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ON ZEROS OF POLYNOMIALS AND INFINITE SERIES WITH SOME BOUNDS

P. N. SHIVAKUMAR[1], YANG ZHANG[1], AND ASHISH GUPTA[2]

[1]Department of Mathematics, University of Manitoba
Winnipeg, MB, R3T 2N2, Canada
[2]Department of Mathematics, RMU Univesity, West Bengal, India

ABSTRACT. In this paper, we consider a given infinite series in x of the form $y(x) = \sum_{k=0}^{\infty} b_k x^k$ expressed formally also by an infinite product as $y(x) = \Pi_{k=1}^{\infty}(1 - \frac{x}{a_k})$ into real positive zeros $a_i, i = 1, 2, \ldots, \infty$ forming a strictly increasing sequence. For consideration of polynomials of degree n, we replace suitably ∞ by n.

Using the known formal solution of a second linear differential $y" = f(x)y, y(0) = y_0, y'(0) = y'_0$ in the form $y(x) = \sum_{k=0}^{\infty} d_k x^k$, we demonstrate that the above infinite product form of $y(x)$ yields the set of infinite equations of the form for a suitable $f(x)$.

$\sum_{k=1}^{\infty}(a_k)^{-p} = c_p$, $p = 1, 2, \ldots, \infty$ with c'_ks depending on $f(x)$, its derivarives at $x = 0$ and b'_ks. Recognizing the infinite matrix as the infinite Vandermonde matrix, some bounds for the zeros are given.

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1. INTRODUCTION

We will be concerned with the zeros a_i, $i = 1, 2, \ldots, \infty$ of the convergent series $y(x)$ given by

$$y(x) = \sum_{k=0}^{\infty} b_k x^k \tag{1.1}$$

also formally expressed as

$$y(x) = \Pi_{k=1}^{\infty}\left(1 - \frac{x}{a_k}\right) \tag{1.2}$$

where the zeros a_k of $y(x)$ are real, positive and

$$a_{i+1} > a_i > 0, i = 1, 2, \ldots, \infty. \tag{1.3}$$

Emails: pappur.shivakumar@umanitoba.ca; yang.zhang@umanitoba.ca.
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Clearly,

$$y(0) = b_0 = 1,\ y'(0) = b_1 = -\sum_{k=1}^{\infty} \frac{1}{a_k}. \tag{1.4}$$

We note that for a polynomial of degree n, we replace suitably ∞ by n. A large number of classical differential equation yield infinite series solutions with the above mentioned properties of the zeros. An example is the Bessel's function, particularly of the first kind. Our plan is to use an earlier result [3] which gives the formal implicit series solution of the differential system.

$$y'' = f(x)y,\ y(0) = y_0,\ y'(0) = y_0' \tag{1.5}$$

in the form

$$y(x) = \sum_{k=0}^{\infty} \alpha_k x^k \tag{1.6}$$

where α_k's depends entirely on $f(x)$ and its derivatives at $x = 0$.

In section 2, we give the details of the coefficients α_k in (1.6) in terms of $f(x)$ and its derivatives. In section 3, we consider the case when

$$f(x) = \left[\sum_{k=1}^{\infty} \frac{1}{a_k - x}\right]^2 - \sum_{k=1}^{\infty} \left(\frac{1}{a_k - x}\right)^2 \tag{1.7}$$

where it is shown that (see Appendix) using (1.7) for $f(x)$ and using (1.2), (1.5) is satisfied.

Now we can equate (1.1) with the resulting $\alpha_k's$ on using (1.7) for $f(x)$ yielding

$$\alpha_k = b_k,\ k = 0, 1, 2, \ldots, \infty. \tag{1.8}$$

Also in section 3, we give expressions for derivatives of $f(x)$. In section 4, we demonstrate that (1.8) reduces to

$$\sum_{k=1}^{\infty} a_k^{-p} \alpha_k = c_p,\ p = 1, 2, \ldots, \infty.$$

with c_p is depending on $f(x)$, its derivatives at $x = 0$ and b_k's.

2. THE SOLUTIONS OF (1.5)

From [3], we have the following theorem and a lemma concerning the formal solution of (1.5).

Theorem 2.1. *For the second order linear differential system (1.5), the formal solution is given by, with $a = 0$,*

$$
\begin{aligned}
y(x) &= \textstyle\sum_{k=0}^{\infty} \alpha_k x^k \\
&= y(0) + xy'(0) \\
&\quad + \textstyle\sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+2)!} \left[\sum_{s_1=0}^{l} \sum_{j=1}^{k+1} P_j(s_1, 2k, 0) \right] x^{2k+2} \right\} \qquad (2.1)\\
&\quad + \textstyle\sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+3)!} \left[\sum_{s_1=0}^{l} \sum_{j=1}^{k} P_j(s_1, 2k+1, 0) \right] x^{2k+3} \right\} \\
&= b_0 + b_1 x + \textstyle\sum_{k=0}^{\infty} b_{2k+2} x^{2k+2} + \sum_{k=0}^{\infty} b_{2k+3} x^{k+3}
\end{aligned}
$$

using (1.6) and where

(2.2)

$$
P_q(s_1, k, 0) =
$$

$$
\textstyle\sum_{s_2=s_1}^{k-2(q-1)} \cdots \sum_{s_q=s_{q-1}}^{k-2(q-1)} \left[\binom{k}{s_q+2(q-1)} \binom{s_q+2(q-2)}{s_{q-1}+2(q-2)} \cdots \binom{s_2}{s_1} f^{(k-2(q-1)-s_q)}(0) f^{(s_{q-1}-s_{q-2})}(0) \right.
$$

$$
\left. \cdots f^{(s_2-s_1)}(0) y^{(s_1)}(0) \right].
$$

The following lemma establishes the recurvese relation between P_q and P_{q-1}.

Lemma 2.2.

$$
P_q(s_q, k, 0) = \sum_{s_2=s}^{k-2(q-1)} f^{k-s_1-2(q-1)}(0) P_{q-1}(s_1, s_2 + 2(q-2), 0)
$$

For demonstration purpose, we give below first few terms of $y(x)$ as solutions of (1.6).

$$
\begin{aligned}
y(x) &= y(0) + xy'(0) + \tfrac{x^2}{2!} y''(0) + \tfrac{x^3}{3!} y^{(3)}(0) + \tfrac{x^4}{4!} y^{(4)}(0) + \tfrac{x^5}{5!} y^{(5)}(0) + \cdots \\
&= y(0) + xy'(0) + \tfrac{x^2}{2!} f(0)y(0) + \tfrac{x^3}{3!} [f'(0)y(0) + f(0)y'(0)] \\
&\quad + \tfrac{x^4}{4!} [y(0)(f''(0) + f^2(0)] + 2y'(0)f'(0)] \qquad (2.3)\\
&\quad + \tfrac{x^5}{5!} [y(0)(f^{(3)}(0) + 4f(0)f'(0) + y'(0)(3f''(0) + f^2(0))] + \cdots \\
&= \textstyle\sum_{k=0}^{\infty} b_k x^k
\end{aligned}
$$

It is noted that in evaluation of b_k, the highest power of the derivatives of $f(x)$ at $x = 0$ is $k - 2$. In summary, we have the infinite numbers of equations given by $k = 0, 1, 2, \ldots, \infty$ with

$$
b_0 = 1, \ b_1 = -\sum_{k=1}^{\infty} \frac{1}{a_k}
$$

and

$$b_{2k+2} = \sum_{s_1=0}^{1} \sum_{j=1}^{k+1} P_j(s_1, 2k, 0), \quad b_{2k+3} = \sum_{s_1=0}^{1} \sum_{j=1}^{k} P_j(s_1, 2k+1, 0),$$

where P_q's are given by (1.12).

Again, we note that b_k evaluation involves $f^{(k-2)}(0)$ which in turn gives an expression of Q_k.

3. PROPERTIES OF $f(x)$

In this section, we develop expressions for $f(x)$ and its derivatives at $x = 0$, where from (1.7), we have

$$f(x) = \left(\sum_{k=1}^{\infty} \frac{1}{\alpha_k - x} \right)^2 - \sum_{k=1}^{\infty} \frac{1}{(\alpha_k - x)^2}. \tag{3.1}$$

Using the identity

$$\sum_{k=1}^{\infty} g_k^2 = (\sum_{k=1}^{\infty} g_k)^2 - \sum_{k=1}^{\infty} \sum_{j=1, j \neq k}^{\infty} g_k g_j. \tag{3.2}$$

We can rewrite $f(x)$ as

(3.3)

$$f(x) = \sum_{k=1}^{\infty} \sum_{j=1, j \neq k}^{\infty} \left\{ \frac{1}{(\alpha_k - x)} \frac{1}{(\alpha_j - x)} \right\} = \sum_{k=1}^{\infty} \sum_{j=1, j \neq k}^{\infty} \left\{ \left[\frac{1}{(\alpha_k - x)} - \frac{1}{(\alpha_j - x)} \right] \frac{1}{(\alpha_j - \alpha_k)} \right\}.$$

For convenience, we will use the notation

$$Q_p = \sum_{k=1}^{\infty} z_k^p, \quad p = 1, 2, 3, \ldots, \infty, Z_k = \frac{1}{\alpha_k} \tag{3.4}$$

which yields on using (3.1)

$$f(0) = Q_1^2 - Q_2 \tag{3.5}$$

and from (3.4)

$$Q_1 = -\sum_{k=1}^{\infty} \frac{1}{\alpha_k} = -\sum_{k=1}^{\infty} Z_k = -b_1 \tag{3.6}$$

Differenting $f(x)$ in (3.3) p times we get

$$f^{(p)}(x) = p! \sum_{k=1}^{\infty} \sum_{j=1, j \neq k}^{\infty} \left[\frac{1}{(\alpha_k - x)^{p+1}} - \frac{1}{(a_j - x)^{p+1}} \right] \frac{1}{a_j - a_k}.$$

yielding on using (3.4)

(3.7)

$$
\begin{aligned}
f^{(p)} &= p!\sum_{k=1}^{\infty}\sum_{j=1,j\neq k}^{\infty}\left[\frac{1}{\alpha_k^{p+1}}-\frac{1}{a_j^{p+1}}\right]\frac{1}{a_j-a_k}\\
&= p!\sum_{k=1}^{\infty}\sum_{j=1,j\neq k}^{\infty}\left[\frac{\alpha_j^{p+1}-\alpha_j^{p+1}}{a_k^{p+1}a_j^{p+1}}\right]\frac{1}{a_j-a_k}\\
&= p!\sum_{k=1}^{\infty}\sum_{j=1,j\neq k}^{\infty}\frac{\alpha_j^{p}+\alpha_j^{p-1}\alpha_k+\cdots+\alpha_j\alpha_k^{p-1}+\alpha_k^{p}}{\alpha_j^{p+1}\alpha_k^{p+1}}\\
&= p!\sum_{k=1}^{\infty}\sum_{j=1,j\neq k}^{\infty}[Z_k^p+Z_k^{p-1}Z_j+\cdots+Z_kZ_j^{p-1}+Z_j^p]Z_kZ_j\\
&= p!\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}[Z_k^p+Z_k^{p-1}Z_j+\cdots+Z_kZ_j^{p-1}+Z_j^p-(p+1)Z^{p+2}]Z_kZ_j\\
&= p![Q_1Q_{p+1}+Q_2Q_p+\cdots+Q_{p+1}-(p+1)Q_{p+2}]\\
&= p!\left\{\sum_{k=1}^{p+1}[Q_kQ_{p+2-k}]-(p+1)Q_{p+2}\right\}
\end{aligned}
$$

4. CASE $n = 3$

In this section, we demonstrated the theory to apply to (1.8) using (1.13), (3.6) and (3.7). We have

$$b_0 = y(0) = 1.$$

(4.1) $$b_1 = y'(0) = -Q_1 \text{ yielding } Q_1 = -b_1$$

$b_2 = f(0)y(0) = Q_1^2 - Q_2$ yields

(4.2) $$Q_2 = Q_1^2 - b_2 = b_1^2 - b_2$$

(4.3)

$$
\begin{aligned}
b_3 &= \tfrac{1}{3!}[f'(0)y(0)_f(0)y'(0)]\\
&= \tfrac{1}{3!}[Q_1Q_2+Q_2^2-2Q_3-Q_1(Q_1^2-Q_1)]\\
&= \tfrac{1}{3!}[2Q_1Q_2+Q_2^2-Q_1^2-2Q_3]
\end{aligned}
$$

yielding

(4.4)

$$
\begin{aligned}
Q_3 &= \tfrac{3!}{2}[2Q_1Q_2+Q_2^2-Q_1^3-b_3]\\
&= \tfrac{3!}{2}[-2b_1(b_1^2-b_3)+(b_1^2-b_2)^2+b_1^3-b_3]\\
&= \tfrac{3!}{2}[-2b_1^3+2b_2b_2+b_1^4+b_2^3-2b_1^2b_2^3+b_1^3-b_3]\\
&= 3[-b_1^3+2b_1b_2+b_1^4+b_2^2-2b_1^2b_2^2-b_3]
\end{aligned}
$$

Equations (4.1)-(4.4) illustrate (1.9) when ∞ is replaced by 3.

Now we will show that

$$y(x) = \Pi_{k=1}^{\infty}(1 - \frac{x}{\alpha_k})$$

satisfies

$$y''(x) = y(x)f(x)$$

where $f(x)$ us given by (1.7). We have

$$y'(x) = \Pi_{k=1}^{\infty}(1 - \frac{x}{\alpha_k})\sum_{k=1}^{\infty}(-\frac{1}{\alpha_k})(\frac{1}{1-\frac{x}{\alpha}}) = -y(x)\sum_{k=1}^{\infty}\frac{1}{\alpha_k - x}$$

giving

$$y''(x) = y(x)\sum_{k=1}^{\infty}\frac{1}{(\alpha_k - x)^2} - y'(x)\sum_{k=1}^{\infty}\frac{1}{\alpha_k - x}$$

Substituting for $y'(x)$, we get

$$y''(x) = y(x)\left\{\sum_{k=1}^{\infty}\frac{1}{(\alpha_k - x)^2} + \left[\sum_{k=1}^{\infty}\frac{1}{\alpha_k - x}\right]^2\right\} = y(x)f(x)$$

5. BOUBDS FOR THE ROOTS OF POLYNOMIAL VANDERMONDE SYSTEM

Our aim in this section is to give suitable bounds for real variables $0 < x_1 < x_2 < \cdots < x_m$ satisfying

$$\begin{aligned} x_1 + x_2 + \cdots + x_m &= c_1, \\ x_1^2 + x_2^2 + \cdots + x_m^2 &= c_2, \\ \cdots \quad \cdots & \quad \cdots \\ x_1^m + x_2^m \cdots + x_m^m &= c_m. \end{aligned}$$

Let $f(Y) \in \mathbb{R}[Y]$ be the monic polynomial of degree m whose roots are the numbers x_i $(i = 1, \cdots, m)$, that is,

$$f(Y) = \prod_{i=1}^{m}(Y - x_i) = \sum_{j=0}^{m} s_j Y^{m-j}. \tag{5.1}$$

By virtue of the well known Newton's identities the coefficients s_j are multivariate polynomial functions $s_j = s_j(c_1, \cdots, c_m)$ of the c_j. For example, if $m = 3$ then it is well-known that $3s_3 = \frac{1}{2}c_1^3 - \frac{3}{2}c_1c_2 + c_3$.

We recall [4] that the (quadratic) norm $\mathrm{N}(g)$ of a polynomial $g(Y) = \sum_{j=0}^{d} a_j Y^{d-j} \in \mathbb{R}[Y]$ of degree d is defined as

$$\mathrm{N}(g) = \sqrt{\sum_{j=0}^{d} a_j{}^2}.$$

By [4, Proposition 2.7.1] the minimal distance $\mathrm{Sep}(g)$ between the roots of g is given by:

$$\mathrm{Sep}(g) > d^{-\frac{d+2}{2}}|\Delta|^{\frac{1}{2}}\mathrm{N}(g)^{1-d}, \tag{5.2}$$

where $\Delta := \Delta(g)$ stands for the discriminant of the polynomial g and may be found as

$$\Delta = (-1)^{\frac{d(d-1)}{2}}\mathrm{Res}(g, g').$$

where g' stands for the formal derivative of g and $\mathrm{Res}(f, f')$ denotes the resultant of the polynomials f and f'. Recall that $\mathrm{Res}(f, f')$ is defined as the determinant of a matrix defined in terms of the coefficients of the polynomials f and f'.

Applying (5.2) to the polynomial f above we thus obtain:

$$\mathrm{Sep}(f) > m^{-\frac{m+2}{2}}|\mathrm{Res}(f, f')|^{\frac{1}{2}}\mathrm{N}(f)^{1-m}.$$

Note that all quantities involved in the RHS of the preceding inequality can be expressed in terms of the coefficients s_j of f and hence in terms of the c_i. By the bound of Cauchy [5, Theorem 1.1.2] we have

$$x_n < \rho := 1 + \max\{|s_j|\}.$$

The following bounds for x_i are now immediate from the discussion above.

$$\begin{aligned}
&x_1 < \rho - (m-1)\mathrm{Sep}(f),\\
&x_2 < \rho - (m-2)\mathrm{Sep}(f),\\
&\ldots\ldots\\
&x_i < \rho - (m-i)\mathrm{Sep}(f),\\
&\ldots\ldots\\
&x_n < \rho.
\end{aligned}$$

REFERENCES

[1] P.N. Shivakumar, K.C. Sivakumar and Yang Zhang, *Infinite Matrices and Their Recent Applications*, Springer, 2016.

[2] P.N. Shivakumar, Y. Wu and Yang Zhang, *Shape of a drum, a constructive approach*, WSEAS Trans. Math. 10, 21–31, 2011.

[3] P.N. Shivakumar and Yang Zhang, *On series solutions of* $y'' = f(x)y$ *and applications*, Advances in Differential equations, 2013:47, Springer.

[4] M. Mignotte, and D. Stefanescu, *Polynomials: An Algorithmic Approach*, Springer 1999.

[5] V.V. Prasolov, *Polynomials*, Springer-Verlag, 2010.