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MULTIPLE SOLUTIONS OF A DISCRETE NONLINEAR BOUNDARY VALUE PROBLEM INVOLVING $P(K)$-LAPLACE KIRCHHOFF TYPE OPERATOR

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ABSTRACT. In this article, we prove the existence of at least one or two nontrivial solutions of a discrete nonlinear boundary value problem of $p(k)$-Laplace Kirchhoff type in a finite dimensional Banach space. Our approach is based on variational methods and on critical point theory.

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1. INTRODUCTION

In this article, we are interested in the following Kirchhoff type problem with the Dirichlet boundary value condition

$$(1.1)\qquad \begin{cases} -M(\delta\,[u])\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) \\ \quad = \lambda f(k,u(k)),\ \ k\in\mathbb{Z}\,[1,T]\,, \\ u(0)=u(T+1)=0, \end{cases}$$

where $T \geq 2$ is a fixed positive integer, $\mathbb{Z}\,[a,b]$ for $a<b$, $a,b\in\mathbb{Z}$ denotes a discrete interval $\{a,a+1,\dots,b-1,b\}$, $\delta[u]$ is a non-local term defined by the following relation

$$\delta[u]=\sum_{k=1}^{T+1}A_0(k-1,|\triangle u(k-1)|),$$

$\lambda>0$ is a real parameter, $\triangle u(k)=u(k+1)-u(k)$ is the forward difference operator, $f:\mathbb{Z}\,[1,T]\times\mathbb{R}\to\mathbb{R}$ is Carathéodory, $a(k,\cdot), M:[0,\infty)\to[0,\infty)$ are two continuous functions for all $k\in\mathbb{Z}\,[1,T]$, $t\in[0,\infty)$ with the function $t\to M(t)$ nondecreasing,

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$A_0 : \mathbb{Z}[1,T] \times [0,\infty) \to [0,\infty)$ which satisfies $A_0(k,s) = \int_0^t a(k,\xi)\xi \, d\xi$ and the function $p : \mathbb{Z}[0,T] \to (1,\infty)$ is bounded.
we denote for short

$$p^+ := \max_{k\in\mathbb{Z}[0,T]} p(k) \quad \text{and} \quad p^- := \min_{k\in\mathbb{Z}[0,T]} p(k).$$

The presence of the non-local term $\delta[u]$ is an important feature of this article. Kirchhoff in 1876 (see [31]) proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{Ea}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

where $\rho > 0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string.

Equation (1.2) takes into account the change of the tension on the string which is caused by the change of its length during the vibration. After that, several physicists also considered such equations for their researches in the theory of nonlinear vibrations theoretically or experimentally [16, 17, 41, 43]. Moreover, Kirchhoff equation received a lot of attention only after Lions in 1978 (see [35]) suggested an abstract framework to the problem which is related to the stationary analogue of the equation of Kirchhoff type. Many authors have investigated Kirchhoff type equations, we refer the readers to [4, 18] and the references therein. For the recent papers of the discrete problems of Kirchhoff type, we refer the readers to [19, 28, 44, 42, 32, 48, 49, 50] and the references therein. For example, in [50] Yucedag obtained, by using variational approach and applying the Mountain Pass theorem, existence of at least one nontrivial solution for an anisotropic discrete boundary value problem of $p(k)$-Kirchhoff type in T-dimensional Hilbert space. In [32], Koné et al. proved, by using minimization method, existence of a weak solution to a family of discrete boundary value problems whose right-hand side belongs to a discrete Hilbert space. More recently, Heidarkhani et *al.* (see [28]) have dealt with the $p(k)$-Kirchhoff type problems by using variational methods and critical point theory.

The importance of problem (1.1) arises mainly from the existence of the nonhomogeneous differential operator

$$\triangle(a(k-1, |\triangle u(k-1)|)\triangle u(k-1)).$$

This operator generalizes the usual operators with variable exponent. For instance, if $a(k,\xi) = \xi^{p(k)-2}$ then we obtain the standard $p(\cdot)$-Laplace difference operator, that is,

$$\triangle_{p(k-1)} u(k-1) := \triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right).$$

The differential equations and variational problems involving non-homogeneous differential operators have been intensively studied in the last few decades since they can model various phenomena arising from the study of elastic mechanics [55], electrorheological fluids [45, 46] and image restoration [20].

In recent years, many authors have discussed the existence and multiplicity of solutions for discrete boundary value problems by using variational methods. For the papers involving the discrete $p(k)$-Laplacian operator, we refer the readers to [5, 21, 22, 23, 24, 29, 36, 38]. In the case where $p(k)$ is a constant called the discrete p-Laplacian operator, we refer to recent works [1, 2, 10, 11, 12, 13, 14] and references therein. The discrete $p(k)$-Laplacian operator has more complicated nonlinearities than the discrete p-Laplacian operator, for example, it is not homogeneous. The nonlinear problems involving the discrete $p(k)$-Laplacian have firstly been discussed by the authors in [33, 39]. For example, in [39] the authors proved, by using critical point theory, existence of a continuous spectrum of eigenvalues for a discrete anisotropic problem. In [33], Koné and Ouaro proved, by using minimization method, the existence and uniqueness of weak solutions for anisotropic discrete boundary value problems. We refer the readers to the recent article done by Kyelem et *al* [34] for the applications of variational methods on difference equations.

Problem (1.1) can be seen as a discrete variant of the following variable exponent anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i} = \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies a Carathéodory condition, p_i continuous on $\overline{\Omega}$ such that $1 < p_i(x)$ for each $x \in \overline{\Omega}$ and every $i \in \{1, 2, \ldots, N\}$, and λ is a positive real parameter and it was recently analysed by I.H. Kim and Y.H. Kim [30].

In this paper, we prove the existence of solutions of the discrete nonlinear boundary value problem of $p(k)$-Kirchhoff type equations, by using variational methods and critical point theory. We apply a result of Ricceri for functionals (see [47]) to obtain the existence of at least one nontrivial solution and also a result of Mawhin and Willem (see [37]), to obtain the existence of at least two solutions.

The rest of the paper is organized as follows. In Section 2, the variational framework associated with problem (1.1) is established, the abstract critical point theorem and our main tools are recalled. Some definitions and lemmas which are essential to show our main results are also stated. In Section 3, we investigate the existence of at least one nontrivial solution for (1.1). Finally, in Section 4, we focus on the existence of at least two nontrivial solutions.

2. PRELIMINARIES

In this section, we provide a variational framework associated with problem (1.1). We consider the following T-dimensional Banach space.

$$W = \{u : \mathbb{Z}[0, T+1] \to \mathbb{R} \text{ such that } u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-}\right)^{1/p^-}.$$

Moreover, we will also use the following norm

$$\|u\|_{p^+} = \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+}\right)^{1/p^+}$$

and by using the discrete Hölder inequality ([25]), one has

$$\begin{aligned}
&\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} \\
&\leq \left(\sum_{k=1}^{T+1} \{1\}^{\frac{p^+}{p^+-p^-}}\right)^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T+1} \left(|\triangle u(k-1)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}} \\
&\leq (T+1)^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+}\right)^{\frac{p^-}{p^+}} \\
&= (T+1)^{\frac{p^+-p^-}{p^+}} \|u\|_{p^+}^{p^-}.
\end{aligned}$$

Consequently,

$$\|u\| \leq (T+1)^{\frac{p^+-p^-}{p^+p^-}} \|u\|_{p^+}. \tag{2.1}$$

On the space W we can also introduce the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \sum_{k=1}^{T+1} \left|\frac{\triangle u(k-1)}{\mu}\right|^{p(k-1)} \leq 1\right\}.$$

For our purpose, since W is of finite dimensional, these norms are equivalent. Therefore there exist two constants $0 < K_1 < K_2$ such that

$$K_1 \|u\|_{p(\cdot)} \leq \|u\| \leq K_2 \|u\|_{p(\cdot)}. \tag{2.2}$$

Next, let $\varphi : W \to \mathbb{R}$ be given by

$$\varphi(u) = \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)}.$$

It is easy to check that for any $u_n, u \in W$, the following relations hold true.

$$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \le \varphi(u) \le \|u\|_{p(\cdot)}^{p^+}, \tag{2.3}$$

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \le \varphi(u) \le \|u\|_{p(\cdot)}^{p^-}, \tag{2.4}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \Leftrightarrow \varphi(u_n - u) \to 0 \;\text{ as }\; n \to \infty. \tag{2.5}$$

We also consider another norm in W, that is

$$\|u\|_\infty = \max_{k \in \mathbb{Z}[1,T]} |u(k)|.$$

For every $u \in W$, there exists $\tau \in \mathbb{Z}\,[1, T]$ such that

$$\|u\|_\infty = |u(\tau)| \le \frac{1}{2} \sum_{k=1}^{T+1} |\triangle u(k-1)| \le \frac{(T+1)^{(p^- -1)/p^-}}{2} \|u\|. \tag{2.6}$$

Let $\Phi, \Psi : W \to \mathbb{R}$ be two functionals defined by

$$\Phi(u) = \widehat{M}(\delta[u]), \tag{2.7}$$

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \tag{2.8}$$

where

$$\widehat{M}(t) = \int_0^t M(\xi)\, d\xi \quad \text{and} \quad F(k,t) = \int_0^t f(k,\xi)\, d\xi.$$

Then, for any $\lambda > 0$, we define the energy functional $I_\lambda : W \to \mathbb{R}$ corresponding to problem (1.1) and given by

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u), \tag{2.9}$$

for every $u \in W$.

We recall that a critical point of I_λ is a point $u \in W$ such that

$$\begin{aligned} M\left(\delta[u]\right) \sum_{k=1}^{T+1} a(k-1, |\triangle u(k-1)|) \triangle u(k-1) \triangle v(k-1) \\ = \lambda \sum_{k=1}^{T} f(k, u(k)) v(k), \end{aligned} \tag{2.10}$$

for any $v \in W$.

It is easy to verify that Φ and Ψ are two functionals of class $C^1(W,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in W$ are given by

$$\langle \Phi'(u), v\rangle = M\left(\delta[u]\right)\sum_{k=1}^{T+1} a(k-1,|\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) \tag{2.11}$$

and

$$\langle \Psi'(u), v\rangle = \sum_{k=1}^{T} f(k,u(k))v(k), \tag{2.12}$$

for all $u, v \in W$.

By (2.11) and (2.12), we observe that I_λ is of class $C^1(W,\mathbb{R})$ and

$$\langle I'_\lambda(u), v\rangle = \langle \Phi'(u), v\rangle - \lambda\langle \Psi'(u), v\rangle,$$

for all $u, v \in W$.
Thus, for every $v \in W$ and taking $v(0) = v(T+1) = 0$ into account, one has

$$\begin{aligned} &M\left(\delta[u]\right)\sum_{k=1}^{T+1} a(k-1,|\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1) \\ &\quad = -M\left(\delta[u]\right)\sum_{k=1}^{T+1} \triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))v(k), \end{aligned}$$

then,

$$\langle I'_\lambda(u), v\rangle = \sum_{k=1}^{T}\left[-M(\delta\left[u\right])\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) - \lambda f(k,u(k))\right]v(k).$$

Consequently, the critical points of I_λ in W are exactly the solutions of problem (1.1).

In the sequel, we will use the following auxiliary result.

Lemma 2.1. *(i) Let $u \in W$ and $\|u\| > 1$. Then,*

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} \geq \|u\|^{p^-} - (T+1).$$

(ii) Let $u \in W$ and $\|u\| < 1$. Then,

$$\sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} \geq (T+1)^{\frac{p^- - p^+}{p^-}}\|u\|^{p^+}.$$

(iii) For any $m \geq 2$, one has

$$\sum_{k=1}^{T} |u(k)|^m \leq \left(\frac{(T+1)^{(p^- -1)/p^-}}{2}\right)^m T\|u\|^m, \text{ for every } u \in W.$$

Proof. Fix $u \in W$ with $\|u\| > 1$. By a similar argument as in [24], we define for each $k \in \mathbb{Z}[0,T]$,

$$\beta_k := \begin{cases} p^+ & \text{if } |\triangle u(k)| \le 1, \\ p^- & \text{if } |\triangle u(k)| > 1. \end{cases}$$

(i) We deduce that

$$\begin{aligned} \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} &\ge \sum_{k=1}^{T+1} |\triangle u(k-1)|^{\beta_{k-1}} \\ &= \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} - \sum_{k=1,\beta_{k-1}=p^+}^{T+1} \left(|\triangle u(k-1)|^{p^-} - |\triangle u(k-1)|^{p^+} \right) \\ &\ge \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^-} - (T+1) = \|u\|^{p^-} - (T+1), \end{aligned}$$

for all $u \in W$ such that $\|u\| > 1$.

(ii) By relation (2.1) as $|\triangle u(k)| < 1$ since $\|u\| < 1$, we deduce that

$$\begin{aligned} \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p(k-1)} &\ge \sum_{k=1}^{T+1} |\triangle u(k-1)|^{p^+} \\ &= \|u\|_{p^+}^{p^+} \ge (T+1)^{\frac{p^- - p^+}{p^-}} \|u\|^{p^+}. \end{aligned}$$

(iii) Note that by relation (2.2), one has

$$|u(k)|^m \le \left(\frac{(T+1)^{(p^- -1)/p^-}}{2} \right)^m \|u\|^m, \ \text{for every } k \in \mathbb{Z}[1,T].$$

Then, summing up k from 1 to T, we obtain

$$\sum_{k=1}^{T} |u(k)|^m \le \left(\frac{(T+1)^{(p^- -1)/p^-}}{2} \right)^m T\|u\|^m,$$

for every $u \in W$ and any $m \ge 2$. Hence, we conclude that Lemma 2.1 holds true. □

We also assume that a and M satisfy the following assumptions.

$(H1)$ $a_1 : \mathbb{Z}[0,T] \to [0,\infty)$ and there exists a constant $a_2 > 0$ such that

$$|a(k,|\xi|)\xi| \le a_1(k) + a_2|\xi|^{p(k)-1},$$

for all $k \in \mathbb{Z}[0,T]$ and $\xi \in \mathbb{R}$.

$(H2)$ For all $k \in \mathbb{Z}[0,T]$ and $\xi > 0$, one has

$$0 \le a(k,|\xi|)\xi^2 \le p^+ \int_0^{|\xi|} a(k,s)s\, ds.$$

$(H3)$ There exists a positive constant c such that

$$\min\left\{a(k,|\xi|),\ |\xi|\frac{\partial a}{\partial \xi}(k,|\xi|)+a(k,|\xi|)\right\}\geq c|\xi|^{p(k)-2},$$

for all $k\in\mathbb{Z}\,[0,T]$ and $\xi\in\mathbb{R}$.

$(H4)$ $M:(0,\infty)\to(0,\infty)$ is continuous, nondecreasing and there exist positive reals numbers A, B with $A\leq B$ and $\alpha\geq 1$ such that

$$As^{\alpha-1}\leq M(s)\leq Bs^{\alpha-1}\ \text{ for }\ s\geq s^{*}>0.$$

Example 2.1 As examples of functions A_0 and a satisfying the above assumptions, we can give the following.

(1) If we take

$$M(A_0(k,|\xi|))=\frac{1}{p(k)}|\xi|^{p(k)}\ \text{ and }\ M(t)=1,$$

then

$$a(k,|\xi|)=|\xi|^{p(k)-2},\ \text{ for all }\ (k,\xi)\in\mathbb{Z}\,[1,T]\times\mathbb{R}.$$

(2) Now, if we take

$$M(A_0(k,|\xi|))=a+\frac{b}{p(k)}\left[\left(1+|\xi|^2\right)^{\frac{p(k)}{2}}-1\right]\ \text{ and }\ M(t)=a+bt,$$

then

$$a(k,|\xi|)=\left(1+|\xi|^2\right)^{\frac{p(k)-2}{2}},\ \text{ for all }\ (k,\xi)\in\mathbb{Z}\,[1,T]\times\mathbb{R}.$$

We now introduce some necessary definitions.

Definition 2.2. An element $u\in E$ is a critical point of the functional $I:E\to\mathbb{R}$ if

$$\langle I'(u),v\rangle=0,\ \text{ for all }\ v\in E.$$

Definition 2.3. We say that I satisfies the Palais-Smale condition ((PS) *condition* for short) if for any sequence $\{u_n\}\subset E$ such that $\{I(u_n)\}$ is bounded and $I'(u_n)\to 0$ as $n\to\infty$, there exists a subsequence of $\{u_n\}$ which is convergent in E.

Definition 2.4. We say that a sequence $\{u_n\}\subset E$ is said to satisfy the $(PS)_c$ condition if

$$I(u_n)\to c\in\mathbb{R}\ \text{ and }\ I'(u_n)\to 0\ \text{ as }\ n\to\infty.$$

Our first tool and approach is based on a local minimum theorem due to Bonanno [7, Theorem 5.1], which is inspired by the Ricceri variational principle (see [47, Theorem 2.5]). We refer the readers to the papers [3, 6, 8, 9, 26, 40] in which Theorem 2.5 below have been successfully employed to get the existence of at least one nontrivial solution for boundary value problems.

For a given nonempty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions.

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup\limits_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u)}{\Phi(v) - r_1},$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

Theorem 2.5. *(*[7, Theorem 5.1] *). Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Further, set $I_\lambda := \Phi - \lambda\Psi$ and assume that there are two constants $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in \left(\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right)$ there is $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_0, \lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Our second main tool and approach are based on theorems 2.6 and 2.7 below. We refer the readers to the papers [15, 53] in which Theorem 2.6 have been applied to obtain multiple solutions for boundary value problems. On the other hand, in [27, 54], theorems 2.6 and 2.7 have been successfully employed to prove the existence of two solutions for boundary value problems.

Theorem 2.6. *(*[37, Theorem 4.10] *). If X is a reflexive Banach space, $I \in C^1(X, \mathbb{R})$ and I satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and*

$$\inf_{u\in\partial\Omega} I(u) > \max\{I(u_0), I(u_1)\},$$

then there exists a critical point $\tilde{u} \in X$ of I such that $J(\tilde{u}) > \max\{I(u_0), I(u_1)\}$.

Theorem 2.7. *(*[52, Theorem 38]*). For the functional $F : M \subseteq X \to (-\infty, \infty)$ with $M \neq \varnothing$, $\min\limits_{u\in M} F(u) = \alpha$ has a solution in case where the following conditions hold.*

(i) X is a real reflexive Banach space,
(ii) M is bounded and weakly sequentially closed,
(iii) F is weakly sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, one has $F(u) \leq \liminf\limits_{n\to\infty} F(u_n)$ holds.

Proposition 2.8. *Assume that the condition $(H3)$ is fulfilled. Then, the following estimate*

$$\begin{aligned}&\langle a(k,|u|)u - a(k,|v|)v, u-v\rangle \\ &\geq \begin{cases} c\left(|u|+|v|\right)^{p(k)-2}|u-v|^2 & \text{if } 1<p(k)<2 \\ 4^{2-p^+}c|u-v|^{p(k)} & \text{if } \quad p(k)\geq 2\end{cases}\end{aligned}$$

holds true for all $u, v \in \mathbb{R}$ and $k \in \mathbb{Z}[1,T]$ such that $(u,v) \neq (0,0)$.

Proof. Let $u, v \in \mathbb{R}$ with $(u,v) \neq (0,0)$. Let us define $\psi(k,u) = a(k,|u|)u$. From condition $(H3)$, we see that, for all $u \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned}\frac{\partial\psi(k,u)}{\partial u} &= |u|\frac{\partial a}{\partial u}(k,|u|) + a(k,|u|) \\ &\geq c|u|^{p(k)-2}.\end{aligned} \tag{2.13}$$

Note that

$$\psi(k,u)-\psi(k,v) = \int_0^1 \frac{\partial\psi(k, v+t(u-v))}{\partial u}(u-v)dt. \tag{2.14}$$

Suppose that $k \in \mathbb{Z}[0,T]$ such that $p(k) \geq 2$. Thus, it follows from (2.13) and (2.14) that

$$\begin{aligned}\langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &= \int_0^1 \frac{\partial\psi}{\partial u}(k, v+t(u-v))(u-v)(u-v)\,dt \\ &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2 dt.\end{aligned}$$

Without loss of generality, we may assume that $|u| \leq |v|$. Thus, $|u-v| \leq 2|v|$.
For any $t \in [0, 1/4]$, we get

$$|v+t(u-v)| \geq |v| - \frac{1}{4}|u-v|,$$

then

$$|v+t(u-v)| \geq \frac{1}{4}|u-v|.$$

Consequently,

$$\begin{aligned}\langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2 dt \\ &\geq 4^{2-p^+}c|u-v|^{p(k)}.\end{aligned}$$

Suppose now that $k \in \mathbb{Z}[0,T]$ such that $1 < p(k) < 2$. By the preceding arguments, we deduce by condition $(H3)$ that, for all $u \in \mathbb{R}\setminus\{0\}$,

$$\begin{aligned}\frac{\partial\psi(k,u)}{\partial u} &= |u|\frac{\partial a}{\partial u}(k,|u|) + a(k,|u|) \\ &\geq c|u|^{p(k)-2}.\end{aligned}$$

Using the fact that $|tu+(1-t)v| \le |u|+|v|$, we clearly obtain

$$\begin{aligned}\langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &\ge \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2 dt \\ &\ge c\,(|u|+|v|)^{p(k)-2}\,|u-v|^2.\end{aligned}$$

This ends the proof. □

Lemma 2.9. *Assume that $(H1)$, $(H3)$ and $(H4)$ are fulfilled. Then, the operator $\Phi' : W \to W^*$ is strictly monotone on W and verifies the (S_+) condition, i.e., for every sequence $\{u_n\} \subset W$ such that $u_n \rightharpoonup u$ in W as $n \to \infty$ and $\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \le 0$, one has $u_n \to u$ in W as $n \to \infty$. Here, $\langle \cdot,\cdot\rangle$ denotes the duality pairing between W and its dual W^*.*

Proof. We prove that Φ' is a strictly monotone operator.
For that, we consider the functional $\phi : W \to \mathbb{R}$ given by

$$\phi(u) = \delta[u] = \sum_{k=1}^{T+1}\int_0^{|\triangle u(k-1)|} a(k-1,\xi)\xi d\xi, \ \text{ for all } u \in W.$$

Then, $\phi \in C^1(W,\mathbb{R})$ and its Gâteaux derivative at the point $u \in W$ is

$$\langle \phi'(u), v\rangle = \sum_{k=1}^{T+1} a(k-1,|\triangle u(k-1)|)\triangle u(k-1)\triangle v(k-1),$$

for all $u, v \in W$.
For all $u, v \in W$ with $u \ne v$, we get
$\langle \phi'(u) - \phi'(v), u-v\rangle$

$$= \sum_{k=1}^{T+1}\left(a(k-1,|\triangle u(k-1)|)\triangle u(k-1) - a(k-1,|\triangle v(k-1)|)\triangle v(k-1)\right)\left(\triangle u(k-1) - \triangle v(k-1)\right).$$

By Proposition 2.8, one obtains

$$\langle \phi'(u) - \phi'(v), u-v\rangle$$

$$\ge \begin{cases} c\displaystyle\sum_{k=1}^{T+1}\left(|\triangle u(k-1)| + |\triangle v(k-1)|\right)^{p(k-1)-2}|\triangle u(k-1) - \triangle v(k-1)|^2 > 0 & \text{if } 1 < p(k-1) < 2, \\ 4^{2-p^+}c\displaystyle\sum_{k=1}^{T+1}|\triangle u(k-1) - \triangle v(k-1)|^{p(k-1)} > 0 & \text{if } \quad p(k-1) \ge 2.\end{cases}$$

Thus, ϕ' is strictly monotone. Therefore, by Proposition 25.10 of [51], it follows that ϕ is strictly convex. Furthermore, since M is nondecreasing, then $\widehat{M}$ is convex in $(0,\infty)$. Thus, for all $u, v \in W$ with $u \ne v$ and every $s, t \in (0,1)$ with $s+t=1$, one has

$$\widehat{M}\left(\phi(su+tv)\right) < \widehat{M}\left(s\phi(u)+t\phi(v)\right) \le s\widehat{M}\left(\phi(u)\right) + t\widehat{M}\left(\phi(v)\right).$$

Then, it follows that Φ is stictly convex and so Φ' is strictly monotone in W.

Now, we claim that the operator Φ' is of type (S_+). Let $\{u_n\} \subset W$ be a sequence such that $u_n \rightharpoonup u$ in W as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0.$$

We will show that $u_n \to u$ in W as $n \to \infty$.
Since $\limsup\limits_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$ and according to the strict monotonicity of Φ', one has

$$\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0.$$

Thus,

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle = 0,$$

which means that

$$\lim_{n\to\infty} M\left(\delta[u_n]\right) \sum_{k=1}^{T+1} a(k-1, |\triangle u_n(k-1)|)\triangle u_n(k-1)\triangle(u_n - u)(k-1) = 0. \tag{2.15}$$

We deduce by hypothesis $(H1)$ that

$$\begin{aligned} \delta[u_n] &= \sum_{k=1}^{T+1} \int_0^{|\triangle u_n(k-1)|} a(k-1,\xi)\xi d\xi \\ &\leq \sum_{k=1}^{T+1} a_1(k-1)|\triangle u_n(k-1)| + \sum_{k=1}^{T+1} \frac{a_2}{p(k-1)} |\triangle u_n(k-1)|^{p(k-1)} \\ &\leq \max_{k\in\mathbb{Z}[1,T]} a_1(k) \sum_{k=1}^{T+1} |\triangle u_n(k-1)| + \frac{a_2}{p^-} \sum_{k=1}^{T+1} |\triangle u_n(k-1)|^{p(k-1)} \\ &\leq K_1 + K_2 \sum_{k=1}^{T+1} |\triangle u_n(k-1)|^{p(k-1)}, \end{aligned}$$

where K_1 and K_2 are positive constants.
It is immediate to see that

$$\sum_{k=1}^{T+1} |\triangle u_n(k-1)|^{p(k-1)} = \|u_n\|^{p^*} = \begin{cases} \|u_n\|^{p^+} & \text{if } \|u_n\| > 1, \\ \|u_n\|^{p^-} & \text{if } \|u_n\| < 1. \end{cases}$$

Then,

$$\delta[u_n] \leq K_1 + K_2 \|u_n\|^{p^*} \leq K\left(1 + \|u_n\|^{p^*}\right).$$

Hence, we infer that $\left(\delta[u_n]\right)_{n\geq 1}$ is bounded.
Since M is continuous, up to a subsequence there is $s_0 \geq 0$ such that

$$M\left(\delta[u_n]\right) \to M(s_0) \geq A s_0^{\alpha-1} \text{ as } n \to \infty. \tag{2.16}$$

Thus, it follows from (2.15) and (2.16) that

$$\lim_{n\to\infty}\sum_{k=1}^{T+1} a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1)\triangle(u_n-u)(k-1)=0.$$

Therefore,

$$\lim_{n\to\infty}\langle \phi'(u_n),u_n-u\rangle=0.$$

Then,

$$\lim_{n\to\infty}\langle \phi'(u_n)-\phi'(u),u_n-u\rangle=0. \tag{2.17}$$

On the other hand, by Proposition 2.8, one has

$$\langle \phi'(u_n)-\phi'(u),u_n-u\rangle$$

$$\geq \begin{cases} c\sum_{k=1}^{T+1}\hat{u}(k-1)^{p(k-1)-2}|\triangle u_n(k-1)-\triangle u(k-1)|^2>0 & \text{if } 1<p(k-1)<2,\\ 4^{2-p^+}c\sum_{k=1}^{T+1}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)}>0 & \text{if } \quad p(k-1)\geq 2.\end{cases} \tag{2.18}$$

where $\hat{u}(k-1)=|\triangle u_n(k-1)|+|\triangle u(k-1)|$.
By using the discrete Hölder inequality (see [25]), we get

$$\begin{aligned}&\sum_{k=1}^{T+1}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)}\\
&=\sum_{k=1}^{T+1}\hat{u}(k-1)^{\frac{p(k-1)(2-p(k-1))}{2}}\left(\hat{u}(k-1)^{\frac{p(k-1)(p(k-1)-2)}{2}}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)}\right)\\
&\leq K'\|\hat{u}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}}\|\hat{u}^{\frac{p(\cdot)(p(\cdot)-2)}{2}}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}}\\
&\leq K'\|\hat{u}\|_{p(\cdot)}^s\left(\sum_{k=1}^{T+1}\hat{u}(k-1)^{p(k-1)-2}|\triangle u_n(k-1)-\triangle u(k-1)|^2\right)^v,\end{aligned}$$

where s is either $p^-(2-\overline{p})/2$ or $\overline{p}(2-p^-)/2$ and v is either $p^-/2$ or $\overline{p}/2$ with $\overline{p}=\max_{\{k\in\mathbb{Z}[0,T]:1<p(k)<2\}}p(k)$. Therefore, taking the above inequality and (2.17)-(2.18) into account, one has

$$\lim_{n\to\infty}\sum_{k=1}^{T+1}|\triangle u_n(k-1)-\triangle u(k-1)|^{p(k-1)}=0. \tag{2.19}$$

Combining (2.5) with (2.19), we see that

$$\lim_{n\to\infty}\|u_n-u\|_{p(\cdot)}=0.$$

Thus, Φ' is of type (S_+). The proof of Lemma 2.9 is complete. □

Lemma 2.10. *Suppose that* $(H1)$, $(H3)$ *and* $(H4)$ *are satisfied. Then,* Φ *is weakly lower semi-continuous, i.e.,* $u_n \rightharpoonup u$ *in* W *as* $n \to \infty$ *implies that* $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$.

Proof. Assuming that $u_n \rightharpoonup u$ in W as $n \to \infty$. Then, it follows from (2.11) and Lemma 2.9 that Φ is convex (see [52, Proposition 42.6]) and we deduce that for any $n \in \mathbb{N}$,

$$\Phi(u_n) \geq \Phi(u) + \langle \Phi'(u), u_n - u \rangle.$$

Then,

$$\liminf_{n\to\infty} \Phi(u_n) \geq \Phi(u) + \liminf_{n\to\infty} \langle \Phi'(u), u_n - u \rangle,$$

which means that

$$\liminf_{n\to\infty} \Phi(u_n) \geq \Phi(u).$$

We conclude that Φ is weakly lower semi-continuous and the proof is complete. □

3. EXISTENCE OF A NONTRIVIAL SOLUTION

In this section, we prove that problem (1.1) has at least one nontrivial solution using Theorem 2.5. Let ϵ and b be two positive constants with

$$\frac{(\epsilon\kappa)^{\alpha p^*} A c^{\alpha} (T+1)^{\alpha(p^- - p^+)/p^-}}{\alpha (p^+)^{\alpha}} \neq \widehat{M}\left(\int_0^b (a(0,\xi) + a(T,\xi))\, \xi \, d\xi \right).$$

We put

$$\Theta_\epsilon(b) = \frac{\displaystyle\sum_{k=1}^{T} \max_{|t|\leq\epsilon} F(k,t) - \sum_{k=1}^{T} F(k,b)}{\displaystyle\frac{(\epsilon\kappa)^{\alpha p*} A c^{\alpha} (T+1)^{\alpha(p^- - p^+)/p^-}}{\alpha (p^+)^{\alpha}} - \widehat{M}\left(\int_0^b (a(0,\xi) + a(T,\xi))\, \xi \, d\xi \right)}.$$

Theorem 3.1. *Assume that there exist three positive constants* ϵ_1, b *and* ϵ_2 *with*

$$\epsilon_1 < \frac{(\alpha (p^+)^{\alpha})^{1/\alpha p^*}}{\kappa (A c^{\alpha})^{1/\alpha p^*} (T+1)^{(p^- - p^+)/p^* p^-}} \left[\widehat{M}\left(\int_0^b (a(0,\xi) + a(T,\xi))\, \xi \, d\xi \right) \right]^{1/\alpha p^*} < \epsilon_2,$$

such that

$$\Theta_{\epsilon_2}(b) < \Theta_{\epsilon_1}(b).$$

Then, for each $\lambda \in \left(\frac{1}{\Theta_{\epsilon_1}(b)}, \frac{1}{\Theta_{\epsilon_2}(b)} \right)$, *problem (1.1) admits at least one nontrivial solution* $\check{u} \in W$ *such that*

$$\frac{(\epsilon_1\kappa)^{\alpha p*} A c^{\alpha} (T+1)^{\alpha(p^- - p^+)/p^-}}{\alpha (p^+)^{\alpha}} < \widehat{M}(\delta[\check{u}]) < \frac{(\epsilon_2\kappa)^{\alpha p*} A c^{\alpha} (T+1)^{\alpha(p^- - p^+)/p^-}}{\alpha (p^+)^{\alpha}}.$$

Proof. We are going to apply Theorem 2.5 to problem (1.1). Take $X = W$ and put Φ, Ψ and I_λ as given in (2.7), (2.8) and (2.9), respectively. Clearly, the regularity assertions are required on Φ and Ψ. Moreover, the critical points of I_λ are exactly the solutions of problem (1.1). Now, put

$$r_1 = \frac{(\epsilon_1 \kappa)^{\alpha p^*} A c^\alpha (T+1)^{\alpha(p^- - p^+)/p^-}}{\alpha (p^+)^\alpha} \quad \text{and} \quad r_2 = \frac{(\epsilon_2 \kappa)^{\alpha p^*} A c^\alpha (T+1)^{\alpha(p^- - p^+)/p^-}}{\alpha (p^+)^\alpha}$$

and define the function $\overline{u} : W \to \mathbb{R}$ as follows.

$$\overline{u}(k) = \begin{cases} b \ \text{ if } k \in \mathbb{Z}[1,T], \\ 0 \ \text{ otherwise.} \end{cases}$$

Then, we deduce that $\Psi(\overline{u}) = \sum_{k=1}^{T} F(k, \overline{u}(k)) = \sum_{k=1}^{T} F(k, b)$ and

$$\Phi(\overline{v}) \ = \ \widehat{M}\left(\int_0^b \left(a(0,\xi) + a(T,\xi) \right) \xi \, d\xi \right).$$

By (2.6), one has

$$\|u\|_\infty := \max_{k \in \mathbb{Z}[1,T]} |u(k)| \le \frac{1}{\kappa} \|u\|,$$

for each $u \in W$, where

$$\kappa := \frac{2}{(T+1)^{(p^- - 1)/p^-}}.$$

We also use the following notation.

$$\beta^{\alpha p*} := \begin{cases} \beta^{\alpha p^+} & \text{if } \beta > 1, \\ \beta^{\alpha p^-} & \text{if } 0 < \beta < 1. \end{cases}$$

Thus, we obtain

$$\|u\|_\infty \le \frac{r_1^{1/\alpha p*}}{\kappa} \left(\frac{\alpha (p^+)^\alpha}{A c^\alpha (T+1)^{\alpha(p^- - p^+)/p^-}} \right)^{1/\alpha p^*} = \epsilon_1,$$

for all $u \in W$ such that

$$\|u\| \le r_1^{1/\alpha p*} \left(\frac{\alpha (p^+)^\alpha}{A c^\alpha (T+1)^{\alpha(p^- - p^+)/p^-}} \right)^{1/\alpha p^*}$$

and

$$\|u\|_\infty \le \frac{r_2^{1/\alpha p*}}{\kappa} \left(\frac{\alpha (p^+)^\alpha}{A c^\alpha (T+1)^{\alpha(p^- - p^+)/p^-}} \right)^{1/\alpha p^*} = \epsilon_2,$$

for all $u \in W$ such that

$$\|u\| \le r_2^{1/\alpha p*} \left(\frac{\alpha (p^+)^\alpha}{A c^\alpha (T+1)^{\alpha(p^- - p^+)/p^-}} \right)^{1/\alpha p^*}.$$

Therefore, one has

$$\sup_{u\in\Phi^{-1}(-\infty,r_1]}\Psi(u)=\sup_{\Phi(u)\leq r_1}\sum_{k=1}^{T}F(k,u(k))\leq\sum_{k=1}^{T}\max_{|t|\leq\epsilon_1}F(k,t)$$

as well as

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)\leq\sum_{k=1}^{T}\max_{|t|\leq\epsilon_2}F(k,t).$$

Hence, it follows that

$$\begin{aligned}\beta(r_1,r_2) &\leq \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)-\Psi(\overline{u})}{r_2-\Phi(\overline{u})}\\ &\leq \frac{\sum\limits_{k=1}^{T}\max\limits_{|t|\leq\epsilon_2}F(k,t)-\sum\limits_{k=1}^{T}F(k,b)}{\frac{(\epsilon_2\kappa)^{\alpha p*}Ac^{\alpha}(T+1)^{\alpha(p^--p^+)/p^-}}{\alpha(p^+)^{\alpha}}-\widehat{M}\left(\int_0^b(a(0,\xi)+a(T,\xi))\,\xi\,d\xi\right)}\\ &= \Theta_{\epsilon_2}(b).\end{aligned}\tag{3.1}$$

Moreover, arguing as before, one has

$$\begin{aligned}\rho(r_1,r_2) &\geq \frac{\Psi(\overline{u})-\sup\limits_{u\in\Phi^{-1}(-\infty,r_1]}\Psi(u)}{\Phi(\overline{u})-r_1}\\ &\geq \frac{\sum\limits_{k=1}^{T}F(k,b)-\sum\limits_{k=1}^{T}\max\limits_{|t|\leq\epsilon_1}F(k,t)}{\widehat{M}\left(\int_0^b(a(0,\xi)+a(T,\xi))\,\xi d\xi\right)-\frac{(\epsilon_1\kappa)^{\alpha p*}Ac^{\alpha}(T+1)^{\alpha(p^--p^+)/p^-}}{\alpha(p^+)^{\alpha}}}\\ &= \Theta_{\epsilon_1}(b).\end{aligned}\tag{3.2}$$

Combining (3.1) and (3.2), we obtain $\beta(r_1,r_2)<\rho(r_1,r_2)$. Furthermore, again from $\Theta_{\epsilon_1}(b)$ and $\Theta_{\epsilon_2}(b)$, one has $\lambda\in\left(\frac{1}{\Theta_{\epsilon_1}(b)},\frac{1}{\Theta_{\epsilon_2}(b)}\right)$. Therefore, the functional I_λ admits at least one critical point $\check{u}$, which is solution of problem (1.1) such that $r_1<\Phi(\check{u})<r_2$, that is

$$\frac{(\epsilon_1\kappa)^{\alpha p*}Ac^{\alpha}(T+1)^{\alpha(p^--p^+)/p^-}}{\alpha(p^+)^{\alpha}}<\widehat{M}(\delta[\check{u}])<\frac{(\epsilon_2\kappa)^{\alpha p*}Ac^{\alpha}(T+1)^{\alpha(p^--p^+)/p^-}}{\alpha(p^+)^{\alpha}},$$

and the proof is complete. □

4. MULTIPLE SOLUTIONS

In this section, one uses theorems 2.6 and 2.7 in order to get the existence results of multiple solutions of problem (1.1). We introduce firstly some additional assumptions.

$(H5)$ $\lim\limits_{|t|\to\infty} \dfrac{f(k,t)}{|t|^{p^--1}} = 0$, for any $k \in \mathbb{Z}[1,T]$.

$(H6)$ There exist two constants $c_1 > 0$ and $s < \alpha p^+$ such that

$$|f(k,t)| \leq c_1 \left(1 + |t|^{s-1}\right) \text{ for all } (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}.$$

$(H7)$ There exist $t_1, \theta, A, B \in (0,\infty)$ and $\alpha \geq 1$, which satisfy $A \leq B$, $\theta > \dfrac{B}{A}\alpha p^+$, such that

$$0 < \theta F(k,t) := \theta \int_0^t f(k,s)\, ds \leq f(k,t)t \text{ for all } k \in \mathbb{Z}[1,T] \text{ and } t \in \mathbb{R} \text{ with } |t| \geq t_1.$$

$(H8)$ $f(k,t) = \circ(|t|^{\alpha p^+ -1})$, as $t \to 0$ for all $k \in \mathbb{Z}[1,T]$ uniformly.

Our main result in this section is the following theorem.

Theorem 4.1. *Assume hypotheses $(H1) - (H8)$ are satisfied. Then, there exists $\lambda^* > 0$ such that for each $\lambda \in (0,\lambda^*)$, problem (1.1) has at least two nonzero solutions.*

The next lemma proves that I_λ has a mountain pass geometry.

Lemma 4.2. *Assume that the hypotheses of Theorem 4.1 are satisfied. Then,*

(i) There exist $\lambda^*, \nu, \varrho, \rho > 0$ such that for each $\lambda \in (0,\lambda^*)$, one has

$$I_\lambda(u) \geq \varrho > 0 \text{ for all } u \in \partial B_\rho \text{ with } \|u\| = \rho.$$

(ii) There exists $e \in W$ with $\|e\| > \rho$ such that

$$I_\lambda(e) < 0.$$

Proof. (i) We will verify that the functional I_λ satisfies the conditions of Theorem 2.6.

From $(H6)$ and $(H8)$, for any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that

$$|F(k,t)| \leq \varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^s \text{ for all } k \in \mathbb{Z}[1,T] \text{ and } t \in \mathbb{R}. \tag{4.1}$$

Let $B_\nu = \{u \in W \text{ such that } \|u\| < \nu\}$. So by (2.6), one has

$$|u(k)| \leq \frac{(T+1)^{(p^--1)/p^-}}{2}\|u\|, \text{ for any } k \in \mathbb{Z}[1,T].$$

Take $u \in B_\nu$ with $\|u\| \leq 1$. Then, we obtain

$$|u(k)| \leq \frac{(T+1)^{(p^--1)/p^-}}{2}\|u\| \leq 1, \text{ for any } k \in \mathbb{Z}[1,T].$$

Put $\nu = 2(T+1)^{(1-p^-)/p^-}$. For $u \in \partial B_\nu$ with $\|u\| \leq \nu$, by (4.1), $(H2)$-$(H4)$ and Lemmas $2.1-(ii)-(iii)$, it follows that

$$\begin{aligned} I_\lambda(u) &= \widehat{M}(\delta[u]) - \lambda \sum_{k=1}^{T} F(k,u(k)) \\ &\geq \frac{A}{\alpha}(\delta[u])^\alpha - \lambda \sum_{k=1}^{T}\left(\varepsilon |u(k)|^{\alpha p^+} + c_\varepsilon |u(k)|^s\right) \\ &\geq \frac{Ac^\alpha}{\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}\|u\|^{\alpha p^+} - \lambda\varepsilon \sum_{k=1}^{T}|u(k)|^{\alpha p^+} - \lambda c_\varepsilon \sum_{k=1}^{T}|u(k)|^s \\ &\geq \frac{Ac^\alpha}{\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}\|u\|^{\alpha p^+} - \lambda\varepsilon\left(\frac{(T+1)^{(p^--1)/p^-}}{2}\right)^{\alpha p^+} T\|u\|^{\alpha p^+} \\ &- \lambda c_\varepsilon \left(\frac{(T+1)^{(p^--1)/p^-}}{2}\right)^{s} T\|u\|^{s}. \end{aligned}$$

Let $\varepsilon > 0$ be small enough such that $\lambda\varepsilon\left(\frac{(T+1)^{(p^--1)/p^-}}{2}\right)^{\alpha p^+} T < \frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}$. Then, it follows that

$$\begin{aligned} I_\lambda(u) &\geq \frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}\|u\|^{\alpha p^+} - \lambda c_\varepsilon\left(\frac{(T+1)^{(p^--1)/p^-}}{2}\right)^{s} T\|u\|^{s} \\ (4.2) \quad &= \|u\|^{\alpha p^+}\left(\frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}} - \lambda c_\varepsilon\left(\frac{(T+1)^{(p^--1)/p^-}}{2}\right)^{s} T\|u\|^{s-\alpha p^+}\right). \end{aligned}$$

We set

$$\eta(\sigma) = b\lambda\sigma^{s-\alpha p^+}, \ \ \sigma \in (0,1),$$

where $b = c_\varepsilon\left(\frac{(T+1)^{(p^--1)/p^-}}{2}\right)^{s} T$. Since $s < \alpha p^+$, we obtain that $\eta(\sigma) \to \infty$ as $\sigma \to 0^+$. Thus, η possesses a minimum at $u_0 \neq 0$ since $I_\lambda(0) = 0$. Now, we will show that $u_0 \in B_\nu$ and therefore according to Thorem 2.7, I_λ has a local minimum $u_0 \in \overline{B_\nu}$. We deduce that for $u \in \partial B_\rho$ with $\|u\| = \rho$, the following inequality holds.

$$\begin{aligned} I_\lambda(u) &\geq \frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}\|u\|^{\alpha p^+} - \lambda b\|u\|^s \\ &= \left(\frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}\rho^{\alpha p^+-s} - \lambda b\right)\rho^s. \end{aligned}$$

Then, there exists $\lambda^* = \frac{Ac^\alpha}{4b\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}\rho^{\alpha p^+-s}$ such that $\eta(u_0) < \frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}}$ for any $\lambda \in (0,\lambda^*)$. On the other hand, relation (4.2) implies that

$$\inf_{u\in B_\nu} I_\lambda(u) \geq \rho^{\alpha p^+}\left(\frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^--p^+)\alpha}{p^-}} - \eta(\rho)\right) = \varrho > 0 = I_\lambda(0) \geq I_\lambda(u_0).$$

Hence, we infer that $u_0 \in B_\nu$ and $I'_\lambda(u_0) = 0$. Choosing $\nu = \rho$, then for each $\lambda \in (0, \lambda^*)$ there exists $\lambda^* = \frac{Ac^\alpha}{4b\alpha(p^+)^\alpha}(T+1)^{\frac{(p^- - p^+)\alpha}{p^-}}\rho^{\alpha p^+ - s}$, $\rho = 2(T+1)^{(1-p^-)/p^-}$ and

$$\varrho = \rho^{\alpha p^+}\left(\frac{Ac^\alpha}{2\alpha(p^+)^\alpha}(T+1)^{\frac{(p^- - p^+)\alpha}{p^-}} - \eta(\rho)\right)$$

such that $I_\lambda(u) \geq \varrho > 0$ for all $u \in \partial B_\rho$ with $\|u\| = \rho$.

(ii) By assumption $(H7)$, one has

$$F(k,t) \geq C|t|^\theta, \tag{4.3}$$

for all $(k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$ and for some constant $C > 0$.

From $(H4)$, we deduce as $t \geq t^* > 0$ since $A \leq B$ and $\alpha \geq 1$, that

$$\widehat{M}(t) \leq \frac{B}{\alpha}t^\alpha \leq \frac{B}{\alpha}t^{\frac{B}{A}\alpha}. \tag{4.4}$$

Thus, it follows from (4.3) and (4.4) that for $v \in W \setminus \{0\}$ and $t > 1$,

$$\begin{aligned} I_\lambda(tv) &= \widehat{M}(\delta[tv]) - \lambda\sum_{k=1}^{T} F(k, tv(k)) \\ &\leq \frac{B}{\alpha}t^{\frac{B}{A}\alpha p^+}(\delta[v])^\alpha - \lambda C t^\theta \sum_{k=1, |v(k)|>t_1}^{T} |v(k)|^\theta + \lambda M T, \end{aligned}$$

with $M := \max\{|F(k,t)| : k \in \mathbb{Z}[1,T],\ |t| \leq t_1\}$.
Since $\theta > \dfrac{B}{A}\alpha p^+$, we infer that $I_\lambda(tv) \to -\infty$ as $t \to \infty$. Thus, there exists a sufficiently large $t_0 > \nu$ such that $u = t_0 v \in W$ and $I_\lambda(u) < 0$ with $\|u\| > \rho$. So, it follows that $\inf\limits_{u \in \partial B_\nu} I_\lambda(u) > \max\{I_\lambda(u_0), I_\lambda(u_1)\}$. Then, by Theorem 2.6, the functional I_λ has a second critical point $\hat{u}$. □

Lemma 4.3. *Assume the hypotheses of Theorem* 4.1 *are satisfied. Then, I_λ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset W$ be a Palais-Smale sequence such that

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to \infty.$$

We first show that $\{u_n\}$ is bounded in W. For this, assume by contradiction that $\{u_n\}$ is unbounded, so $\|u_n\| \to \infty$ as $n \to \infty$. Thus, assuming that $\|u_n\| > 1$ for n large enough, we deduce by condition $(H2)$ that

$$\begin{aligned} & I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle \\ & \geq A\int_0^{\delta[u_n]} \xi^{\alpha-1}d\xi - \frac{Bp^+}{\theta}(\delta[u_n])^\alpha + \lambda\sum_{k=1}^{T}\left(\frac{1}{\theta}f(k,u_n(k))u_n(k) - F(k,u_n(k))\right) \\ & \geq \left(\frac{A}{\alpha} - \frac{Bp^+}{\theta}\right)(\delta[u_n])^\alpha + \lambda\sum_{k=1}^{T}\left(\frac{1}{\theta}f(k,u_n(k))u_n(k) - F(k,u_n(k))\right). \end{aligned}$$

Put

$$M=\max\left\{\left|\frac{1}{\theta}f(k,t)t-F(k,t)\right|:k\in\mathbb{Z}\left[1,T\right],\ |t|\leq t_1\right\}.$$

Thus, we get by $(H7)$ that

$$\begin{aligned}\left(\frac{A}{\alpha}-\frac{Bp^+}{\theta}\right)(\delta[u_n])^\alpha &\leq I_\lambda(u_n)-\frac{1}{\theta}\left\langle I'_\lambda(u_n),u_n\right\rangle\\ &- \lambda\sum_{k=1,|u_n(k)|>t_1}^{T}\left(\frac{1}{\theta}f(k,u_n(k))u_n(k)-F(k,u_n(k))\right)+\lambda MT\\ &\leq I_\lambda(u_n)-\frac{1}{\theta}\left\langle I'_\lambda(u_n),u_n\right\rangle+\lambda MT,\end{aligned}$$

for n large enough. From $(H2)$, $(H3)$ and Lemma 2.1(i), we obtain

$$\left(\frac{A}{\alpha}-\frac{Bp^+}{\theta}\right)\frac{c^\alpha}{(p^+)^\alpha}\|u_n\|^{\alpha p^-}\leq I_\lambda(u_n)-\frac{1}{\theta}\left\langle I'_\lambda(u_n),u_n\right\rangle+K(\alpha,T)T^\alpha+\lambda MT.$$

Since $\theta>\dfrac{B}{A}\alpha p^+$ and $\alpha p^->1$, this last inequality is absurd and so $\{u_n\}$ is bounded in W. Thus, up to a subsequence $u_n\rightharpoonup u$ weakly in W and

$$\delta[u_n]=\sum_{k=1}^{T+1}\int_0^{|\triangle u_n(k-1)|}a(k-1,\xi)\xi d\xi\to m_0 \text{ as } n\to\infty. \tag{4.5}$$

If $m_0=0$, then $\|u_n\|\to 0$ as $n\to\infty$ and the proof is complete. Assume that $m_0>0$ and show that $\|u_n-u\|\to 0$ as $n\to\infty$. From $I_\lambda=\Phi-\lambda\Psi$, one has

$$\langle\Phi'(u_n),u_n-u\rangle=\langle I'_\lambda(u_n),u_n-u\rangle+\lambda\sum_{k=1}^{T}f(k,u_n(k))(u_n(k)-u(k)).$$

Therefore,

(4.6)

$$\begin{cases}M(\delta[u_n])\displaystyle\sum_{k=1}^{T+1}(a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1)-a(k-1,|\triangle u_n(k-1)|)\\ \times\triangle u_n(k-1))\triangle(u_n-u)(k-1)\\ =\langle I'_\lambda(u_n),u_n-u\rangle-M\left(\delta[u_n]\right)\displaystyle\sum_{k=1}^{T+1}a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1)\triangle(u_n-u)(k-1)\\ +\lambda\displaystyle\sum_{k=1}^{T}f(k,u_n(k))(u_n(k)-u(k)).\end{cases}$$

Since $u_n\rightharpoonup u$ in W as $n\to\infty$, we get

$$\begin{cases}\langle I'_\lambda(u_n),u_n-u\rangle\to 0,\\ \displaystyle\sum_{k=1}^{T+1}a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1)\triangle(u_n-u)(k-1)\to 0.\end{cases} \tag{4.7}$$

Besides, by $(H5)$ there exists a constant $c_1 > 0$ such that

$$|f(k,t)| \leq c_2 \left(1 + |t|^{p^- -1}\right) \text{ for all } (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}.$$

Then, by the discrete Hölder inequality and lemma 2.1(iii), we obtain

$$\begin{aligned}
&\sum_{k=1}^{T} |f(k,u_n(k))||(u_n(k)-u(k))| \\
&\leq c_2 \sum_{k=1}^{T} |u_n(k)-u(k)| + c_2 \sum_{k=1}^{T} |u_n(k)|^{p^- -1}|u_n(k)-u(k)| \\
&\leq c_2 \frac{(T+1)^{(p^- -1)/p^-}}{2} T \left(1 + \left(\frac{(T+1)^{(p^- -1)/p^-}}{2}\right)^{p^- -1} \|u_n\|^{p^- -1}\right) \|u_n - u\|.
\end{aligned}$$

Since $u_n \rightharpoonup u$ in W as $n \to \infty$, one has

$$\lim_{n\to\infty} \sum_{k=1}^{T} |f(k,u_n(k))||u_n(k)-u(k)| = 0. \tag{4.8}$$

On the other hand, by the continuity of M, it follows from (4.5)-(4.8) that

$$\begin{aligned}
&\lim_{n\to\infty} M(\delta[u_n]) \sum_{k=1}^{T+1} (a(k-1,|\triangle u_n(k-1)|)\triangle u_n(k-1) - a(k-1,|\triangle u_n(k-1)|) \\
&\times \triangle u_n(k-1))\triangle(u_n-u)(k-1) = 0,
\end{aligned}$$

which means that

$$\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0.$$

Therefore, by the above equality, we deduce that

$$\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0.$$

But the operator Φ' has the (S_+) property and so it follows that $u_n \to u$ strongly in W. This complete the proof of Lemma 4.3. □

Proof of Theorem 4.1 From lemmas 4.2, 4.3 and the fact that $I_\lambda(0) = 0$, the functional I_λ satisfies the assumptions of Theorem 2.6. Consequently, the functional I_λ admits at least two critical points u_0, $\hat{u}$, which are solutions of problem (1.1) and the proof is complete.

REFERENCES

[1] R.P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, *Adv. Differ. Equ.*, 2:93–99, 2005.

[2] R.P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Anal.*, 58:69–73, 2004.

[3] G.A. Afrouzi, A. Hadjian and S. Heidarkhani, Non-trivial solutions for a two-point boundary value problem, *Ann. Polo. Math.*, 108:75–84, 2013.

[4] A. Arosio and S. Pannizi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.*, 348:305–330, 1996.

[5] C. Bereanu, P. Jebelean and C. Serban, Periodic and Neumann problems for discrete $p(\cdot)$-Laplacian, *J. Math. Anal. Appl.*, 399: 75–87, 2013.

[6] G. Bonanno, S. Heidarkhani and D. O'Regan, Non-trivial solutions for Sturm-Liouville systems via a local minimum theorem for functional, *Bull. Aust. Math. Soc.*, 89:8–18, 2014.

[7] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.*, 75:2992–3007, 2012.

[8] G. Bonanno and P.F. Pizzimenti, Neumann boundary value problems with not coercive potential, *Mediterr. J. Math.*, 9:601–609, 2012.

[9] G. Bonanno, B. DI Bella and D. O'Regan, Non-trivial solutions for nonlinear fourth-order elastic beam equations, *Comput. Math. Appl.*, 62:1862–1869, 2011.

[10] G. Bonanno and P. Candito, Infinitely many solutions for a class of discrete nonlinear boundary value problems, *Appl. Anal.*, 884:605–616, 2009.

[11] A. Cabada, A. Iannizzotto and S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.*, 356:418–428, 2009.

[12] P. Candito and G. D'Agui, Three solutions for a discrete nonlinear Neumann problem involving the p-Laplacian, *Adv. Differ. Equ.*, 2010:Article ID 862016, 2010.

[13] P. Candito and N. Giovannelli, Multiple solutions for a discrete boundary value problem involving the p-Laplacian, *Math. Appl. Comput.*, 56:959–964, 2008.

[14] X. Cai and J. Yu, Existence theorems for second-order discrete boundary value problems, *J. Math. Anal. Appl.*, 320:649–661, 2006.

[15] G. Caristi, S. Heidarkhani, A. Salari and S.A. Tersian, Multiple solutions for degenerate nonlocal problems,*Appl. Math. Lett.*, 84:26–33, 2018.

[16] G.F. Carrier, A note on the vibrating string, *Quart. Appl. Math.*, 7:97–101, 1949.

[17] G.F. Carrier, On the nonlinear vibration problem of the elastic string, *Quart. Appl. Math.*, 3:157–165, 1945.

[18] M.M. Cavalcanti, V.N. Cavalcanti and J.A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation, *Adv. Differ. Equ.*, 6:701–730, 2001.

[19] O. Chakrone, E.M. Hssini, M. Rahmani and O. Darhouche, Multiplicity results for a p-Laplacian discrete problems of Kirchhoff type, *Appl. Math. Comput.*, 276:310–315, 2016.

[20] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.*, 66(4):1383–1406, 2006.

[21] M. Galewski and R. Wieteska, Existence and multiplicity results for boundary value problems connected with the discrete $p(\cdot)$-Laplacian on weighted finite graphs, *Appl. Math. Comput.*, 290:376–391, 2016.

[22] M. Galewski, G.M. Bisci and R. Wieteska, Existence and multiplicity of solutions to discrete inclusions with the $p(k)$-Laplacian problem, *J. Differ. Equ. Appl.*, 21(10):887–903, 2015.

[23] M. Galewski and S. Glab, On the discrete boundary value problem for anisotropic equation, *Math. Anal. Appl.*, 386(2):956–965, 2012.

[24] A. Guiro, I. Nyanquini and S. Ouaro, On the solvability of discrete nonlinear Neumann problems involving the $p(x)$-Laplacian, *Adv. Differ. Equ.*, 32:1–14, 2011.

[25] A. Guiro, B. Koné and S. Ouaro, Weak homoclinic solutions of anisotropic difference equation with variable exponents, *Adv. Differ. Equ.*, 154:1–13, 2012.

[26] A. Hadjian and M. Bagheri, Non-trivial solutions for a discrete nonlinear boundary value problem with ϕ_c-Laplacian, *CJMS.*, 10(2):235–243, 2021.

[27] S. Heidarkhani, A. Ghobadi and G. Caristi, Critical Point Approaches for Discrete Anisotropic Kirchhoff Type Problems, *Adv. Appl. Math.*, 13(4):58–65, 2022.

[28] S. Heidarkhani, G. Caristi and A. Salari, Perturbed Kirchhoff-type p-Laplacian discrete problems, *Collect. Math.*, 68, No. 3:401–418, 2017.

[29] P. Jebelean and C. Serban, Ground state periodic solutions for difference equations with discrete $p(x)$-Laplacian, *Appl. Math. Comput.*, 217:9820–9827, 2011.

[30] I.H. Kim and Y.H. Kim, Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents, *manuscripta math.*, 147:169–191, 2015.

[31] G. Kirchhoff, Vorlesungen uber mathematische Physik: Mechanik, *Teubner*, Leipzig, 1876.

[32] B. Koné, I. Nyanquini and S. Ouaro, Weak solutions to discrete nonlinear two-point boundary value problems of Kirchhoff type, *Electron. J. Differ. Equ.*, 2015(105):1–10, 2015.

[33] B. Koné and S. Ouaro, Weak solutions for anisotropic discrete boundary value problems, *J. Differ. Equ. Appl.*, 16(2):1–11, 2010.

[34] B.A. Kyelem, S. Ouaro and M. Zoungrana, Classical solutions for discrete potential boundary value problems with generalized Leray-Lions type operator and variable exponent, *Electron. J. Differ. Equ.*, 2017(109):1–16, 2017.

[35] J.L. Lions, On some questions in boundary value problems of mathematical physics. In Contemporary Developpements in Continuum Mechanics and Partial Differential Equations, *Proc. Internat. Sympos. Inst. Mat. Univ. Fed. Rio de Janeiro, North-Holland Math Stud.*, 30:284–345, 1978.

[36] R.A. Mashiyev, Z. Yucedag and S. Ogras, Existence and multiplicity of solutions for a Dirichlet problem involving the discrete $p(x)$-Laplacian operator, *Electron. J. Qual. Theory Differ. Equ.*, 67:1–10, 2011.

[37] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, *Springer, Berlin*, 1989.

[38] M. Mihailescu, V. Radulescu and S. Tersian, Homoclinic solutions of difference equations with variable exponents, *Topol. Methods Nonlinear Anal.*, 38:277–289, 2011.

[39] M. Mihailescu, V. Radulescu and S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems, *J. Differ. Equ. Appl.*, 15(6):557–567, 2009.

[40] M.K. Moghadam and S. Heidarkhani, Existence of a non-trivial solution for nonlinear difference equations, *J. Differ. Equ. Appl.*, 6(4):517–525, 2014.

[41] R. Narashima, Nonlinear vibration of an elastic string, *J. Sound Vibration.*, 8:134–146, 1968.

[42] S. Ouaro and M. Zoungrana, Multiplicity of solutions to discrete inclusions with the p(k)-Laplace Kirchhoff type equations, *Asia Pacif. J. Math.*, 5(1):27–49, 2018.

[43] D.W. Oplinger, Frequency response of a nonlinear stretched string, *J. Acoustic Soc. Amer.*, 32:1529-1538, 1960.

[44] A. Ourraoui and A. Ayoujil, On a class of non-local discrete boundary value problem, *Arab J. Math. Sci.*, 28:No. 2, 130–141, 2022.

[45] K.R. Rajagopal and M. Ruzicka, Mathematical modeling of electrorheological materials, *Contin. Mech. Thermodyn.*, 13:59–78, 2001.

[46] M. Ruzicka, Electrorheological Fluids: Modelling and Mathematical Theory, *Lecture Notes in Mathematics 1748.*, Springer, Berlin, 2002.

[47] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.*, 113:401–410, 2000.

[48] J. Yang and J. Liu, Nontrivial solutions for discrete Kirchhoff type problems with resonance via critical groups, *Adv. Differ. Equ.*, 2013(1):1–14, 2013.

[49] Z. Yucedag, Solutions for a discrete boundary value problem involving Kirchhoff type equation via variational methods, *TWMS J. App. Eng. Math.*, 8(1):144–154, (2018).

[50] Z. Yucedag, Existence of solutions for anisotropic discrete boundary value problems of Kirchhoff type, *Int. J. Differ. Equ. Appl.*, 13(1):1–15, 2014.

[51] E. Zeidler, Nonlinear Functional Analysis and its Applications II/B, *Berlin-Heidelberg-New York*, 1990.

[52] E. Zeidler, Nonlinear Functional Analysis and its Applications III, *Springer*, New York, 1985.

[53] D. Zhang, Multiple solutions of nonlinear impulsive differential equations with Dirichlet boundary conditions via variational method, *Results. Math.*, 63:611–628, 2013.

[54] D. Zhang and B. Dai, Infinitely many solutions for a class of nonlinear impulsive differential equations with periodic boundary conditions, *Comput. Math. Appl.*, 61:3153–3160, 2011.

[55] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR. Izv.*, 29:33–66, 1987.

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