathbb{E}\,[1+\sigma_i g(X_i)\xi_{i+1}]=1$, it has a finite limit as $n\to+\infty$ (thanks MCT) and is uniformly bounded with respect to $n \in \mathbb{N}$. The second product $\prod_{i=0}^{n}(1-\alpha_i f(X_i)/(1+\sigma_i g(X_i)\xi_{i+1}))$ decreases monotonically due to nonnegativity of $\alpha_i f(X_i)$ and $1+\sigma_i g(X_i)\xi_{i+1}$.

To show that $\lim_{n\to+\infty} X_n = 0$, we use an indirect proof. Suppose that

$$\lim_{n\to+\infty} |X_n(\omega)| \;=\; c(\omega) > 0 \tag{5.6}$$

for $\omega \in \Omega_1$ with positive probability $\mathbb{P}\{\Omega_1\} = \beta_1 > 0$. Therefore, there exists an integer $N = N(\omega)$ such that we can put

$$\Omega_1 \;=\; \left\{\omega \in \Omega : |X_n(\omega)| \geq \frac{c(\omega)}{2} > 0, \forall n > N(\omega)\right\} \;\subseteq\; \Omega. \tag{5.7}$$

We also note that, from condition (5.2) with $\varepsilon = c(\omega)/2$, we can find an expression $H\left(c(\omega)/2\right)$ such that, for $n \geq N_1(\omega)$, the estimates

$$f(x_n) \geq H\left(\frac{c(\omega)}{2}\right), \quad g^2(x_n) \geq H^2\left(\frac{c(\omega)}{2}\right) \tag{5.8}$$

hold. For all $n \geq N_1(\omega)$, $\omega \in \Omega_1$, this leads to

$$\left\{\begin{array}{rcl} \sum_{i=0}^{n} \alpha_i f(X_i) & \geq & H\left(\frac{c(\omega)}{2}\right)\cdot \sum_{i=0}^{n} \alpha_i, \\ \sum_{i=0}^{n} \sigma_i^2 g^2(X_i)\eta_i & \geq & H^2\left(\frac{c(\omega)}{2}\right)\cdot \sum_{i=0}^{n} \sigma_i^2\eta_i \end{array}\right. . \tag{5.9}$$

Now, we distinguish between two cases. First, suppose that, for $\omega \in \Omega_1$, we have divergence of the series $\sum_{i=0}^{+\infty}\sigma_i^2\eta_i$ from condition (3.3) and, second, its convergence. In the second case, it means that condition (4.5) is fulfilled.

In the first case, while using the representation (5.5) and estimates (5.9) and acting as in the proofs of Theorems 3.1 and 4.2 (e.g. see 4.12), we obtain that, for all $n \geq N_1(\omega)$, $\omega \in \Omega_1$,

$$\begin{aligned} |X_n| \;&\leq\; |X_0| H_1^{\frac{1}{2e^k}} \exp\left(-\frac{H\left(\frac{c(\omega)}{2}\right)}{1+k}\sum_{i=0}^{n}\alpha_i\right)\cdot \exp\left(-\rho(k)H^2\left(\frac{c(\omega)}{2}\right)\sum_{i=0}^{n}\sigma_i^2\eta_i\right) \\ &\leq\; |X_0| H_7 \exp\left(-\frac{\exp(-k^2-2e^k)}{4}H^2\left(\frac{c(\omega)}{2}\right)\sum_{i=0}^{n}\sigma_i^2\eta_i\right) \end{aligned} \tag{5.10}$$

where $\rho(k) = (\exp(-k^2-2e^k))/4$ and $H_7 \leq H_1^{(1/(2e^k))}$, hence $X_n \to 0$ ($\mathbb{P}$-a.s.) as $n \to +\infty$. Similarly, the second case yields that

$$|X_n| \;\leq\; |X_0| H_8 \exp\left(-\frac{H\left(\frac{c(\omega)}{2}\right)}{1+k}\sum_{i=0}^{n}\alpha_i\right) \tag{5.11}$$

with finite random variable $H_8 \leq H_1^{(1/(2e^k)}$, hence $X_n \to 0$ ($\mathbb{P}$-a.s.) as $n \to +\infty$. Thus, we arrived at contradictions in both cases.

It remains to prove necessity of condition (4.4) for $\lim_{n\to+\infty} X_n = 0$ ($\mathbb{P}$-a.s.). Suppose that $\lim_{n\to+\infty} X_n = 0$ ($\mathbb{P}$-a.s.). Then, due to the continuity of functions f and g, there exist $\mathbb{P}$-a.s. finite random variables $H_9(\omega) > 0$ and $N_3(\omega)$ on Ω such that

$$0 \le f(X_n(\omega)) \le H_9(\omega), \quad 0 \le g^2(X_n(\omega)) \le H_9^2(\omega) \tag{5.12}$$

hold for $n \ge N_3(\omega)$.

Acting as in the last part of proof of Theorem 4.2, combining the observations above with the estimates (4.9) and (4.13) implies that

$$|X_n| \ \ge \ |X_0| \frac{1}{H_6} \exp\left(-\frac{H_9(\omega)}{(1-k)^2} \sum_{i=0}^{n} \alpha_i - \frac{H_9^2(\omega)}{(1-k)^3} \sum_{i=0}^{n} \sigma_i^2 \eta_i \right). \tag{5.13}$$

This also means that at least one of the expressions $\sum_{i=0}^{n} \alpha_i$ or $\sum_{i=0}^{n} \sigma_i^2 \eta_i$ has to tend to $+\infty$ as $n \to +\infty$. This completes the proof of Theorem 5.2. □

Remark 5.3. Consider the retarded delay-type stochastic difference equation

$$X_{n+1} = X_n(1 - \alpha_n f(X_n, X_{n-1}, \ldots, X_{n-l}) + \sigma_n g(X_n, X_{n-1}, \ldots, X_{n-l})\xi_{n+1} \tag{5.14}$$

with nonrandom initial data $X_0, X_{-1}, \ldots, X_{-l}$ (or at least independent of $\mathcal{F}_n$ for all $n \in \mathbb{N}$), some nonrandom constant delay-length parameter $l \in \mathbb{N}$, nonrandom continuous functions $f, g : \mathbb{R}^{l+1} \to \mathbb{R}^1$ such that

$$\forall \varepsilon > 0\, \exists H(\varepsilon) > 0 \ : \ \inf_{\|v\|_{\mathbb{R}^{l+1}} > \varepsilon} f(v) \ge H(\varepsilon), \ \inf_{\|v\|_{\mathbb{R}^{l+1}} > \varepsilon} |g(v)| \ge H(\varepsilon), \tag{5.15}$$

$$\forall v \in \mathbb{R}^{l+1} \ : \ 0 \le f(v), |g(v)| \le 1. \tag{5.16}$$

Then, the conclusion of Theorem 5.2 remains valid for equation (5.14) under conditions (5.15) and (5.16), instead of (5.2) and (5.3).

Indeed, in the proof of Theorem 5.2, when we choose number N_1 for expression (5.8), we take $N^* = N_1 + l$ instead of N_1. Then, instead of condition (5.8), we have that, for all $n \ge \max\{N^*, N\}$, the estimates

$$f(X_n, X_{n-1}, \ldots, X_{n-l}) \ge H\left(\frac{c(\omega)}{2}\right), \ g^2(X_n, X_{n-1}, \ldots, X_{n-l}) \ge H^2\left(\frac{c(\omega)}{2}\right).$$

hold. Analogously, instead of estimates (5.12), we arrive at

$$\begin{cases} 0 \le f(X_n, X_{n-1}, \ldots, X_{n-l}) \le H_9(\omega), \\ 0 \le g^2(X_n, X_{n-1}, \ldots, X_{n-l}) \le H_9^2(\omega) \end{cases} \tag{5.17}$$

for $n \ge N_3 + \delta$. Acting as in the proofs above, we find that, together with $|X_n| \to 0$ ($\mathbb{P}$-a.s.), the expression $\max\{|X_n|, |X_{n-1}|, \ldots, |X_{n-l}|\}$ must converge to 0 ($\mathbb{P}$-a.s.).

Remark 5.4. For equations with polynomial-type coefficients, we can immediately derive estimates of the decay rate of solutions of equation (5.1). More precisely, let

$\mu_1 > 0$ and $\mu_2 > 0$ be real constants such that

$$\lim_{u \to 0} \frac{f(u)}{|u|^{\mu_1}} = c_1 > 0 \quad \text{and} \quad \alpha_n = \left(\frac{1}{n+1}\right)^{\beta_1}, \beta_1 \geq 0, \tag{5.18}$$

$$\lim_{u \to 0} \frac{g^2(u)}{|u|^{\mu_2}} = c_2 > 0 \quad \text{and} \quad \sigma_n^2 = \left(\frac{1}{n+1}\right)^{\beta_2}, \beta_2 \geq 0, \tag{5.19}$$

and $|\xi_{n+1}| \leq k < 1$. Assume that conditions (5.2), (5.3), (5.4) are fulfilled. Then, for all exponents γ satisfying $\gamma > \max\{\frac{1-\beta_1}{\mu_1}, \frac{1-\beta_2}{\mu_2}\} > 0$, we may conclude that

$$\limsup_{n \to +\infty} (|X_n| n^{\gamma}) \;=\; +\infty. \tag{5.20}$$

For the proof, we only note that all conditions of Theorem 5.2 are fulfilled. Therefore, we must have $\lim_{n\to+\infty} X_n = 0$ ($\mathbb{P}$-a.s.).

Suppose that, for some exponent $\gamma_0 > \max\{\frac{1-\beta_1}{\mu_1}, \frac{1-\beta_2}{\mu_2}\}$, conclusion (5.20) is not true. Because of $\gamma_0 > \frac{1-\beta_1}{\mu_1}$ there exist ($\mathbb{P}$-a.s.) finite numbers $H_{10} = H_{10}(\omega) > 0$ and $\varepsilon_{10} > 0$ such that

$$0 \;\leq\; \limsup_{n \to +\infty} \frac{|X_n|}{n^{\frac{-1-\varepsilon_{10}+\beta_1}{\mu_1}}} \;\leq\; H_{10} \tag{5.21}$$

(i.e. one may take $\varepsilon_{10} = \mu_1\gamma_0 - 1 + \beta_1 > 0$). Because of $\lim_{n\to+\infty} X_n = 0$ and Abel's series test, expression (5.21) together with the condition (5.18) imply that there exists ($\mathbb{P}$-a.s.) a finite random variable $N_{10} = N_{10}(\omega) > 0$ such that

$$\begin{aligned} \sum_{i=N_{10}}^{+\infty} \alpha_i f(X_i) \;&\leq\; (H_{10})^{\mu_1} c_1 \sum_{i=N_{10}}^{+\infty} (i+1)^{-\beta_1} (i+1)^{-1-\varepsilon_{10}+\beta_1} \\ &=\; H_{11} \sum_{i=N_{10}}^{+\infty} (i+1)^{-1-\varepsilon_{10}} \;<\; +\infty \end{aligned} \tag{5.22}$$

with finite random variable $H_{11} = (H_{10})^{\mu_1} c_1$. On the other hand, when $\gamma_0 > \frac{1-\beta_2}{\mu_2}$, there exists ($\mathbb{P}$-a.s.) finite numbers $H_{12} = H_{12}(\omega) > 0$ and $\varepsilon_{12} > 0$ such that

$$\limsup_{n \to +\infty} \frac{|X_n|}{n^{\frac{-1-\varepsilon_{12}+\beta_2}{\mu_2}}} \;\leq\; H_{12} \tag{5.23}$$

(take $\varepsilon_{12} = \mu_2\gamma_0 - 1 + \beta_2 > 0$). Because of $\lim_{n\to+\infty} X_n = 0$ and Abel's series test, the expression (5.23) together with condition (5.19) imply that there exists (a.s.) finite number $N_{12} = N_{12}(\omega) > 0$ such that

$$\begin{aligned} \sum_{i=N_{12}}^{+\infty} \sigma_i^2 g^2(X_i) \;&\leq\; (H_{12})^{\mu_1} c_2 \sum_{i=N_{12}}^{+\infty} (i+1)^{-\beta_2} i^{-1-\varepsilon_{12}+\beta_2} \\ &=\; H_{13} \sum_{i=N_{12}}^{+\infty} (i+1)^{-1-\varepsilon_{12}} \;<\; +\infty \end{aligned} \tag{5.24}$$

where $H_{13} = (H_{12})^{\mu_1} c_2$ is a finite random variable. Finally, from representation (5.5), estimates (5.22) and (5.24), and using Lemma 2.4, we find some $\mathbb{P}$-a.s. finite random variables $H_{14}(k) = H_{14}(k)(\omega)$, $H_{15}(k) = H_{15}(k)(\omega)$, $K_2(k) = K_2(k)(\omega)$ and $K_3(k) = K_3(k)(\omega) > 0$ such that

$$\begin{aligned} |X_n| &\geq |X_0| H_{15}(k) \exp\left(-K_2(k)\sum_{i=0}^{n}\alpha_i - K_3(k)\sum_{i=0}^{n}\sigma_i^2\eta_i\right) \\ &\geq |X_0| H_{15}(k) \exp\left(-K_2(k)\sum_{i=0}^{n}(i+1)^{-1-\varepsilon_{10}} - K_3(k)\sum_{i=0}^{n}(i+1)^{-1-\varepsilon_{12}}\right). \end{aligned}$$

Using this estimate from below, we can conclude that the limit $\lim_{n\to+\infty} X_n$ cannot be equal to 0 - a fact which contradicts to the already established conclusion from Theorem 4.2.

Example. Fix nonrandom real constants $p_1, p_2, \alpha_0, |\sigma_0|, \beta_1, \varepsilon > 0$ and $\beta_2 \geq 0$. Consider the nonlinear stochastic difference equation

$$X_{n+1} = X_n - \alpha_0 \frac{X_n |X_n|^{p_1}}{(n+1)^{1+\varepsilon}(1+|X_n|^{p_1})} + \sigma_0 \frac{X_n |X_n|^{p_2} \xi_{n+1}}{(n+1)^{\beta_2}(1+|X_n|^{p_2})} \tag{5.25}$$

driven by square-integrable independent random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ with $|\xi_{n+1}| \leq 1$ for all $n \in \mathbb{N}$ and $\inf_{i\in\mathbb{N}} \mathbb{E}\,[\xi_{i+1}]^2 > 0$. Now, we may take $\mu_1 = p_1$, $\mu_2 = 2p_2$, $c_1 = \alpha_0$, $c_2 = \sigma_0^2$, $\beta_1 = 1+\varepsilon$. Suppose that $0 \leq \beta_2 < 1/2$ and $0 < |\sigma_0| < 1$. Then, from the previous remark 5.4, it follows that, for all $\varepsilon > 0$ and all exponents $\gamma > (1-2\beta_2)/(2p_2)$, we have ($\mathbb{P}$-a.s.) that $\limsup_{n\to+\infty}(|X_n| n^\gamma) = +\infty$. As a matter of fact, if $0 < \beta_2 \leq 1/2$ then we do not need to require for $\lim_{n\to+\infty} X_n = 0$ ($\mathbb{P}$-a.s.) that $|\sigma_0| < 1$ (because of $|\sigma_0|(n+1)^{-\beta_2} \to 0$ as $n \to +\infty$ here). However, the requirements $|\sigma_0| > 0$ and $\sum_{i=0}^{+\infty} \mathbb{E}\,[\xi_{i+1}]^2/(i+1)^{2\beta_2} = +\infty$ are essential for a.s. asymptotic stability in above example. Interestingly, this example explains how noise can asymptotically stabilize ($\mathbb{P}$-a.s.) the dynamics of its solution $X = (X_n)_{n\in\mathbb{N}}$, in contrast to the related non-asymptotically stable deterministic subclass with $\sigma_0 = 0$ and small $0 \leq \alpha_0 < 1$ for all $\varepsilon > 0$. Note that, when $\sigma_0 = 0$ and $0 \leq \alpha_0 < 1$, the monotonicity relation $|X_n| \geq |X_{n+1}|$ for $(X_n)_{n\in\mathbb{N}}$ governed by (5.25) and the estimation

$$\begin{aligned} |X_{n+1}| &= |X_n| \cdot \left(1 - \alpha_0 \frac{|X_n|^{p_1}}{(n+1)^{1+\varepsilon}(1+|X_n|^{p_1})}\right) \\ &\geq |X_n| \left(1 - \alpha_0 \frac{|X_0|^{p_1}}{(n+1)^{1+\varepsilon}(1+|X_0|^{p_1})}\right) \geq |X_0| \prod_{i=0}^{n}\left(1 - \frac{\alpha_0}{(i+1)^{1+\varepsilon}}\right) \\ &= |X_0| \exp\left(\sum_{i=0}^{n} \ln\left[1 - \frac{\alpha_0}{(i+1)^{1+\varepsilon}}\right]\right) \geq |X_0| \exp\left(-\frac{\alpha_0}{1-\alpha_0}\sum_{i=0}^{n}\frac{1}{(i+1)^{1+\varepsilon}}\right) \\ &\geq |X_0| \exp\left(-\frac{\alpha_0}{1-\alpha_0}\sum_{i=0}^{+\infty}\frac{1}{(i+1)^{1+\varepsilon}}\right) \end{aligned}$$

can be established, hence $\lim_{n\to+\infty} |X_n| > 0$ for all $|X_0| > 0$ ($\mathbb{P}$-a.s.) - a fact which obviously follows from taking the logarithm in the above estimate and applying the

Taylor expansion to $\ln(1-u) = -u/(1-\theta) > -|u|/(1-\alpha_0)$ for $|u| \leq \alpha_0 < 1$ afterwards. Furthermore, in the asymptotically stable case, an increasing parameter β_2 of its diffusion term ranging between $0 < \beta_2 < 1/2$ decreases the polynomial decay rate γ. The same is true for increasing exponents p_2.

6. ON SOME RELAXATION OF CONDITION (3.2)

The conditions (3.2) of uniformly bounded noise turn out to be too restrictive in some cases. Let us relax these conditions a little. For this purpose, recall that a nonnegative random variable $N : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{N}, \mathcal{B}(\mathbb{N}))$ is called a *Markov moment* if the following events satisfy $\{N = n\} \in \mathcal{F}_n$ (or equivalently $\{n < N\} \in \mathcal{F}_n$) and are independent of $\sigma(X_0)$ for all $n \in \mathbb{N}$, where $\mathcal{F}_n = \sigma(\xi_0, \xi_1, \ldots, \xi_n)$ is the naturally underlying filtration belonging to the square-integrable sequence of independent random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ with the same moment properties as in previous sections. Now, suppose that, instead of condition (3.2) we require that there exist finite nonrandom numbers $k \in (0,1)$ and $c^* \in \mathbb{R}^1$, and a $\mathbb{P}$-a.s. finite Markov moment $N = N(\omega)$ such that, for all $\omega \in \Omega$, we have

$$\forall i \in \mathbb{N} : \quad |\sigma_i \xi_{i+1}(\omega)| \leq c^*, \tag{6.1}$$

$$\forall i > N(\omega) : \quad |\sigma_i \xi_{i+1}(\omega)| \leq k < 1, \tag{6.2}$$

$$\sum_{i=0}^{\infty} p_i < +\infty, \quad \text{where} \quad p_i = \mathbb{P}(\{\omega \in \Omega : i < N(\omega)\}). \tag{6.3}$$

We already know that a solution of equation (4.1) exists for all $n \in \mathbb{N}$. For all $\omega \in \Omega$ and $n > N(\omega)$, we can represent solutions of (4.1) in the form

$$X_{n+1} = X_N \cdot \prod_{i=N}^{n} (1 - \alpha_i + \sigma_i \xi_{i+1}) = X_N \cdot \prod_{i=N}^{n} (1 + \sigma_i \xi_{i+1}) \cdot \prod_{i=N}^{n} \Big(1 - \frac{\alpha_i}{1 + \sigma_i \xi_{i+1}}\Big). \tag{6.4}$$

Define

$$\bar{\xi}_{i+1}(\omega) := \frac{\sigma_i \xi_{i+1}(\omega)}{(i+2)^2 |\sigma_i \xi_{i+1}(\omega)|}$$

for all $i \in \mathbb{N}$ on Ω. Let $\zeta_{i+1}^N(\omega) = \bar{\xi}_{i+1}(\omega)$ if $i < N(\omega)$ and $\zeta_{i+1}^N(\omega) = \sigma_i \xi_{i+1}(\omega)$ if $i \geq N(\omega)$. Hence, the decomposition

$$\zeta_{i+1}^N(\omega) = \bar{\xi}_{i+1}(\omega) I_{\{\omega : i < N(\omega)\}} + \sigma_i \xi_{i+1}(\omega) I_{\{\omega : i \geq N(\omega)\}}$$

holds for all $i \in \mathbb{N}$, where I_S denotes the indicator functions of the subscribed random set S. Set $\zeta_0^N = 0$. Then, $(\zeta_n^N)_{n\in\mathbb{N}}$ is a sequence of $(\mathcal{F}_n, \mathcal{B}(\mathbb{R}^1))$-measurable independent random variables satisfying

$$-1 < -\max\{1/(i+2)^2, k\} \leq \zeta_{i+1}^N(\omega) \leq \max\{1/(i+2)^2, k\} < 1. \tag{6.5}$$

for all $i \in \mathbb{N}$. Hence, the two-sided estimate of (6.5) leads to

$$\sup_{i\in\mathbb{N}} |\kappa_{i+1}| \;<\; 1 \tag{6.6}$$

where $\kappa_{i+1} = \mathbb{E}\,[\zeta^N_{i+1}]$. Because of $\mathbb{E}\,[\sigma_i\xi_{i+1}] = 0$ one can estimate

$$\begin{aligned}
\kappa_{i+1} \;&=\; \mathbb{E}\,[\zeta^N_{i+1}] \;=\; \int_{\{\omega\in\Omega: i<N(\omega)\}} \bar{\xi}_{i+1}\, d\mathbb{P}(\omega) + \int_{\{\omega\in\Omega: i\geq N(\omega)\}} \xi_{i+1}\, d\mathbb{P}(\omega)\\
&=\; \int_{\{\omega\in\Omega: i<N(\omega)\}} \bar{\xi}_{i+1}\, d\mathbb{P}(\omega) - \int_{\{\omega\in\Omega: i<N(\omega)\}} \sigma_i\xi_{i+1}\, d\mathbb{P}(\omega) + \int_{\Omega} \sigma_i\xi_{i+1}\, d\mathbb{P}(\omega)\\
&\leq\; \int_{\{\omega\in\Omega: i<N(\omega)\}} \frac{d\mathbb{P}(\omega)}{(i+2)^2} + \int_{\{\omega\in\Omega: i<N(\omega)\}} c^*\, d\mathbb{P}(\omega) \;=\; p_i\left(\frac{1}{(i+2)^2} + c^*\right). \qquad (6.7)
\end{aligned}$$

From exploiting the $(\mathcal{F}_{i+1}, \mathcal{B}([-1,1]))$-measurability of random variables ζ^N_{i+1} due to its construction based on the Markov moment N, and estimates (6.5) and (6.6), we may conclude that

$$M^N_{n+1} \;=\; \prod_{i=0}^{n} \frac{1+\zeta^N_{i+1}}{1+\kappa_{i+1}} \tag{6.8}$$

started at $M^N_0 = 1.0$ forms a nonnegative martingale with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Moreover, by applying Lemma 2.4 (or Lemma 2.5 similarly), we estimate

$$\begin{aligned}
\prod_{i=0}^{n=N-1} (1+\bar{\xi}_{i+1}) \;&\leq\; \exp\left(\sum_{i=0}^{+\infty} \bar{\xi}_{i+1}\right) \leq \exp\left(\sum_{i=0}^{+\infty} \frac{1}{(i+2)^2}\right) = H_0 < +\infty\\
\prod_{i=0}^{n=N-1} (1+\bar{\xi}_{i+1}) \;&\geq\; \exp\left(\sum_{i=0}^{+\infty} \bar{\xi}_{i+1} - \frac{1}{2(1-k)^2}\sum_{i=0}^{+\infty} \bar{\xi}^2_{i+1}\right)\\
&\geq\; \exp\left(-\sum_{i=0}^{+\infty} \frac{1}{(i+2)^2} - \frac{1}{2(1-k)^2}\sum_{i=0}^{+\infty} \frac{1}{(i+2)^4}\right) = \bar{H}_0 > 0, \qquad (6.9)
\end{aligned}$$

and

$$\begin{aligned}
\prod_{i=0}^{n=N-1} (1+\kappa_{i+1}) &\leq \exp\left(\sum_{i=0}^{+\infty} \kappa_{i+1}\right) \leq \exp\left(\sum_{i=0}^{+\infty} p_i\left(\frac{1}{(i+2)^2} + c^*\right)\right) = H_1 < +\infty,\\
\prod_{i=0}^{n=N-1} (1+\kappa_{i+1}) &\geq \exp\left(\sum_{i=0}^{+\infty} \kappa_{i+1} - \frac{1}{2(1-k)^2}\sum_{i=0}^{+\infty} \kappa^2_{i+1}\right)\\
&\geq \exp\left(-\sum_{i=0}^{+\infty} p_i\left(\frac{1}{(i+2)^2} + c^*\right) - \frac{1}{2(1-k)^2}\sum_{i=0}^{+\infty} p_i^2\left(\frac{1}{(i+2)^2} + c^*\right)^2\right) = \bar{H}_1 > 0, \qquad (6.10)
\end{aligned}$$

since the series $\sum_{i=0}^{+\infty} p_i^2$ satisfying $0 \leq \sum_{i=0}^{+\infty} p_i^2 \leq \sum_{i=0}^{+\infty} p_i < +\infty$ converges too.

Now, while returning to representation (6.4) of X^N and using estimates (6.9) and (6.10), one can carry out a martingale-based approach similar to the ones used in the proofs of Theorems 3.1, 4.2 and 5.2. This leads to the following result for equation (4.1).

Theorem 6.1. *Assume that conditions (4.3), (4.6) (or (4.14)), (6.1), (6.2) and (6.3) hold. Then, condition (4.4) is fulfilled if and only if the limit* $\lim_{n\to+\infty} X_n = 0$ *holds* $\mathbb{P}$*-a.s. for all solutions* $X = (X_n)_{n\in\mathbb{N}}$ *of equation (4.1).*

In a similar way we can prove an analogous result for equation (5.1).

Theorem 6.2. *Assume that conditions (4.6), (5.2), (5.3), (5.4), (6.1), (6.2) and (6.3) hold. Then, condition (4.4) is fulfilled if and only if the limit* $\lim_{n\to+\infty} X_n = 0$ *holds* $\mathbb{P}$*-a.s. for all solutions* X_n *of equation (5.1).*

7. LOCAL ALMOST SURE ASYMPTOTIC STABILITY

Conditions (5.3) and (5.4) seem to be rather restrictive in some nonlinear situations. However, this kind of boundedness can be relaxed in view of local asymptotic stability as follows (but this is not necessarily true for global stability). Let the nonlinear equations (5.1) be driven by the square-integrable sequence of independent random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ with $\mathbb{E}[\xi_{n+1}] = 0$ and $\eta_n = \mathbb{E}[\xi_{n+1}]^2$ for $n \in \mathbb{N}$. Moreover, the functions $f, g : \mathbb{R}^1 \to \mathbb{R}^1$ are supposed to be continuous and $f(x) \geq 0$ for all $x \in \mathbb{R}^1$. Furthermore, in the statement of Theorem 7.2 below, we also refer to the set of conditions that the nonrandom real constants k, K_α, K_σ, K_ξ and $\varepsilon > 0$ can be chosen such that, for all $n \in \mathbb{N}$,

$$0 \le \alpha_n \le K_\alpha, \quad |\sigma_n| \le K_\sigma, \quad |\xi_n| \le K_\xi, \tag{7.1}$$

$$-k \le \frac{\sigma_n \xi_{n+1}}{K_\sigma K_\xi} \le k, k \in (0,1), \quad 1 - \frac{\alpha_n}{K_\alpha} + \frac{\sigma_n \xi_{n+1}}{K_\sigma K_\xi} > 0, \tag{7.2}$$

$$\forall x \in \mathbb{R} \text{ with } |x| \le \varepsilon : \quad K_\alpha f(x) + K_\sigma K_\xi |g(x)| < 1, \tag{7.3}$$

$$\sup_{x\in\mathbb{R}:|x|\le\varepsilon} \frac{|g(x)|}{f(x)} \cdot K_\xi \le \inf_{n\in\mathbb{N}} \frac{\alpha_n}{|\sigma_n|}. \tag{7.4}$$

Remark 7.1. Condition (7.4) can be substituted by the requirement ($\mathbb{P}$-a.s.)

$$\exists \varepsilon > 0 : \sup_{n\in\mathbb{N}, x\in\mathbb{R}:|x|\le\varepsilon} [-\alpha_n f(x) + g(x)\sigma_n \xi_{n+1}] \le 0,$$

and then the conclusion of Theorem 7.2 below remains still valid. One recognizes from this relation that the divergence of $\sum_{n=0}^{+\infty} \alpha_n = +\infty$ is essential in order to obtain local a.s. asymptotic stability, and that the case $\sum_{n=0}^{+\infty} \sigma_n^2 \eta_n = +\infty$ is not so important for the local asymptotic stability as established by Theorem 7.2. This fact is due to the a.s. monotonicity relation which we exploit in the proof below.

Theorem 7.2 (Local Asymptotic Stability)**.** *Assume that the conditions (5.2), (7.1) – (7.4) are satisfied with positive constants* K_α*,* K_σ*,* K_ξ *and* $\varepsilon > 0$*. Then, condition (4.4) is fulfilled if and only if the limit* $\lim_{n\to+\infty} X_n = 0$ *holds* $\mathbb{P}$*-a.s. for all solutions* X_n *of equation (5.1) with initial values* X_0 *bounded by* $|X_0| \le \varepsilon$*.*

Proof. First of all, equation (5.1) is equivalently rewritten as

$$X_{n+1} \;=\; X_n\Big(1-\hat{\alpha}_n\hat{f}(X_n)+\hat{\sigma}_n\hat{g}(X_n)\hat{\xi}_{n+1}\Big),\ \text{where} \tag{7.5}$$

$$\hat{\alpha}_n=\frac{\alpha_n}{K_\alpha},\ \hat{\sigma}_n=\frac{\sigma_n}{K_\sigma},\ \hat{\xi}_{n+1}=\frac{\xi_{n+1}}{K_\xi},\ \hat{f}(x)=K_\alpha f(x),\ \hat{g}(x)=K_\sigma K_\xi g(x) \tag{7.6}$$

for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$. Therefore, the splitting of equation (7.5) with new coefficients (7.6) satisfies all assumptions of Theorem 5.2 for small values of $|X_0|\le\varepsilon$. This fact is supported by the monotonicity relation

$$|X_{n+1}| \;\le\; |X_n| \;\le\; ... \;\le\; |X_0| \;\le\; \varepsilon$$

by induction on $n\in\mathbb{N}$, provided that $|X_0|\le\varepsilon$ ($\mathbb{P}$-a.s.) with $\varepsilon>0$ resulting from (7.3) and (7.4). To see the origin of this relation more clearly, take $|X_0|\le\varepsilon$ and estimate

$$|X_{n+1}| \;\;=\;\; |X_n|\Big|1-\alpha_n f(X_n)+g(X_n)\sigma_n\xi_{n+1}\Big| \;\le\; |X_n|$$

since, due to (7.3) and (7.4), we find that

$$\begin{aligned}-1 \;&\le\; -K_\alpha f(X_n)-K_\sigma K_\xi|g(X_n)| \;\le\; -\alpha_n f(X_n)+|g(X_n)||\sigma_n\xi_{n+1}|\\ &\le\; -\alpha_n f(X_n)+|g(X_n)||\sigma_n|K_\xi \;\le\; 0\end{aligned}$$

whenever $|X_n|\le\varepsilon$. Now, we may repeat the same proof-procedure as for Theorem 5.2 while restricting to initial values X_0 with $|X_0|\le\varepsilon$. Hence, the equivalence-statement of Theorem 7.2 is verified. □

Remark 7.3. If f,g are continuous and $f(0)=g(0)=0$ then the condition (7.3) is trivially satisfied for sufficiently small $\varepsilon>0$, and hence not restrictive.

Example. Fix nonrandom real constants $c_1>0$, $c_2\neq 0$, $p\ge 0$, $r_2\ge r_1\ge 0$ and $\gamma>0$. Consider

$$X_{n+1} \;\;=\;\; X_n\left(1-\frac{c_1}{(n+1)^{r_1}}|X_n|^p+\frac{c_2}{(n+1)^{r_2}}|X_n|^{p+\gamma}\xi_{n+1}\right)$$

driven by independent random variables $(\xi_{n+1})_{n\in\mathbb{N}}$ with $\mathbb{E}\left[\xi_{n+1}\right]=0$ and $|\xi_{n+1}|\le 1$ for all $n\in\mathbb{N}$. Suppose that $r_1\ge 0$ is not greater than 1 (hence, the series $\sum_{i=0}^n\alpha_i$ diverges). Then, due to Theorem 7.2, there exists $\varepsilon>0$ satisfying (7.3) - (7.4) such that the trivial solution $X\equiv 0$ of equation (7.7) is locally a.s. asymptotically stable for all initial values $|X_0|\le\varepsilon$. To verify this statement, take $\alpha_n=c_1/(n+1)^{r_1}$, $f(x)=|x|^p$, $\sigma_n=c_2/(n+1)^{r_2}$, $g(x)=|x|^{p+\gamma}$, $K_\alpha=c_1$, $K_\sigma=|c_2|$, $K_\xi=1$ and choose $\varepsilon\le\min(1,(c_1/|c_2|)^{1/\gamma},1/(c_1+|c_2|)^{1/p})$ such that

$$\begin{aligned}0 \;<\; \sup_{u\in\mathbb{R}:|u|\le\varepsilon}\frac{|g(u)|}{f(u)}\cdot K_\xi \;=\; \varepsilon^\gamma \;\le\; \inf_{n\in\mathbb{N}}\frac{\alpha_n}{|\sigma_n|} \;=\; \frac{c_1}{|c_2|}\inf_{n\in\mathbb{N}}(n+1)^{r_2-r_1} \;=\; \frac{c_1}{|c_2|}\\ \text{and}\;\; K_\alpha f(u)+K_\sigma K_\xi|g(u)| \;=\; c_1|u|^p+|c_2||u|^{p+\gamma} \;<\; (c_1+|c_2|)\varepsilon^p \;<\; 1\end{aligned}$$

are ensured (cf. requirements (7.3) and (7.4)).

Remark 7.4. It is possible to relax the uniform boundedness assumptions (3.2) on noise terms $\sigma_n\xi_{n+1}$ or to equations (3.1) driven by more general martingale-differences $(\xi_{n+1})_{n\in\mathbb{N}}$ further. Another way to relaxation of (3.2) can be the application of techniques as described in Schurz [42] which are aiming at an introduction of additional terms stabilizing the original nonlinear equation (5.1). However, details of studying such relaxations of conditions are beyond the scope of this paper.

Remark 7.5. There are possible extensions of martingale approach to systems of stochastic difference equations (see [45]) and or systems of stochastic numerical methods (for an application to drift-implicit stochastic Theta methods, see [44]) in any dimension. Even systems of delay equations with memory effects can be studied by our approach (cf. [39]). The martingale approach applied to asymptotic stability of continuous stochastic differential equations is also exploited in [2]. This paper represents just an attempt to nonlinear discrete stochastic difference equations using the martingale approach. So, one can reckon that the martingale approach to clarify and verify a.s. asymptotic stability is a very powerful tool for the qualitative study of stochastic dynamical systems.

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