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PHASE PORTRAITS OF THE COMPLEX RICCATI EQUATION WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we characterize the phase portrait of the complex Riccati quadratic polynomial differential systems

$$\frac{dz}{dt} = \dot{z} = a(z-b)(z-c),$$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$ with $a \neq 0$ and $t \in \mathbb{R}$. Taking $z = x + iy$, and writing the Riccati equation as the differential system $(\dot{x}, \dot{y})$ in the plane, we give the complete description of their phase portraits in the Poincaré disk (i.e. in the compactification of $\mathbb{R}^2$ adding the circle $\mathbb{S}^1$ of the infinity) modulo topological equivalence.

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1. Introduction and statement of the main results

Numerous problems of applied mathematics are modeled by quadratic polynomial differential systems. Excluding linear systems, such systems are the ones with the lowest degree of complexity, and the large bibliography on the subject proves its relevance. We refer for example to the books of Ye Yanqian et al. [16], Reyn [14], and Artes, Llibre, Schlomiuk, Vulpe [1], and the surveys of Coppel [6], and Chicone and Jinghuang [5] are excellent introductory readings to the quadratic polynomial differential systems.

Among such quadratic polynomial differential systems we emphasize the Riccati polynomial systems. Since their introduction at the end of the seventeenth century with the crucial work of Jacobo F. Riccati based on the variable separable technique, they have been studied intensively by renown mathematicians with numerous

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approaches underlying all the facets of their richness. Their field of applications is widespread ranging from continued fractions to some useful applications in control system theory, see for instance [9, 12, 13, 15].

In this paper we characterize the phase portraits of the complex Riccati differential equation

$$\dot{z} = a(z-b)(z-c) \tag{1.1}$$

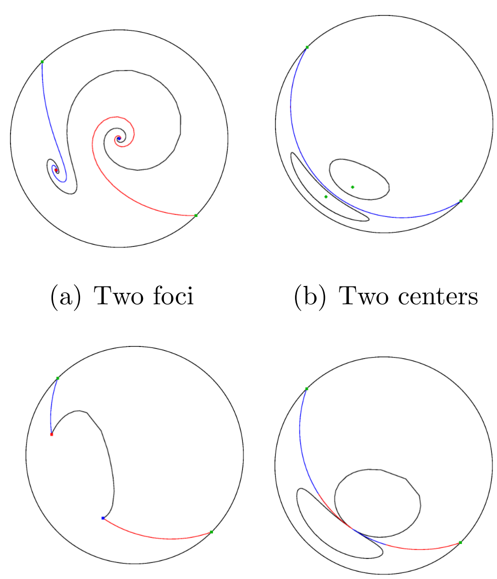
with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$ with $a \neq 0$ and the dot means derivative with respect to $t \in \mathbb{R}$. The Riccati equation is a particular complex differential equation in one variable. A good reference on relevant general results on complex differential equations in one variable see [3].

We write

$$z = x + iy, \quad a = a_1 + ia_2, \quad b = b_1 + ib_2, \quad c = c_1 + ic_2$$

with $x, y \in \mathbb{R}$ and $a_i, b_i, c_i \in \mathbb{R}$ for $i = 1, 2$ so that $(a_1, a_2) \neq (0, 0)$. System (1.1) becomes the real system

$$\begin{aligned}
\dot{x} &= a_1b_1c_1 - a_2b_2c_1 - a_2b_1c_2 - a_1b_2c_2 - (a_1b_1 - a_2b_2 + a_1c_1 - a_2c_2)x \\
&\quad + (a_2b_1 + a_1b_2 + a_2c_1 + a_1c_2)y + a_1x^2 - 2a_2xy - a_1y^2, \\
\dot{y} &= a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2 - a_2b_2c_2 - (a_2b_1 + a_1b_2 + a_2c_1 + a_1c_2)x \\
&\quad - (a_1b_1 - a_2b_2 + a_1c_1 - a_2c_2)y + a_2x^2 + 2a_1xy - a_2y^2.
\end{aligned} \tag{1.2}$$



(a) Two foci (b) Two centers

(c) Two nodes (d) Equilibrium with two elliptic sectors

FIGURE 1. The four different topological phase portraits of the complex Riccati equation with constant coefficients in the Poincaré disc.

The objective of this work is to classify the phase portraits of the quadratic polynomial differential systems (1.2) in the Poincaré disk modulo topological equivalence. As any polynomial differential system, system (1.2) can be extended to an analytic system on a closed disk of radius one, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity of $\mathbb{R}^2$. This closed disk is denoted by $\mathbb{D}^2$ and called the *Poincaré disk*, because the technique for doing such an extension is precisely the *Poincaré compactification* for polynomial differential systems in $\mathbb{R}^2$, which is described in details in chapter 5 of [8]. In this paper we shall use the notation of that chapter. By using this compactification technique the dynamics of system (1.2) in a neighborhood of the infinity can be studied and we have the following result.

Theorem 1.1. *The phase portraits of the complex Riccati equation* (1.1) *in the Poincaré disk are topologically equivalent to one of the four phase portraits presented in Figure 1.*

2. **Infinite equilibrium points**

For a complete description of the Poincaré compactification method we refer to chapter 5 of [8]. In what follows we remember some formulas.

Consider the polynomial differential system in $\mathbb{R}^2$ with degree 2

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{2.1}$$

or equivalently its associated polynomial vector field $X = (P, Q)$. As we said before, any polynomial differential system can be extended to an analytic differential system on a closed disk of radius one centered at their origin of coordinates, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity. We consider four open charts covering the disk $\mathbb{D}^2$:

$$\phi_1 : \mathbb{R}^2 \longrightarrow U_1, \quad \phi_1(x, y) = (1/v, u/v),$$

$$\phi_2 : \mathbb{R}^2 \longrightarrow U_2, \quad \phi_1(x, y) = (u/v, 1/v)$$

and

$$\psi_k : \mathbb{R}^2 \longrightarrow V_k, \quad \psi_k(x, y) = -\phi_k(x, y), \quad k = 1, 2$$

with

$$U_1 = \{(u, v) \in \mathbb{D}^2 : u > 0\}, \quad U_2 = \{(u, v) \in \mathbb{D}^2 : v > 0\},$$

$$V_1 = \{(u, v) \in \mathbb{D}^2 : u < 0\}, \quad V_2 = \{(u, v) \in \mathbb{D}^2 : v < 0\}.$$

The extended vector field of X from $\mathbb{R}^2$ to $\mathbb{D}^2$, i.e. the Poincaré compactification is denoted by $p(X)$. The expression of $p(X)$ in the chart U_1 is

$$\dot{u} = v^2(-uP + Q), \quad \dot{v} = -v^3 P, \tag{2.2}$$

where P and Q are evaluated at $(1/v, u/v)$.

The expression of $p(X)$ in the chart U_2 is

$$\dot{u} = v^2(P - uQ), \quad \dot{v} = -v^3 Q, \tag{2.3}$$

where P and Q are evaluated at $(u/v, 1/v)$. Moreover in all these local charts the points (u, v) of the infinity have its coordinate $v = 0$.

The expression for the extend differential system in the local chart V_i, $i = 1, 2$ is the same as in U_i multiplied by -1.

The following result summarizes the information at infinity.

Lemma 2.1. *The complex Riccati equation* (1.1) *at infinity has a unique pair of infinite equilibrium points which are saddles.*

Proof. First we analyze the phase portrait in the local chart U_1. The expression of the system in this chart is

$$\begin{aligned}
\dot{u} &= a_2 + a_1 u - (a_2 b_1 + a_1 b_2 + a_2 c_1 + a_1 c_2) v + a_2 u^2 \\
&\quad + (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2 - a_2 b_2 c_2) v^2 + a_1 u^3 \\
&\quad - (a_2 b_1 + a_1 b_2 + a_2 c_1 + a_1 c_2) u^2 v - (a_1 b_1 c_1 - a_2 b_2 c_1 - a_2 b_1 c_2 - a_1 b_2 c_2) u v^2, \\
\dot{v} &= -a_1 v + 2 a_2 u v + (a_1 b_1 - a_2 b_2 + a_1 c_1 - a_2 c_2) v^2 + a_1 u^2 v \\
&\quad - (a_2 b_1 + a_1 b_2 + a_2 c_1 + a_1 c_2) u v^2 - (a_1 b_1 c_1 - a_2 b_2 c_1 - a_2 b_1 c_2 - a_1 b_2 c_2) v^3.
\end{aligned} \tag{2.4}$$

The equilibrium points at infinity in the local chart U_1 satisfy $v = 0$ and are the real solutions of

$$a_2 + a_1 u + a_2 u^2 + a_1 u^3 = 0.$$

Since the solutions of the above equation are $u = \pm i$ and $u = -a_2/a_1$, we conclude that if $a_1 = 0$ there are no equilibrium points in the local chart U_1 and if $a_1 \neq 0$ the unique equilibrium point in the local chart U_1 is $p = (-a_2/a_1, 0)$. Computing the eigenvalues of the Jacobian matrix at this point we obtain that they are $\pm a_1\big(1 + (a_2/a_1)^2\big)$ and so it is a saddle. Hence we have another saddle in the local chart V_1 diametrally opposite to p.

Now we analyze the phase portrait in the local chart U_2, we need to study only the origin of U_2, the others infinite equilibria have been studied in the local charts

$U_1 \cup V_1$. The expression of the system in this chart is

$$
\begin{aligned}
\dot{u} = \; & -a_1 - a_2 u + (a_2 b_1 + a_1 b_2 + a_2 c_1 + a_1 c_2) v - a_1 u^2 \\
& +(a_1 b_1 c_1 - a_2 b_2 c_1 - a_2 b_1 c_2 - a_1 b_2 c_2) v^2 - a_2 u^3 + (a_2 b_1 + a_1 b_2 \\
& +a_2 c_1 + a_1 c_2) u^2 v - (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2 - a_2 b_2 c_2) u v^2, \\
\dot{v} = \; & a_2 v - 2 a_1 u v + (a_1 b_1 - a_2 b_2 + a_1 c_1 - a_2 c_2) v^2 - a_2 u^2 v + (a_2 b_1 + a_1 b_2 \\
& +a_2 c_1 + a_1 c_2) u v^2 - (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2 - a_2 b_2 c_2) v^3.
\end{aligned} \tag{2.5}
$$

Note that if $a_1 \neq 0$ the origin is not an equilibrium point in the local chart U_2, and if $a_1 = 0$ then the origin of the local chart U_2 is an equilibrium point whose eigenvalues of the Jacobian matrix at this point are $\pm a_2$, and so the origin is a saddle. In short there is a unique pair of infinite equilibrium points which are saddles. □

3. **Finite equilibrium points**

The finite equilibrium points of system (1.2) are the real solutions of $\dot{x} = \dot{y} = 0$. Computing such solutions the real equilibrium points are (b_1, b_2) and (c_1, c_2). The Jacobian matrix at the point (b_1, b_2) becomes

$$
J = \begin{pmatrix} a_1 b_1 - a_2 b_2 - a_1 c_1 + a_2 c_2 & -a_2 b_1 - a_1 b_2 + a_2 c_1 + a_1 c_2 \\ a_2 b_1 + a_1 b_2 - a_2 c_1 - a_1 c_2 & a_1 b_1 - a_2 b_2 - a_1 c_1 + a_2 c_2 \end{pmatrix}.
$$

whose eigenvalues are

$$
\lambda_\pm = a_1 b_1 - a_1 c_1 - a_2 b_2 + a_2 c_2 \pm i(a_2 b_1 + a_1 b_2 - a_2 c_1 - a_1 c_2).
$$

The Jacobian matrix at the point (c_1, c_2) becomes $-J$ whose eigenvalues are $-\lambda_\pm$.

We consider different cases.

Case 1: $a_2 b_1 + a_1 b_2 - a_2 c_1 - a_1 c_2 \neq 0$ and $a_1 b_1 - a_1 c_1 - a_2 b_2 + a_2 c_2 \neq 0$. In this case the finite singular points (b_1, b_2) and (c_1, c_2) are both foci with opposite stability. The focus (b_1, b_2) is stable whenever $a_1 b_1 - a_1 c_1 - a_2 b_2 + a_2 c_2 < 0$ and unstable whenever $a_1 b_1 - a_1 c_1 - a_2 b_2 + a_2 c_2 > 0$.

Lemma 3.1. *Under these assumptions system* (1.2) *has no limit cycles.*

Proof. We translate the finite singular point (b_1, b_2) at the origin of system (1.2). Then system (1.2) becomes

$$
\begin{aligned}
\dot{x} &= (a_2(c_2 - b_2) + a_1(b_1 - c_1))x + (a_2(c_1 - b_1) + a_1(c_2 - b_2))y + a_1 x^2 \\
&\quad - 2a_2 xy - a_1 y^2, \\
\dot{y} &= (a_1(b_2 - c_2) + a_2(b_1 - c_1))x + (a_1(b_1 - c_1) + a_2(c_2 - b_2))y + a_2 x^2 \\
&\quad + 2a_1 xy - a_2 y^2.
\end{aligned}
$$

The function

$$
H = \frac{(b_1 - c_1)(b_1 - c_1 + 2x) + (b_2 - c_2)(b_2 - c_2 + 2y)}{x^2 + y^2}
$$

satisfies

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y}\\ &= -2\frac{(a_1b_1 - a_1c_1 - a_2b_2 + a_2c_2)((b_1 - c_1 + x)^2 + (b_2 - c_2 + y)^2)}{x^2 + y^2}\end{aligned}$$

Since the function H along the orbits does not change sign (and is zero or not defined at the equilibria), we conclude that there are no periodic solutions in this case and so there cannot be limit cycles. □

Case 2: $a_2b_1 + a_1b_2 - a_2c_1 - a_1c_2 \neq 0$ and $a_1b_1 - a_1c_1 - a_2b_2 + a_2c_2 = 0$. In this case the finite singular points (b_1, b_2) and (c_1, c_2) are both either weak focus or centers. Since we have $(b_1 - c_1)^2 + (b_2 - c_2)^2 \neq 0$ (otherwise the first condition is not satisfied), we can assume that $b_1 \neq c_1$ because the other case is done exactly in the same way interchanging the roles of (b_1, b_2) by (c_1, c_2). In this case we have that

$$a_1 = \frac{a_2(b_2 - c_2)}{b_1 - c_1}.$$

Lemma 3.2. *Under these assumptions the finite singular points* (b_1, b_2) *and* (c_1, c_2) *are both centers.*

Proof. We study the finite singular point (b_1, b_2), the other finite singular point is studied exactly in the same way and so we will not do it. We translate the finite singular point (b_1, b_2) at the origin and we get

$$\begin{aligned}\dot{x} &= -\frac{a_2}{b_1 - c_1}((b_1 - c_1)^2 + (b_2 - c_2)^2)y + \frac{a_2(b_2 - c_2)}{b_1 - c_1}x^2 - 2a_2xy\\ &\quad - \frac{a_2(b_2 - c_2)}{b_1 - c_1}y^2,\\ \dot{y} &= \frac{a_2}{b_1 - c_1}((b_1 - c_1)^2 + (b_2 - c_2)^2)x + a_2x^2 + 2\frac{a_2(b_2 - c_2)}{b_1 - c_1}xy - a_2y^2.\end{aligned} \tag{3.1}$$

In order to write it as in Theorem 4.1 (see the appendix) we introduce the rescaling by the quantity $a_2((b_1 - c_1)^2 + (b_2 - c_2)^2)/(b_1 - c_1)$ and system (3.1) becomes

$$\begin{aligned}\dot{x} &= -y + \frac{b_2 - c_2}{(b_1 - c_1)^2 + (b_2 - c_2)^2}x^2 - 2\frac{b_1 - c_1}{(b_1 - c_1)^2 + (b_2 - c_2)^2}xy\\ &\quad - \frac{b_2 - c_2}{(b_1 - c_1)^2 + (b_2 - c_2)^2}y^2,\\ \dot{y} &= x + \frac{b_1 - c_1}{(b_1 - c_1)^2 + (b_2 - c_2)^2}x^2 + 2\frac{b_2 - c_2}{(b_1 - c_1)^2 + (b_2 - c_2)^2}xy\\ &\quad - \frac{b_1 - c_1}{(b_1 - c_1)^2 + (b_2 - c_2)^2}y^2.\end{aligned} \tag{3.2}$$

Note that

$$b = -\frac{b_2 - c_2}{(b_1 - c_1)^2 + (b_2 - c_2)^2} \quad \text{and} \quad d = \frac{b_2 - c_2}{(b_1 - c_1)^2 + (b_2 - c_2)^2}.$$

Hence $b+d=0$ and it follows from Theorem 4.1 that the origin of system (3.2) is a center. This concludes the proof of the lemma. $\square$

Case 3: $a_2b_1+a_1b_2-a_2c_1-a_1c_2=0$ and $a_1b_1-a_1c_1-a_2b_2+a_2c_2\neq 0$. In this case the finite singular points (b_1,b_2) and (c_1,c_2) are both nodes with opposite stability. The node (b_1,b_2) is stable whenever $a_1b_1-a_1c_1-a_2b_2+a_2c_2<0$ and unstable whenever $a_1b_1-a_1c_1-a_2b_2+a_2c_2>0$. Note that it follows from Theorem 4.2 (see the appendix) that there cannot be limit cycles in this case.

Case 4: $a_2b_1+a_1b_2-a_2c_1-a_1c_2=0$ and $a_1b_1-a_1c_1-a_2b_2+a_2c_2=0$. Taking into account that $(a_1,a_2)\neq(0,0)$ the unique solution of both equations is $c_1=b_1$ and $c_2=b_2$. In this case both finite equilibrium points collide in the unique equilibrium point (b_1,b_2) and the matrix J at this point is identically zero.

Lemma 3.3. *Under these assumptions the finite equilibrium point* (b_1,b_2) *is formed by two elliptic sectors.*

Proof. We translate the equilibrium point at the origin and we get

$$\dot{x}=a_1x^2-2a_2xy-a_1y^2,\quad \dot{y}=a_2x^2+2a_1xy-a_2y^2. \tag{3.3}$$

This system is homogeneous of degree two. Such systems have been completely studied by Date in [7], and it follows from his results that the origin is formed by two elliptic sectors. We prove this result here using polar coordinates for completeness. We introduce polar coordinates (r,θ) through $x=r\cos\theta$, $y=r\sin\theta$, and system (3.3) becomes

$$\dot{r}=r^2(a_1\cos\theta-a_2\sin\theta),\quad \dot{\theta}=r(a_2\cos\theta+a_1\sin\theta).$$

Note that $\dot{\theta}=0$ on the straight line of slope $\theta=\arctan(-a_2/a_1)$. The endpoints of this invariant straight line are the unique pair of infinite singular points which are saddles. Therefore, since the origin is the unique finite singular point, taking into account the invariant straight line, by the Poincaré-Bendixson Theorem (see [8, Theorem 1.25]) it follows that the origin is formed by two elliptic sectors. $\square$

Lemma 3.4. *System* (1.2) *has an invariant straight line if and only if either* $a_2b_1+a_1b_2-a_2c_1-a_1c_2=0$ *or* $a_1b_1-a_1c_1-a_2b_2+a_2c_2=0$.

Proof. System (1.2) has an invariant straight line $Ax+By+C=0$ with $A,B,C\in\mathbb{R}$ if and only if

$$A\dot{x}+B\dot{y}=(k_0+k_1x+k_2y)(Ax+By+C),$$

that is

$$\begin{aligned}
&A(a_1b_1c_1 - a_2b_2c_1 - a_2b_1c_2 - a_1b_2c_2 - (a_1b_1 - a_2b_2 + a_1c_1 - a_2c_2)x \\
&\quad + (a_2b_1 + a_1b_2 + a_2c_1 + a_1c_2)y + a_1x^2 - 2a_2xy - a_1y^2) \\
&\quad + B(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2 - a_2b_2c_2 - (a_2b_1 + a_1b_2 + a_2c_1 + a_1c_2)x \\
&\quad + (a_1b_1 - a_2b_2 + a_1c_1 - a_2c_2)y + a_2x^2 + 2a_1xy - a_2y^2) \\
&= (k_0 + k_1x + k_2y)(Ax + By + C),
\end{aligned}$$

for some constants $k_0, k_1, k_2 \in \mathbb{R}$.

Doing this computation we obtain that the straight line is:

(a) $a_2x + a_1y - a_2c_1 - a_1c_2 = 0$ if $a_2b_1 + a_1b_2 - a_2c_1 - a_1c_2 = 0$.
(b) $2a_1a_2x + 2a_1^2y - 2a_1a_2c_1 - a_2^2(b_2 - c_2) - a_1^2(b_2 + c_2) = 0$ if $a_1b_1 - a_1c_1 - a_2b_2 + a_2c_2 = 0$ and $a_1^2 + (b_2 - c_2)^2 \neq 0$.
(c) $x - (b_1 + c_1)/2 = 0$ if $a_1b_1 - a_1c_1 - a_2b_2 + a_2c_2 = 0$ and $a_1^2 + (b_2 - c_2)^2 = 0$.

□

4. Proof of Theorem 1.1

We separate the proof into the previous four cases.

In Case 1 taking into account the previous information on the finite and infinite equilibria together with Lemmas 3.1 and 3.4 we conclude that the only possible phase portrait is the first phase portrait in Figure 1.

In Case 2 taking into account the previous information on the finite and infinite equilibria together with Lemmas 3.2 and 3.4 we conclude that the only possible phase portrait is the second phase portrait in Figure 1.

In Case 3 taking into account the previous information on the finite and infinite equilibria together with Lemma 3.4 we conclude that the only possible phase portrait is the third phase portrait in Figure 1.

In Case 4 taking into account the previous information on the finite and infinite equilibria together with Lemmas 3.3 and 3.4 we conclude that the only possible phase portrait is the fourth phase portrait in Figure 1.

Appendix: auxiliary results

In this appendix we introduce some auxiliary results. The first one is proved in [2, 10, 11].

Theorem 4.1 (Kapteyn–Bautin Theorem)**.** *Any quadratic system candidate to have a center can be written in the form*

$$\dot{x} = -y - bx^2 - Cxy - dy^2, \qquad \dot{y} = x + ax^2 + Axy - ay^2. \tag{4.1}$$

This system has a center at the origin if and only if one of the following conditions holds
$A - 2b = C + 2a = 0$,
$C = a = 0$,
$b + d = 0$,
$C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$.

An easy proof of the Kapteyn–Bautin Theorem using the Darboux theory of integrability can be found in [4].

The second result, proved in [6] is the following

Theorem 4.2. *Any limit cycle of a quadratic polynomial differential system must surround a focus.*

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REFERENCES

[1] J.C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe, *Geometric configurations of singularities of planar polynomial differential systems. A global classification in the quadratic case*, in Birkhäuser, 2021.

[2] N.N. Bautin, On the number of limit cycles which appear with the variation of the coefficients from an equilibrium position of focus or center type, *Math. USSR-Sb.* 100:397–413, 1954.

[3] B. Branner and K. Dias, Classification of complex polynomial vector fields in one complex variable, *J. Difference Equ. Appl.* 16(4-5):463–517, 2010.

[4] L. Cairó, M.R. Feix and J. Llibre, Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems, *Resenhas do Instituto de Matematica e Estatistica da Universidade de Sao Paulo* 4:127–161, 1999.

[5] C. Chicone and T. Jinghuang, On General Properties of Quadratic Systems, *The American Mathematical Monthly*, 89(3):167–178, 1982.

[6] W.A. Coppel, A Survey of Quadratic Systems, *J. Differential Equations* 2:293–304, 1966.

[7] T. Date, Classification and analysis of two-dimensional real homogeneous quadratic differential equation systems, *J. Differential Equations* 32:311–334, 1979.

[8] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer, 2006.

[9] P. Figueroa and M. Pinto, Riccati equations and nonoscillatory solutions of third order differential equations, *Dynam. Systems Appl.* 17(3-4):459–475, 2008.

[10] W. Kapteyn, On the midpoints of integral curves of differential equations of the first degree, *Nederl. Akad. Wetensch. Verslag. Afd. Natuurk.*, 19:1446–1457 1911, (Dutch).

[11] W. Kapteyn, New investigations on the midpoints of integrals of differential equations of the first degree, *Nederl. Akad. Wetensch. Verslag Afd. Natuurk.*, 20:1354–1365, 1912; 21:27–33, 1912, (Dutch).

[12] J. Llibre, R. Oliveira and C. Valls, Phase portraits for $\mathbb{Z}_2$-symmetric cubic Riccati polynomial differential equations, Topology and its Applications, 234:220–237, 2018.

[13] S.D. Maharaj, A.K. Tiwari, R. Mohanlal and R. Narain, Riccati equations for bounded radiating systems, *J. Math. Phys.*, 57(9):092501, 12 pp., 2016.

[14] J.W. Reyn, *Phase portraits of planar quadratic systems*, Mathematics and its Applications, 583, Springer, 2007.

[15] A.G. Wu, F.J. Sun and Y. Zhang, A novel iterative algorithm for solving coupled Riccati equations, *Appl. Math. Comput.*, 364:124645, 14 pp., 2020.

[16] Ye Yanqian el al., *Theory of limit cycles*, Trans. of Mathematical Monographs 66, Amer. Math. Soc., Providence, RI, 2 edition, 1984.

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