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DECOMPOSING A CONJUGATE FIXED-POINT PROBLEM INTO MULTIPLE FIXED-POINT PROBLEMS

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ABSTRACT. Converting nonlinear boundary value problems to fixed point problems of an integral operator with a Green's function kernal is a common technique to find or approximate solutions of boundary value problems. It is often difficult to apply Banach's Theorem since it is challenging to find an initial estimate with a contractive constant less than one. We decompose the integral operator associated to a conjugate boundary value problem creating multiple fixed point problems which have contractive constants less than one. We then provide conditions for the original boundary value problem to have a solution that can be found by iteration using the decomposition through a fixed point of a real valued function which matches the fixed points of our decomposition.

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1. Introduction

The Banach Fixed Point Theorem [2] is a powerful tool that can be used to find solutions of nonlinear initial and boundary value problems that have been converted to fixed point problems. The Picard-Lindelöf Theorem (see the fixed point books by Zeidler [5] or Dugundji-Granas [3]) is used to find unique solutions for a first order nonlinear initial value problem where the key is to restrict the interval so the operator whose fixed points are solutions on the interval is k-contractive. This is the first manuscript that we are aware of that follows an approach similar to the initial value problem approach by Picard-Lindelöf for boundary value problems, that is, restricting the interval so an associated operator is k-contractive.

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Consider the second-order conjugate boundary value problem given by

$$y''(t) + h(y(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

$$y(0) = 0 = y(1), \tag{1.2}$$

where $h : [0,\infty) \to [0,\infty)$ is differentiable. The Green's function for (1.1), (1.2) is given by

$$H(t,s) = \begin{cases} t(1-s) & \text{if } 0 \le t \le s \le 1, \\ s(1-t) & \text{if } 0 \le s \le t \le 1; \end{cases}$$

and the fixed points of the operator T defined by

$$Ty(t) = \int_0^1 H(t,s)h(y(s))\,ds \tag{1.3}$$

are the solutions of (1.1), (1.2). Define the cone P of $C[0,1]$ by

$$P = \{y \in C[0,1] \ : \ y(0) = 0 = y(1),\ y \text{ is concave and symmetric}\}.$$

Following an argument similar to that of Avery-Henderson [1], we will show that T is symmetric then we will decompose the matrix T using symmetry arguments. For $y \in P$ and $t \in [0,1]$ we have

$$\begin{aligned}
(Ty)(1-t) &= \int_0^1 H(1-t,s)h(y(s))\,ds \\
&= \int_0^{1-t} H(1-t,s)h(y(s))\,ds + \int_{1-t}^1 H(1-t,s)h(y(s))\,ds \\
&= \int_0^{1-t} sth(y(s))\,ds + \int_{1-t}^1 (1-t)(1-s)h(y(s))\,ds \\
&= \int_0^{1-t} sth(y(1-s))\,ds + \int_{1-t}^1 (1-t)(1-s)h(y(1-s))\,ds \\
&= \int_1^t -(1-u)th(y(u))\,du + \int_t^0 -(1-t)ug(y(u))\,du \\
&= \int_t^1 (1-s)th(y(s))\,ds + \int_0^t s(1-t)h(y(s))\,ds \\
&= (Ty)(t),
\end{aligned}$$

thus we have that $T : P \to P$ since for all $y \in P$ we just verified that T is symmetric, that is $Ty(1-t) = Ty(t)$ for all $t \in [0,1]$, and clearly Ty is concave since $(Ty)''(t) = -h(y(t) < 0$. Also notice that if $y \in P$ with $y(t) = (Ty)(t)$ for all $t \in \left[0, \frac{1}{2}\right]$ then we have that y is a fixed point of T by the symmetry of both y and Ty. For $y \in P$ define the concave functional $\alpha : P \to [0,\infty)$ by

$$\alpha(y) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} y(t) = y\left(\frac{1}{4}\right), \tag{1.4}$$

and for $0 < r < R$ define

$$P(\alpha, r, R) = \{y \in P \ : \ r \leq \alpha(y), \|y\| \leq R\},$$

which is referred to as a Leggett-Williams [4] functional wedge. We will search for fixed points of T in $P(\alpha, r, R)$. Note that functional wedges are closed, convex subsets of P. For $t, s \in [0, 1]$ define

$$D(t, s) = \min\{t, s\}$$

and note that for $y \in P$ we can write Ty as

$$\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)h(y(s))\,ds \\
&= \int_0^{\frac{1}{4}} H(t,s)h(y(s))\,ds + \int_{\frac{1}{4}}^{\frac{3}{4}} H(t,s)h(y(s))\,ds + \int_{\frac{3}{4}}^1 H(t,s)h(y(s))\,ds
\end{aligned}$$

and for $t \in [0, \frac{1}{4}]$ we can define the operator

$$\begin{aligned}
(Jy)(t) &= \int_0^{\frac{1}{4}} H(t,s)h(y(s))\,ds + \int_{\frac{3}{4}}^1 H(t,s)h(y(s))\,ds \\
&= \int_0^t s(1-t)h(y(s))\,ds + \int_t^{\frac{1}{4}} t(1-s)h(y(s))\,ds + \int_{\frac{3}{4}}^1 t(1-s)h(y(1-s))\,ds \\
&= \int_0^t s(1-t)h(y(s))\,ds + \int_t^{\frac{1}{4}} t(1-s)h(y(s))\,ds + \int_{\frac{1}{4}}^0 -tuh(y(u))\,du \\
&= \int_0^t s(1-t)h(y(s))\,ds + \int_t^{\frac{1}{4}} t(1-s)h(y(s))\,ds + \int_0^{\frac{1}{4}} sth(y(s))\,ds \\
&= \int_0^t sh(y(s))\,ds + \int_t^{\frac{1}{4}} th(y(s))\,ds \\
&= \int_0^{\frac{1}{4}} D(t,s)\ h(y(s))\,ds
\end{aligned}$$

and for $t \in [\frac{1}{4}, \frac{1}{2}]$ we can define the operator

$$\begin{aligned}
(Ky)(t) &= \int_{\frac{1}{4}}^{\frac{3}{4}} H(t,s)h(y(s))\,ds \\
&= \int_{\frac{1}{4}}^{t} s(1-t)h(y(s))\,ds + \int_{t}^{\frac{1}{2}} t(1-s)h(y(s))\,ds + \int_{\frac{1}{2}}^{\frac{3}{4}} t(1-s)h(y(1-s))\,ds \\
&= \int_{\frac{1}{4}}^{t} s(1-t)h(y(s))\,ds + \int_{t}^{\frac{1}{2}} t(1-s)h(y(s))\,ds + \int_{\frac{1}{2}}^{\frac{1}{4}} -tuh(y(u))\,du \\
&= \int_{\frac{1}{4}}^{t} s(1-t)h(y(s))\,ds + \int_{t}^{\frac{1}{2}} t(1-s)h(y(s))\,ds + \int_{\frac{1}{4}}^{\frac{1}{2}} tsh(y(s))\,ds \\
&= \int_{\frac{1}{4}}^{t} sh(y(s))\,ds + \int_{t}^{\frac{1}{2}} th(y(s))\,ds \\
&= \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y(s))\,ds.
\end{aligned}$$

Utilizing the operators J and K as well as symmetry we can write the operator T in the form

$$(Ty)(t) = \begin{cases} (Jy)(t) + 4t(Ky)(\frac{1}{4}) & 0 \le t \le \frac{1}{4} \\ (Jy)(\frac{1}{4}) + (Ky)(t) & \frac{1}{4} \le t \le \frac{1}{2} \\ (Ty)(1-t) & \frac{1}{2} < t \le 1 \end{cases}$$

and in what follows we will show how fixed points of operators associated to J and K will lead to a fixed point of the operator T which is a solution of our original boundary value problem (1.1), (1.2). Moreover we will show how one can use the bisection method to create an iterative scheme to approximate a solution of the conjugate boundary value problem (1.1), (1.2).

2. **Preliminaries**

Let

$$Q = \left\{ y \in C\left[\frac{1}{4}, \frac{1}{2}\right] \;:\; y \text{ is non-negative and non-decreasing} \right\},$$

which is a cone in the Banach Space $B_u = C\left[\frac{1}{4}, \frac{1}{2}\right]$ with the sup norm, that is, for $y \in B_u$ let

$$\|y\|_u = \max_{t \in \left[\frac{1}{4}, \frac{1}{2}\right]} |y(t)|.$$

Furthermore, let

$$S = \left\{ y \in C\left[0, \frac{1}{4}\right] \;:\; y \text{ is non-negative and increasing with } y(0) = 0 \right\},$$

which is a cone in the Banach Space $B_\nu = C\left[0, \frac{1}{4}\right]$ with the sup norm, that is, for $y \in B_\nu$ let

$$\|y\|_\nu = \max_{t\in\left[0,\frac{1}{4}\right]} |y(t)|.$$

Let

$$Q[r,R] = \left\{ y \in Q \;:\; r \le y(t) \le R \text{ for all } t \in \left[\frac{1}{4}, \frac{1}{2}\right] \right\}$$

and

$$S_R = \left\{ y \in S \;:\; y(t) \le R \text{ for all } t \in \left[0, \frac{1}{4}\right] \right\}.$$

Our decomposition will involve operators $A_l : S \to S$ defined by

$$A_l y(t) = \int_0^{\frac{1}{4}} D(t,s)h(y(s))\,ds + 4tl = Jy(t) + 4tl \tag{2.1}$$

for each non-negative real number l, and operators $D_m : Q \to Q$ defined by

$$D_m y(t) = m + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y(s))\,ds = m + Ky(t) \tag{2.2}$$

for each non-negative real number m.

Lemma 2.1. *Let τ, R be positive real numbers, $l \in [0,R]$, and*

(A1) $h : [0,R] \to [0,8R]$ be differentiable;
(A3) $|h'(a)| \le \tau < 32$ for all $a \in [0,R]$.

For $a(l,0) \equiv 0$, define the recursive sequence

$$a(l,n+1) = A_l a(l,n)$$

for A_l given in (2.1), then

$$a(l,n) \to a^*(l) \in S_R$$

and for $k_a = \frac{\tau}{32}$,

$$\|a^*(l) - a(l,n)\|_\nu \le \frac{Rk_a^n}{1-k_a}.$$

Proof. Let $y, z \in S_R$ and for each $s \in \left[0, \frac{1}{4}\right]$, let $w(s)$ be between $y(s)$ and $z(s)$ such that

$$h(y(s)) - h(z(s)) = h'(w(s))(y(s) - z(s)).$$

Hence

$$\begin{aligned}
\|A_l y - A_l z\|_\nu &= \max_{t\in[0,\frac{1}{4}]} \left| \int_0^{\frac{1}{4}} D(t,s)h(y(s))\,ds + 4tl - \int_0^{\frac{1}{4}} H(t,s)h(z(s))\,ds - 4tl \right| \\
&\le \max_{t\in[0,\frac{1}{4}]} \int_0^{\frac{1}{4}} D(t,s)\left|h(y(s)) - h(z(s))\right|\,ds \\
&= \int_0^{\frac{1}{4}} D(\frac{1}{4},s)\left|h(y(s)) - h(z(s))\right|\,ds \\
&\le \int_0^{\frac{1}{4}} s\left|h'(w(s))(y(s)-z(s))\right|\,ds \\
&\le \tau \int_0^{\frac{1}{4}} s\|y-z\|_\nu\,ds = \frac{\tau\|y-z\|_\nu}{32}
\end{aligned}$$

and

$$\begin{aligned}
\|A_l y\|_\nu &= \max_{t\in[0,\frac{1}{4}]} \left| \int_0^{\frac{1}{4}} D(t,s)h(y(s))\,ds + 4tl \right| \\
&= \int_0^{\frac{1}{4}} D(\frac{1}{4},s)h(y(s))\,ds + \frac{l}{4} \\
&\le \int_0^{\frac{1}{4}} 8Rs\,ds + \frac{R}{4} \\
&\le \frac{R}{4} + \frac{R}{4} = \frac{R}{2}.
\end{aligned}$$

Therefore $A_l : S_R \to S_R$ is a contraction since $\frac{\tau}{32} < 1$ and S_R is a closed, convex subset of the Banach space B_ν. Therefore by the Banach contraction principle there is an $a^*(l) \in S_R$ such that $a(l,n) \to a^*(l)$. Thus

$$a^*(l)(t) = \int_0^{\frac{1}{4}} H(t,s)h(a^*(l)(s))\,ds + 4tl, \quad t \in \left[0, \frac{1}{4}\right].$$

Also, for any natural numbers n and j by mathematical induction we have

$$\begin{aligned}
\|a(l,n+j+1) - a(l,n+j)\|_\nu &= \|A_l a(l,n+j) - A_l a(l,n+j-1)\|_\nu \\
&\le k_a\|a(l,n+j) - a(l,n+j-1)\|_\nu \\
&\le \cdots \le k_a^j\|a(l,n+1) - a(l,n)\|_\nu
\end{aligned}$$

hence, for any natural numbers n and p, applying the triangle inequality, we have

$$\begin{aligned}
\|a(l,n+p)-a(l,n)\|_\nu &\le \sum_{j=0}^{p-1}\|a(l,n+j+1)-a(l,n+j)\|_\nu \\
&\le \sum_{j=0}^{p-1} k_a^j\|a(l,n+1)-a(l,n)\|_\nu \\
&\le \sum_{j=0}^{\infty} k_a^j\|a(l,n+1)-a(l,n)\|_\nu \\
&= \left(\frac{1}{1-k_a}\right)\|a(l,n+1)-a(l,n)\|_\nu \\
&\le \left(\frac{k_a^n}{1-k_a}\right)\|a(l,1)-a(l,0)\|_\nu \\
&= \left(\frac{k_a^n}{1-k_a}\right)\|a(l,1)\|_\nu \\
&\le \frac{Rk_a^n}{1-k_a}.
\end{aligned}$$

Hence letting $p\to\infty$ we have that

$$\|a^*(l)-a(l,n)\|_\nu \le \frac{Rk_a^n}{1-k_a}.$$

This ends the proof. □

Lemma 2.2. *Let μ, r, R be positive real numbers with $0<r<R$, $m\in\left[0,\frac{R}{4}\right]$, and*

(A1) $h:[0,R]\to[0,8R]$ be differentiable;
(A2) $h(x)\ge 16r$ for $x\in[r,R]$;
(A4) $|h'(b)|\le\mu<\frac{32}{3}$ for all $b\in[0,R]$.

For $b_0\equiv r$ define the recursive sequence

$$b(m,n+1)=D_m b(m,n)$$

for D_m given in (2.2), then

$$b(m,n)\to b^*(m)\in Q[r,R]$$

and for $k_b=\frac{3\mu}{32}$,

$$\|b^*(m)-b(m,n)\|_u \le \frac{Rk_b^n}{1-k_b}.$$

Proof. Let $y,z\in Q[r,R]$ and for each $s\in\left[\frac14,\frac12\right]$, let $w(s)$ be between $y(s)$ and $z(s)$ such that

$$h(y(s))-h(z(s))=h'(w(s))(y(s)-z(s)).$$

Hence

$$\begin{aligned}
\|D_m y - D_m z\|_u &= \max_{t\in[\frac{1}{4},\frac{1}{2}]} \left| m + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y(s))\,ds - m - \int_{\frac{1}{4}}^{\frac{1}{2}} H(t,s)h(z(s))\,ds \right| \\
&\leq \max_{t\in[\frac{1}{4},\frac{1}{2}]} \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)\left|h(y(s)) - h(z(s))\right|\,ds \\
&= \int_{\frac{1}{4}}^{\frac{1}{2}} D(\frac{1}{2},s)\left|h(y(s)) - h(z(s))\right|\,ds \\
&\leq \int_{\frac{1}{4}}^{\frac{1}{2}} s\left|h'(w(s))(y(s) - z(s))\right|\,ds \\
&\leq \mu \int_{\frac{1}{4}}^{\frac{1}{2}} s\|y - z\|_u\,ds = \frac{3\mu\|y - z\|_u}{32},
\end{aligned}$$

$$\begin{aligned}
\|D_m y\|_u &= \max_{t\in[\frac{1}{4},\frac{1}{2}]} \left| m + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y(s))\,ds \right| \\
&= m + \int_{\frac{1}{4}}^{\frac{1}{2}} D(\frac{1}{2},s)h(y(s))\,ds \\
&\leq \frac{R}{4} + \int_{\frac{1}{4}}^{\frac{1}{2}} 8Rs\,ds \\
&= \frac{R}{4} + \frac{24R}{32} = R,
\end{aligned}$$

and

$$\begin{aligned}
\alpha(D_m y) &= \min_{t\in[\frac{1}{4},\frac{1}{2}]} \left| m + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y(s))\,ds \right| \\
&= m + \int_{\frac{1}{4}}^{\frac{1}{2}} D(\frac{1}{4},s)h(y(s))\,ds \\
&\geq \left(\frac{1}{4}\right) \int_{\frac{1}{4}}^{\frac{1}{2}} 16r\,ds = r.
\end{aligned}$$

Therefore $D_m : Q[r,R] \to Q[r,R]$ is a contraction since $\frac{3\mu}{32} < 1$ and $Q[r,R]$ is a closed, convex subset of the Banach space B_u. Therefore by the Banach contraction principle there is an $b^*(m) \in Q[r,R]$ such that $b(m,n) \to b^*(m)$. Thus

$$b^*(m)(t) = m + \int_{\frac{1}{4}}^{\frac{1}{2}} H(t,s)h(b^*(m)(s))\,ds, \quad t \in \left[\frac{1}{4},\frac{1}{2}\right].$$

Also, for any natural numbers n and j by mathematical induction we have

$$\begin{aligned}\|b(m,n+j+1)-b(m,n+j)\|_u &= \|D_m b(m,n+j)-D_m b(m,n+j-1)\|_u \\ &\leq k_b\|b(m,n+j)-b(m,n+j-1)\|_u \\ &\leq \cdots \leq k_b^j\|b(m,n+1)-b(m,n)\|_u\end{aligned}$$

hence, for any natural numbers n and p, applying the triangle inequality, we have

$$\begin{aligned}\|b(m,n+p)-b(m,n)\|_u &\leq \sum_{j=0}^{p-1}\|b(m,n+j+1)-b(m,n+j)\|_u \\ &\leq \sum_{j=0}^{p-1}k_b^j\|b(m,n+1)-b(m,n)\|_u \\ &\leq \sum_{j=0}^{\infty}k_b^j\|b(m,n+1)-b(m,n)\|_u \\ &= \left(\frac{1}{1-k_b}\right)\|b(m,n+1)-b(m,n)\|_u \\ &\leq \left(\frac{k_b^n}{1-k_b}\right)\|b(m,1)-b(m,0)\|_u \\ &\leq \frac{Rk_b^n}{1-k_b}.\end{aligned}$$

Hence letting $p\to\infty$ we have that

$$\|b^*(m)-b(m,n)\|_u \leq \frac{Rk_b^n}{1-k_b}.$$

This ends the proof. □

For $l\in[0,R]$ let

$$m(l)=\int_0^{\frac{1}{4}} D\left(\frac{1}{4},s\right)h(a^*(l)(s))\,ds=\int_0^{\frac{1}{4}} sh(a^*(l)(s))\,ds,$$

and define the real valued function g by

$$g(l)=\int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4},s\right)h(b^*(m(l))(s))\,ds. \tag{2.3}$$

Theorem 2.3. *If $l\in[0,R]$ and $l=g(l)$, then*

$$y_*(t)=\begin{cases} a^*(l)(t) & 0\leq t\leq \frac{1}{4} \\ b^*(m(l))(t) & \frac{1}{4}\leq t\leq\frac{1}{2} \\ y_*(1-t) & \frac{1}{2}<t\leq 1\end{cases}$$

is a solution of (1.1), (1.2).

Proof. Since

$$l = g(l) = \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(l))(s))\, ds$$

and

$$m(l) = \int_{0}^{\frac{1}{4}} D\left(\frac{1}{4}, s\right) h(a^*(l)(s))\, ds,$$

we have that for $t \in \left[0, \frac{1}{2}\right]$

$$\begin{aligned}
y_*(t) &= \begin{cases} a^*(l)(t) & 0 \le t \le \frac{1}{4} \\ b^*(m(l))(t) & \frac{1}{4} \le t \le \frac{1}{2} \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{4}} D(t,s)h(a^*(l)(s))\, ds + 4tl & 0 \le t \le \frac{1}{4} \\ m(l) + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(b^*(m(l))(s))\, ds & \frac{1}{4} \le t \le \frac{1}{2} \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{4}} D(t,s)h(a^*(l)(s))\, ds + 4t \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(l))(s))\, ds & 0 \le t \le \frac{1}{4} \\ \int_0^{\frac{1}{4}} D\left(\frac{1}{4}, s\right) h(a^*(l)(s))\, ds + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(b^*(m(l))(s))\, ds & \frac{1}{4} \le t \le \frac{1}{2} \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{4}} D(t,s)h(a^*(l)(s))\, ds + \int_{\frac{1}{4}}^{\frac{1}{2}} t\, h(b^*(m(l))(s))\, ds & 0 \le t \le \frac{1}{4} \\ \int_0^{\frac{1}{4}} D(t,s)\, h(a^*(l)(s))\, ds + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(b^*(m(l))(s))\, ds & \frac{1}{4} \le t \le \frac{1}{2} \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{4}} D(t,s)h(y_*(s))\, ds + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y_*(s))\, ds & 0 \le t \le \frac{1}{4} \\ \int_0^{\frac{1}{4}} D(t,s)\, h(y_*(s))\, ds + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)h(y_*(s))\, ds & \frac{1}{4} \le t \le \frac{1}{2} \end{cases} \\
&= \begin{cases} \int_0^{\frac{1}{2}} D(t,s)h(y_*(s))\, ds & 0 \le t \le \frac{1}{4} \\ \int_0^{\frac{1}{2}} D(t,s)\, h(y_*(s))\, ds & \frac{1}{4} \le t \le \frac{1}{2} \end{cases} \\
&= Ty_*(t)
\end{aligned}$$

and since $Ty_*(t) = Ty_*(1-t)$ for $t \in \left[\frac{1}{2}, 1\right]$ we have that

$$Ty_*(t) = y_*(t)$$

for all $t \in [0,1]$. Therefore y_* is a fixed point of the operator T and thus a solution of the boundary value problem (1.1), (1.2). □

3. **Main Results**

At this stage we have verified the existence of a solution of the boundary value problem (1.1), (1.2) using iterative techniques, provided we can find a fixed point of the real valued function g by applying Theorem 2.3. In our main results we will show that the real valued function g under the conditions in Theorem 2.3 has a fixed point, so we know that our boundary value problem will have a solution and we'll show how to use the power of iteration to get as close to a solution as desired iteratively. Note that the quantity

$$m(l) = \int_0^{\frac{1}{4}} D\left(\frac{1}{4}, s\right) h(a^*(l)(s))\, ds = \int_0^{\frac{1}{4}} sh(a^*(l)(s))\, ds$$

is calculated from the $a^*(l)$ part of our solution on the interval $\left[0, \frac{1}{4}\right]$, which we will want to approximate. For each natural number n, from Lemma 2.1 we have that $a(l,n) \in S_R$ with

$$\|a^*(l) - a(l,n)\|_\nu \le \frac{Rk_a^n}{1-k_a}$$

where $k_a = \frac{\tau}{32}$ and our approximation of $m(l)$ will be derived from the approximations of $a^*(l)$ by the elements $a(l,n)$. Let

$$m(l,p) = \int_0^{\frac{1}{4}} D\left(\frac{1}{4}, s\right) h(a(l,p)(s))\, ds = \int_0^{\frac{1}{4}} sh(a(l,p)(s))\, ds,$$

the next lemma gives an error bound on our approximation of $m(l)$ by $m(l,p)$.

Lemma 3.1. *Let μ, τ, r, R be positive real numbers with $0 < r < R$, such that*

(A1) $h : [0,R] \to [0,8R]$ be differentiable;
(A2) $h(x) \ge 16r$ for $x \in [r,R]$;
(A3) $|h'(a)| \le \tau < 32$ for all $a \in [0,R]$;
(A4) $|h'(b)| \le \mu < \frac{32}{3}$ for all $b \in [0,R]$.

For $k_a = \frac{\tau}{32}$ and a natural number p,

$$\|b^*(m(l)) - b^*(m(l,p))\|_u \le \frac{\tau R k_a^p}{(32-3\mu)(1-k_a)}$$

and

$$|m(l) - m(l,p)| \le \frac{\tau R k_a^p}{32(1-k_a)}.$$

Proof. Let p be a natural number and for each $s \in \left[0, \frac{1}{4}\right]$, let $w(s)$ be between $a^*(l)(s)$ and $a(l,p)(s)$ such that

$$h(a^*(l)(s)) - h(a(l,p)(s)) = h'(w(s))(a^*(l)(s) - a(l,p)(s))$$

by the mean value theorem, thus from Lemma 2.1 we have

$$\begin{aligned}
|m(l) - m(l,p)| &= \left| \int_0^{\frac{1}{4}} sh(a^*(l)(s))\, ds - \int_0^{\frac{1}{4}} sh(a(l,p)(s))\, ds \right| \\
&\leq \int_0^{\frac{1}{4}} s \left| h(a^*(l)(s)) - h(a(l,p)(s)) \right|\, ds \\
&\leq \int_0^{\frac{1}{4}} s \left| h'(w(s))(a^*(l)(s) - a(l,p)(s)) \right|\, ds \\
&\leq \tau \int_0^{\frac{1}{4}} s \|a^*(l) - a(l,p)\|_\nu\, ds \\
&= \frac{\tau \|a^*(l) - a(l,p)\|_\nu}{32} \\
&\leq \frac{\tau R k_a^p}{32(1-k_a)}.
\end{aligned}$$

By Lemma 2.2 there exist $b^*(m(l)), b^*(m(l,p)) \in Q[r,R]$ such that

$$b^*(m(l)) = D_{m(l)} b^*(m(l)) \quad \text{and} \quad b^*(m(l,p)) = D_{m(l,p)} b^*(m(l,p)).$$

For each $s \in \left[\frac{1}{4}, 1\right]$, let $z(s)$ be between $b^*(m(l))(s)$ and $b^*(m(l,p))(s)$ such that

$$h(b^*(m(l))(s)) - h(b^*(m(l,p))(s)) = h'(z(s))(b^*(m(l))(s) - b^*(m(l,p))(s))$$

by the mean value theorem, hence

$$\begin{aligned}
&\|b^*(m(l)) - b^*(m(l,p))\|_u \\
= &\max_{t \in [\frac{1}{4}, 1]} \left| m(l) + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s) h(b^*(m(l))(s))\, ds \right. \\
&\left. - m(l,p) - \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s) h(b^*(m(l,p))(s))\, ds \right| \\
\leq &|m(l) - m(l,p)| + \max_{t \in \left[\frac{1}{4}, \frac{1}{2}\right]} \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s) \left| h(b^*(m(l))(s)) - h(b^*(m(l,p))(s)) \right|\, ds \\
\leq &|m(l) - m(l,p)| + \int_{\frac{1}{4}}^{\frac{1}{2}} s \left| h'(z(s))(b^*(m(l))(s) - b^*(m(l,p))(s)) \right|\, ds \\
\leq &|m(l) - m(l,p)| + \mu \int_{\frac{1}{4}}^{\frac{1}{2}} s \|b^*(m(l)) - b^*(m(l,p))\|_u\, ds \\
= &|m(l) - m(l,p)| + \frac{3\mu \|b^*(m(l)) - b^*(m(l,p))\|_u}{32} \\
\leq &\frac{\tau R k_a^p}{32(1-k_a)} + \frac{3\mu \|b^*(m(l)) - b^*(m(l,p))\|_u}{32}.
\end{aligned}$$

Therefore

$$\|b^*(m(l)) - b^*(m(l,p))\|_u \leq \frac{\tau R k_a^p}{(32 - 3\mu)(1-k_a)}.$$

This ends the proof. □

In the following theorem we will show that the function g is continuous.

Lemma 3.2. *Let μ, τ, r, R be positive real numbers with $0 < r < R$, such that*

(A1) $h : [0, R] \to [0, 8R]$ *be differentiable;*
(A2) $h(x) \geq 16r$ *for* $x \in [r, R]$*;*
(A3) $|h'(a)| \leq \tau < 32$ *for all* $a \in [0, R]$*;*
(A4) $|h'(b)| \leq \mu < \frac{32}{3}$ *for all* $b \in [0, R]$.

Then the function g given in (2.3) *is uniformly continuous on* $\left[0, \frac{3R}{2}\right]$.

Proof. If we let $l, j \in \left[0, \frac{3R}{2}\right]$, then by Lemma 2.1 there exist $a^*(l), a^*(j) \in S_R$ such that

$$a^*(l) = A_l a^*(l) \quad \text{and} \quad a^*(j) = A_j a^*(l).$$

For each $s \in \left[0, \frac{1}{4}\right]$, let $w(s)$ be between $a^*(l)(s)$ and $a^*(j)(s)$ such that

$$h(a^*(l)(s)) - h(a^*(j)(s)) = h'(w(s))(a^*(l)(s) - a^*(j)(s))$$

by the mean value theorem, thus

$$\begin{aligned}
& \|a^*(l) - a^*(j)\|_\nu \\
= & \max_{t\in[0,\frac{1}{4}]} \left| \int_0^{\frac{1}{4}} D(t,s)h(a^*(l)(s))\, ds + 4tl - \int_0^{\frac{1}{4}} D(t,s)h(a^*(j)(s))\, ds - 4tj \right| \\
\leq & \max_{t\in[0,\frac{1}{4}]} \int_0^{\frac{1}{4}} D(t,s) \left| h(a^*(l)(s)) - h(a^*(j)(s)) \right| \, ds + |l - j| \\
\leq & \int_0^{\frac{1}{4}} s \left| h'(w(s))(a^*(l)(s) - a^*(j)(s)) \right| \, ds + |l - j| \\
\leq & \tau \int_0^{\frac{1}{4}} s\|a^*(l) - a^*(j)\|_\nu \, ds + |l - j| \\
= & \frac{\tau\|a^*(l) - a^*(j)\|_\nu}{32} + |l - j|.
\end{aligned}$$

Therefore

$$\|a^*(l) - a^*(j)\|_\nu \leq \frac{32|l - j|}{32 - \tau},$$

and for

$$m(l) = \int_0^{\frac{1}{4}} sh(a^*(l)(s))\, ds \quad \text{and} \quad m(j) = \int_0^{\frac{1}{4}} sh(a^*(j)(s))\, ds$$

we have

$$\begin{aligned}
|m(l) - m(j)| &= \left| \int_0^{\frac{1}{4}} sh(a^*(l)(s))\, ds - \int_0^{\frac{1}{4}} sh(a^*(j)(s))\, ds \right| \\
&\leq \int_0^{\frac{1}{4}} s\, |h(a^*(l)(s)) - h(a^*(j)(s))|\, ds \\
&\leq \int_0^{\frac{1}{4}} s\, |h'(w(s))(a^*(l)(s) - a^*(j)(s))|\, ds \\
&\leq \tau \int_0^{\frac{1}{4}} s \|a^*(l) - a^*(j)\|_\nu\, ds \\
&= \frac{\tau \|a_{l*} - a_{j*}\|_\nu}{32} \\
&\leq \frac{\tau |l - j|}{32 - \tau}.
\end{aligned}$$

By Lemma 2.2 there exist $b^*(m(l)), b^*(m(j)) \in Q[r, R]$ such that

$$b^*(m(l)) = D_{m(l)} b^*(m(l)) \quad \text{and} \quad b^*(m(j)) = D_{m(j)} b^*(m(j)).$$

For each $s \in \left[\frac{1}{4}, 1\right]$, let $z(s)$ be between $b^*(m(l))$ and $b^*(m(j))$ such that

$$h(b^*(m(l))(s)) - h(b^*(m(j))(s)) = h'(z(s))(b^*(m(l))(s) - b^*(m(j))(s))$$

by the mean value theorem, hence

$$\begin{aligned}
&\|b^*(m(l)) - b^*(m(j))\|_u \\
= \;& \max_{t \in \left[\frac{1}{4}, \frac{1}{2}\right]} \left| m(l) + \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s) g(b^*(m(l))(s))\, ds - m(j) - \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s) g(b^*(m(j))(s))\, ds \right| \\
\leq \;& |m(l) - m(j)| + \max_{t \in [\frac{1}{4}, \frac{1}{2}]} \int_{\frac{1}{4}}^{\frac{1}{2}} D(t,s)\, |h(b^*(m(l))(s)) - h(b^*(m(j))(s))|\, ds \\
\leq \;& |m(l) - m(j)| + \int_{\frac{1}{4}}^{\frac{1}{2}} s\, |g'(z(s))(b^*(m(l))(s) - b^*(m(j))(s))|\, ds \\
\leq \;& |m(l) - m(j)| + \mu \int_{\frac{1}{4}}^{\frac{1}{2}} s \|b^*(m(l)) - b^*(m(j))\|_u\, ds \\
= \;& |m(l) - m(j)| + \frac{3\mu \|b^*(m(l)) - b^*(m(j))\|_u}{32} \\
\leq \;& \frac{\tau |l - j|}{32 - \tau} + \frac{3\mu \|b^*(m(l)) - b^*(m(j))\|_u}{32}.
\end{aligned}$$

Therefore

$$\|b^*(m(l)) - b^*(m(j))\|_u \leq \frac{32\tau |l - j|}{(32 - \tau)(32 - 3\mu)},$$

and from the work embedded the argument above we have

$$\begin{aligned}|g(l)-g(j)| &= \left|\int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4},s\right) h(b^*(m(l))(s))\,ds - \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4},s\right) h(b^*(m(j))(s))\,ds\right| \\ &\leq \frac{3\mu\|b^*(m(l))-b^*(m(j))\|_u}{32} \\ &\leq \frac{3\mu\tau|l-j|}{(32-\tau)(32-3\mu)}.\end{aligned}$$

Therefore g is uniformly continuous on $\left[0,\frac{3R}{2}\right]$. □

We have shown that if $l = g(l)$ then there is a solution given by y_* in Theorem 2.3. In the following Theorem we show how the bisection method can be used to iterate to a fixed point of the real valued function g.

Theorem 3.3. *Let μ,τ,r,R be positive real numbers with $0<r<R$, such that*

(A1) $h:[0,R]\to[0,8R]$ be differentiable;
(A2) $h(x)\geq 16r$ for $x\in[r,R]$;
(A3) $|h'(a)|\leq\tau<32$ for all $a\in[0,R]$;
(A4) $|h'(b)|\leq\mu<\frac{32}{3}$ for all $b\in[0,R]$.

Then there exists a $\psi\in\left[0,\frac{R}{2}\right]$ such that $g(\psi)=\psi$ for g in (2.3)*, and thus*

$$y_*(t)=\begin{cases} a^*(\psi)(t) & 0\leq t\leq\frac{1}{4}\\ b^*(m(\psi))(t) & \frac{1}{4}\leq t\leq\frac{1}{2}\\ y_*(1-t) & \frac{1}{2}\leq t\leq 1\end{cases}$$

is a solution of (1.1), (1.2)*. Moreover, there is a sequence $\{\psi_n\}_{n=0}^{\infty}\subseteq\left[0,\frac{R}{2}\right]$ such that*

$$\psi_n\to\psi$$

with

$$|\psi-\psi_n|\leq\frac{R}{2^{n+2}}.$$

Proof. If we let $l\in\left[0,\frac{R}{2}\right]$, then

$$g(l)=\int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4},s\right) h(b^*(m(l))(s))\,ds \geq \left(\frac{1}{4}\right)\int_{\frac{1}{4}}^{\frac{1}{2}} 16r\,ds = r>0$$

and

$$g(l)=\int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4},s\right) h(b^*(m(l))(s))\,ds \leq \left(\frac{1}{4}\right)\int_{\frac{1}{4}}^{\frac{1}{2}} 8R\,ds = \frac{R}{2}.$$

Hence $g:\left[0,\frac{R}{2}\right]\to\left[0,\frac{R}{2}\right]$ is a continuous real valued function. By the intermediate value theorem applied to

$$f(x)=g(x)-x,$$

there exists a $\psi \in \left[0, \frac{R}{2}\right]$ such that $f(\psi) = 0$, which implies that

$$g(\psi) = \psi$$

and by Theorem 2.3

$$y_*(t) = \begin{cases} a^*(\psi)(t) & 0 \leq t \leq \frac{1}{4} \\ b^*(m(\psi))(t) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ y_*(1-t) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a solution of (1.1), (1.2). Let

$$c_0 = 0, \ d_0 = \frac{R}{2} \text{ and } \ \psi_0 = \frac{c_0 + d_0}{2}$$

then recursively define the sequences $\{c_n\}_{n=0}^{\infty}$, $\{d_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ by

$$c_{n+1} = \psi_n, d_{n+1} = d_n \text{ and } \psi_{n+1} = \frac{c_{n+1} + d_{n+1}}{2}$$

if $g(\psi_n) \geq \psi_n$ and

$$c_{n+1} = c_n, d_{n+1} = \psi_n \text{ and } \psi_{n+1} = \frac{c_{n+1} + d_{n+1}}{2}$$

if $g(\psi_n) < \psi_n$. Observe that for each natural number n that

$$h(c_n) \geq c_n \ \text{ and } \ h(d_n) \leq d_n$$

thus by the intermediate value theorem there is $\psi \in [c_n, d_n]$ such that $h(\psi) = \psi$. By induction we have that

$$d_n - c_n = \frac{d_{n-1} - c_{n-1}}{2} = \frac{d_0 - c_0}{2^n} = \frac{R}{2^{n+1}}$$

and since ψ_n is the midpoint of the interval $[c_n, d_n]$ and $\psi \in [c_n, d_n]$ we have that

$$|\psi - \psi_n| \leq \frac{R}{2^{n+2}}.$$

This ends the proof. □

Our first approximation of the real valued function $g(l)$ given by

$$g(l) = \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(l))(s))\, ds \tag{3.1}$$

will be by the function $g(l, p)$ defined by

$$g(l, p) = \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(l, p))(s))\, ds \tag{3.2}$$

and this will be approximated by the real valued function

$$g(l, p, p) = \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b(m(l, p), p)(s))\, ds. \tag{3.3}$$

Below we provide the tools to determine the sequence ψ_n which converges to ψ where $g(\psi) = \psi$. The key to find the sequence $\{\psi_n\}$ is being able to provide a condition that when verified tells us that $g(\psi_n, p, p) - \psi_n$ and $g(\psi_n) - \psi_n$ are both non-negative or are both non-positive.

Lemma 3.4. *Let n be a whole number, p be a natural number and suppose that*

$$|g(\psi_n) - g(\psi_n, p, p)| \leq |g(\psi_n, p, p) - \psi_n|$$

then

$$\textit{if } g(\psi_n, p, p) \geq \psi_n \textit{ then } g(\psi_n) \geq \psi_n$$

and

$$\textit{if } g(\psi_n, p, p) < \psi_n \textit{ then } g(\psi_n) \leq \psi_n.$$

Proof. Either $g(\psi_n, p, p) \geq \psi_n$ or $g(\psi_n, p, p) < \psi_n$.

Claim 1: if $g(\psi_n, p, p) \geq \psi_n$ then $g(\psi_n) \geq \psi_n$. Since

$$\psi_n - g(\psi_n, p, p) \leq g(\psi_n) - g(\psi_n, p, p) \leq g(\psi_n, p, p) - \psi_n$$

we have $\psi_n < g(\psi_n)$.

Claim 2: if $g($
$psi_n, p, p) < \theta_n$ then $g(\psi_n) < \psi_n$. Since

$$g(\psi_n, p, p) - \psi_n \leq g(\psi_n) - g(\psi_n, p, p) \leq \psi_n - g(\psi_n, p, p)$$

we have $g(\psi_n) \leq \psi_n$. □

In the following lemma we provide the justification for $g(\psi_n) - g(\psi_n, p, p) \to 0$ as $p- > \infty$, hence the left side of the inequality

$$|g(\psi_n) - g(\psi_n, p, p)| \leq |g(\psi_n, p, p) - \psi_n|$$

from Lemma 3.4 goes to zero as p goes to infinity.

Lemma 3.5. *Let n be a whole number and p be a natural number then*

$$|g(\psi_n) - g(\psi_n, p, p)| \leq \frac{(64 - 3\mu)\tau R k_a^p}{32(32 - 3\mu)(1 - k_a)} + \frac{R k_b^{p+1}}{1 - k_b}.$$

Proof. From Lemma 2.2 we have

$$\begin{aligned}
& |g(\psi_n, p) - g(\psi_n, p, p)| \\
= & \left| \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(\psi_n, p))(s))\, ds - \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b(m(\psi_n, p), p)(s))\, ds \right| \\
= & \left| m(\psi_n, p) + \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(\psi_n, p))(s))\, ds \right. \\
& \left. - m(\psi_n, p) - \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b(m(\psi_n, p), p)(s))\, ds \right| \\
= & \left| b^*(m(\psi_n, p)) \left(\frac{1}{4}\right) - b(m(\psi_n, p), p+1) \left(\frac{1}{4}\right) \right| \\
\leq & \, \|b^*(m(\psi_n, p) - b(m(\psi_n, p), p+1)\|_u \\
\leq & \, \frac{Rk_b^{p+1}}{1 - k_b},
\end{aligned}$$

where $k_b = \frac{3\mu}{32}$ and from Lemma 3.1 we have

$$\begin{aligned}
& |g(\psi_n) - g(\psi_n, p)| \\
= & \left| \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(\psi_n))(s))\, ds - \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(\psi_n, p))(s))\, ds \right| \\
= & \left| m(\psi_n) + \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(\psi_n))(s))\, ds \right. \\
& \left. - m(\psi_n, p) - \int_{\frac{1}{4}}^{\frac{1}{2}} D\left(\frac{1}{4}, s\right) h(b^*(m(\psi_n, p))(s))\, ds - (m(\psi_n) - m(\psi_n, p)) \right| \\
= & \left| b^*(\psi_n) \left(\frac{1}{4}\right) - b^*(m(\psi_n, p)) \left(\frac{1}{4}\right) - (m(\psi_n) - m(\psi_n, p)) \right| \\
\leq & \, \|b^*(m(\psi_n)) - b^*(m(\psi_n, p))\|_u + |(m(\psi_n) - m(\psi_n, p))| \\
\leq & \, \frac{\tau R k_a^p}{(32 - 3\mu)(1 - k_a)} + \frac{\tau R k_a^p}{32(1 - k_a)} = \frac{(64 - 3\mu)\tau R k_a^p}{32(32 - 15\mu)(1 - k_a)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|g(\psi_n) - g(\psi_n, p, p)| & \leq |g(\psi_n) - g(\psi_n, p)| + |g(\psi_n, p) - g(\psi_n, p, p)| \\
& \leq \frac{(64 - 3\mu)\tau R k_a^p}{32(32 - 3\mu)(1 - k_a)} + \frac{R k_b^{p+1}}{1 - k_b}.
\end{aligned}$$

This ends the proof. □

Note that for every whole number n we have that

$$\lim_{p \to \infty} |g(\psi_n) - g(\psi_n, p, p)| = 0.$$

In the Theorem below we summarize the iterative scheme which will converge to a solution of (1.1), (1.2).

Theorem 3.6. *Let* μ, τ, r, R *be positive real numbers with* $0 < r < R$*, such that*

(*A1*) $h : [0, R] \to [0, 8R]$ *be differentiable;*
(*A2*) $h(x) \geq 16r$ *for* $x \in [r, R]$*;*
(*A3*) $|h'(a)| \leq \tau < 32$ *for all* $a \in [0, R]$*;*
(*A4*) $|h'(b)| \leq \mu < \frac{32}{3}$ *for all* $b \in [0, R]$*.*

Then there exists an iterative scheme converging to a solution of (1.1), (1.2)*.*

Proof. For natural numbers n and p let

$$y_{n,p}(t) = \begin{cases} a(\psi_n, p)(t) & 0 \leq t \leq \frac{1}{4} \\ b(m(\psi_n, p), p)(t) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ y_{n,p}(1-t) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

From the work in Lemma 3.2 we have

$$\|a^*(\psi) - a^*(\psi_n)\|_\nu \leq \frac{32|\psi - \psi_n|}{32 - \tau}$$

and from the work on Lemma 2.1 we have

$$\|a^*(\psi_n) - a(\psi_n, p)\|_\nu \leq \frac{Rk_a^p}{1 - k_a}$$

for $k_a = \frac{\tau}{32}$, thus we have

$$\begin{aligned} \|a^*(\psi) - a(\psi_n, p)\|_\nu &\leq \|a^*(\psi) - a^*(\psi_n)\|_\nu + \|a^*(\psi_n) - a(\psi_n, p)\|_\nu \\ &\leq \frac{32|\psi - \psi_n|}{32 - \tau} + \frac{Rk_a^p}{1 - k_a}. \end{aligned}$$

From the work in Lemma 3.2 we have

$$\|b^*(m(\psi)) - b^*(m(\psi_n))\|_u \leq \frac{32\tau|\psi - \psi_n|}{(32 - \tau)(32 - 3\mu)}$$

and from the work in Lemma 3.1 we have

$$\|b^*(m(\psi_n)) - b^*(m(\psi_n, p))\|_u \leq \frac{\tau Rk_a^p}{(32 - 3\mu)(1 - k_a)}$$

and from the work in Lemma 2.2 we have

$$\|b^*(m(\psi_n, p)) - b(m(\psi_n, p), p)\|_u \leq \frac{Rk_b^p}{1 - k_b}$$

thus we have

$$\begin{aligned}
& \|b^*(m(\psi)) - b(m(\psi_n, p), p)\|_u \\
\leq\ & \|b^*(m(\psi)) - b^*(m(\psi_n))\|_u + \|b^*(m(\psi_n)) - b^*(m(\psi_n, p))\|_u \\
& + \|b^*(m(\psi_n, p)) - b(m(\psi_n, p), p)\|_u \\
\leq\ & \frac{32\tau|\psi - \psi_n|}{(32-\tau)(32-3\mu)} + \frac{\tau R k_a^p}{(32-3\mu)(1-k_a)} + \frac{R k_b^p}{1-k_b}.
\end{aligned}$$

Therefore

$$\|y_* - y_{n,p}\| \leq \max\{\|a^*(\psi) - a(\psi_n, p)\|_\nu, \|b^*(m(\psi)) - b(m(\psi_n, p), p)\|_u\}.$$

For $\epsilon_n = \frac{1}{n}$ let N_n be a natural number such that

$$\max\left\{\frac{32\tau|\psi - \psi_n|}{(32-\tau)(32-3\mu)}, \frac{32|\psi - \psi_n|}{32-\tau}\right\} < \frac{\epsilon_n}{2}$$

and let P_n be a natural number such that

$$\max\left\{\frac{\tau R k_a^p}{(32-3\mu)(1-k_a)} + \frac{R k_b^p}{1-k_b}, \frac{R k_a^p}{1-k_a}\right\} < \frac{\epsilon_n}{2}.$$

For every natural number n define

$$z_n = y_{N_n, P_n}$$

thus

$$\|y_* - z_n\| \leq \max\{\|a^*(\psi) - a(\psi_{N_n}, P_n)\|_\nu, \|b^*(m(\psi)) - b(m(\psi_{N_n}, P_n), P_n)\|_u\} < \epsilon_n$$

so $\{z_n\}$ is a sequence of functions that converges to y_* a solution of (1.1), (1.2). This ends the proof. □

REFERENCES

[1] R. I. Avery and J. Henderson. Three symmetric positive solutions for a second-order boundary value problem. Appl. Math. Lett., 13(3):1–7, 2000.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.* **3** (1922), 133–181.

[3] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[4] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.* **28** (1979), 673–688.

[5] E. Zeidler, *Nonlinear Functional Analysis and its Applications I, Fixed Point Theorems*, Springer-Verlag, New York, 1986.