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FIXED POINTS OF ENRICHED CONDENSING OPERATORS IN ORDERED BANACH SPACES

RIZWAN ANJUM[1], MUJAHID ABBAS[2], AND RAVI P. AGARWAL[3]

[1]Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan

[2]Department of Mathematics, Government College University Katchery Road, Lahore 54000, Pakistan & Department of Mathematics and Applied Mathematics, University of Pretoria Hatfield 002, Pretoria, South Africa

[3]Department of Mathematics, Texas A&M University-Kingsville 700 University Blvd., MSC 172 Kingsville, Texas 78363-8202

ABSTRACT. The aim of this paper is to obtain common fixed point results for a pair of condensing and weakly isotone self-mappings on ordered Banach spaces by the technique of enrichment. We present a new generalization of Darbo fixed point theorem in the setting of Banach spaces. These results unify and complement various known results in the existing literature on common fixed point theory. Some examples are given to support the concepts and results presented in this paper. Existence of the solution of two nonlinear differential equations is proved as an application of the result presented herein.

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1. Introduction and Preliminaries

The notations $\mathbb{N}$ and $\mathbb{R}$ will denote the set of all natural numbers and the set of all real numbers, respectively.

Suppose that X is a real Banach space equipped with the norm $\|\cdot\|$. If A is a subset of X, then convex hull of A and the closed convex hull of A are denoted by $\mathrm{co}A$ and $\overline{co}A$, respectively. Denote by $\Gamma(X)$, the family of nonempty bounded subsets of X and by $\Lambda(X)$, the subfamily consisting of all relatively compact subsets of X. Let A and B be two sets in X. Then

$$A+B = \{a+b : a \in A \text{ and } b \in B\} \text{ and}$$
$$\lambda A = \{\lambda a : a \in A\}, \text{ where } \lambda \in \mathbb{R}.$$

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Let A be any nonempty subset of X. Define

$$diam(A) = \sup_{x,y\in A} \|x - y\| .$$

A set A is bounded if $diam(A)$ is finite.

Let A be any nonempty bounded subset of X and $\epsilon > 0$.
A collection $C = \{A_1, A_2, A_3, ..., A_n\}$ is called a finite ϵ-covering of A if

$$A \subseteq \cup_{k=1}^{n} A_k$$

and $diam(A_k) < \epsilon$ for each $k \in \{1, 2, 3, ..., n\}$.
A collection $B = \{B_1, B_2, B_3, ..., B_n\}$ of balls is called a finite ϵ-ball covering of A if

$$A \subseteq \cup_{k=1}^{n} B_k$$

and radius of B_k is strictly less than ϵ for each $k \in \{1, 2, 3, ..., n\}$.
Kuratowski [30] introduced the notion of a measure of noncompactness (MNC). This concept is being applied in the study of existence of solutions for ordinary and partial differential equations, integral, and integro-differential equations.
Measure of noncompactness introduced and studied in [30] is given below.

Definition 1.1. [32] The Kuratowski measure of noncompactness of a nonempty and bounded subset A of X, denoted by $\alpha(A)$, is defined as

$$\inf_{C} \epsilon$$

where C is ϵ-covering of A.

In certain cases, finding $\alpha(A)$ with the help of above definition is not straightforward. Therefore, another measure of noncompactness known as the ball measure of noncompactness, which is more applicable in many cases was introduced and studied by Goldenštein, Gohberg and Markus (see, [26, 27, 32] and references mentioned therein).

Definition 1.2. [32] The ball measure of noncompactness of a nonempty and bounded subset A of X, denoted by $\Xi(A)$, is defined as

$$\inf_{B} \epsilon$$

where B is ϵ-ball covering of A.

These measures share several useful properties [13]. In many classical texts, this concept has been defined axiomatically to unify some of important common properties of the measures α and Ξ.
In this direction, Banaś, and Goebel [13] gave the following definition (see also, [6]) which requires that kernal of the measure of noncompactness is nonempty.

Definition 1.3. [13] A mapping $\mu : \Gamma(X) \to [0, \infty)$ is said to be measure of noncompactness if for any $A, B \in \Gamma(X)$, following conditions are satisfied:

1. The family $\Pi = Ker\mu = \{A \in \Gamma(X) : \mu(A) = 0\}$ is nonempty and $Ker\mu \subset \Lambda(X)$.
2. $A \subset B$ implies that $\mu(A) \leq \mu(B)$.
3. $\mu(\overline{A}) = \mu(A)$.
4. $\mu(\overline{co}A) = \mu(A)$.
5. $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B),\ \lambda \in [0, 1]$.
6. If (A_n) is a sequence of nonincreasing closed sets in $\Gamma(X)$ such that $\lim_{n\to\infty} \mu(A_n) = 0$, then

$$\lim_{n\to\infty} A_n = \cap_{k=1}^{\infty} A_k \tag{1.1}$$

is nonempty.

We now give some well known definitions needed in the sequel.

Definition 1.4. [9] Let C be a nonempty subset of a Banach space X, μ a measure of noncompactness on X and $T : C \to C$.

(i) Given $k \in [0, 1)$, the mapping T is called μ-k-set-contraction if

$$\mu(T(A)) \leq k\mu(A),\ \forall A \in \Gamma(C). \tag{1.2}$$

(ii) The mapping T is called μ-condensing if

$$\mu(T(A)) < \mu(A), \tag{1.3}$$

for any nonempty bounded subset A of C with $\mu(A) > 0$.

(iii) The mapping T is called k-contraction if the inequality

$$||Tx - Ty|| \leq k||x - y||, \tag{1.4}$$

holds for every $x, y \in C$.

Now, we state the following important fixed point results in the setting of Banach space.

Theorem 1.5. *(Banach* [15]*) Let C be nonempty closed subset of Banach space X. If $T : C \to C$ is k-contraction mapping, then T has a unique fixed point in C.*

Theorem 1.6. *(Schauder* [10]*) Let C be nonempty closed bounded and convex subset of Banach space X. If $T : C \to C$ is continuous and compact map, then T has at least one fixed point in C.*

The concept of measure of noncompactness has played a basic role in nonlinear functional analysis, especially in metric and topological fixed point theory (see e.g. [3, 5, 6, 7, 8, 14] and references therein).
In 1955, Darbo [24] employed the concept of measure of noncompactness to prove a fixed point theorem which generalizes classical Schauder's fixed point theorem [22] and a special variant of Banach contraction principle [31].
The Darbo theorem is stated as follows.

Theorem 1.7. *(Darbo* [24]*) Let C be a nonempty closed bounded and convex subset of a Banach space X. If $T : C \to C$ is continuous and μ-k-set-contraction mapping, then T has at least one fixed point in C.*

Note that $\mu : \Gamma(X) \to [0, \infty)$ defied by $\mu(A) = diamA$ is a classical example of measure of noncompactness in Banach space X ([13]) called diameter measure. Employing this measure of noncompactness, we can easily see that the Darbo fixed point theorem is a generalization of the Banach fixed point theorem.

In this paper, we use the following definition of measure of noncompactness due to Dhage [25].

Definition 1.8. [25] A mapping $\Theta : \Gamma(X) \to [0, \infty)$ is said to be a measure of noncompactness, if for any $A, B \in \Gamma(X)$, following conditions are satisfied:

1. $Ker\Theta = \Lambda(X)$, and
2. $\Theta(coA) = \Theta(\overline{co}A) = \Theta(A)$.
3. $A \subset B \Rightarrow \Theta(A) \leq \Theta(B)$,
4. $\Theta(\{A \cup B\}) = \max\{\Theta(A), \Theta(B)\}$, and
5. $\Theta(\lambda A) = |\lambda|\, \Theta(A)$.

Remark 1.9. Note that, the Kuratowski's measure α and the ball measure Ξ defined previously are the measure of noncompactness where Π coincides with $\Lambda(X)$. The simplest example of measure of noncompactness with $\Pi \neq \Lambda(X)$ is the diamter measure. Clearly, the kernal of a diameter measure is the family of all one-point sets.

For more properties of the measure of noncompactness, we refer to [9, 12, 13, 21, 30, 32].

Consistent with [9] and [24], the following definitions and results will be needed in the sequel.

A cone C is nonempty closed convex subset of X with (i) $\lambda C \subseteq C$ ($\lambda \geq 0$), and (ii) $C \cap (-C) = \{0\}$. Then the relation $x \leq y$ if and only if $y - x \in C$ defines the partial ordering in X.
Let X be a real ordered Banach space ordered by a cone C.

Definition 1.10. [9] Let M be a nonempty subset of an ordered Banach space $(X, \leq)$. A mapping $T : M \to M$ is said to be isotone increasing if $x, y \in M$ with $x \leq y$ implies $Tx \leq Ty$.

Definition 1.11. [9] Let M be a nonempty subset of an ordered Banach space $(X, \leq)$. Two mappings $S, T : M \to M$ are said to be:
(a) weakly isotone increasing if $Sx \leq TSx$ and $Tx \leq STx$ hold for all $x \in M$.
(b) weakly isotone decreasing if $Tx \geq STx$ and $Sx \geq TSx$ hold for all $x \in M$.
The mappings S and T are said to be weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

The problem of existence of common fixed points of a pair of nonlinear mappings has become an active area of research. By using the MNC, Dhage [25] proved some common fixed point results for a pair of condensing mappings in ordered Banach spaces and obtained some interesting applications in establishing the existence of solution of the system of differential and integral equations.

The fixed point theorem in [25] reads as follows:

Theorem 1.12. [25] *Let E be a nonempty closed bounded and convex subset of ordered Banach space X and $S, T : E \to E$ two continuous and condensing mappings. If S and T are weakly isotone, then they have a common fixed point.*

On the other hands, the technique of enriching contractive mappings has its root in the concept of asymptotic regularity in connection with the study of fixed points of nonexpansive mappings [23]. The same property was used in 1955 by Krasnoselskij [29] to prove the following fact:
If K is a compact convex subset of a uniformly convex Banach space and $T : K \to K$ is a nonexpansive, then for any $x_0 \in K$, the sequence

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ n \geq 0, \tag{1.5}$$

converges to fixed point of T. Note that, an averaged mapping (a term coined in [11]) is mapping of the form $T_\lambda = (1 - \lambda)I + \lambda T$, where $\lambda \in [0, 1]$ and I is the identity operator.
Krasnoselskij used the fact that nonexpansive mapping T which in general, is not an asymptotically regular, the averaged mapping $T_{\frac{1}{2}}$ involved in (1.5) is asymptotically regular.
Therefore, an averaged operator T_λ enriched the class of nonexpansive mappings with respect to the asymptotic regularity. This infect suggests the way to enrich the classes of contractive mappings in metrical fixed point theory by imposing the contractive condition on T_λ instead of T.
In this way, the following classes of mappings were introduced and studied: enriched

contractions and enriched ϕ-contractions [17], modified Kannan enriched contraction pair [1], enriched cyclic contraction [2], enriched Kannan contractions [18], enriched Chatterjea mappings [19], enriched nonexpansive mappings in Hilbert spaces [16], enriched multivalued contractions [4], enriched Ćirić-Reich-Rus contractions [20], etc. For examples, Abbas et al. [4] proved fixed point theorem by imposing the condition that T_λ-orbital subset is a complete subset, (see, Theorem 3 of [4]). Similarly, in [28], Gornicki and Bisht, considered the enriched Ćirić-Reich-Rus contractions and proved fixed point theorem by imposing the condition T_λ is asmymptotically regular mapping, (see, Theorem 3.1 of [28]).

A mapping $T : X \to X$ is called an enriched contraction or (b, θ)-enriched contraction [17] if there exist two constants, $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that for all $x, y \in X$,

$$\|b(x - y) + Tx - Ty\| \leq \theta \|x - y\| . \tag{1.6}$$

As shown in [17], several well-known contractive conditions existing in the literature on fixed point theory imply the (b, θ)-enriched contraction.
In particular, if $b = 0$ and $\theta = c$ then T called a c-contraction. It was proved that any enriched contraction mapping defined on a Banach space has a unique fixed point, which can be approximated by means of the Krasnoselskij iterative scheme.
Similar result for enriched nonexpansive mapping is obtained in [16].
A mapping $T : X \to X$ is called an enriched nonexpansive or b-enriched nonexpansive if there exists $b \in [0, \infty)$ such that for all $x, y \in X$,

$$\|b(x - y) + Tx - Ty\| < (b + 1) \|x - y\| . \tag{1.7}$$

We now give an example of the pair of mappings which are not weakly isotone, but the corresponding averaged operators are weakly isotone, for some $\lambda \in [0, 1]$.
The example of this pair is as follows.

Example 1.13. Let $X = \mathbb{R}$ be endowed with the usual norm. Define mappings $S, T : X \to X$ by

$$\begin{aligned} Sx &= -x, \text{ and} \\ Tx &= 2x^2 - x. \end{aligned}$$

Note that T and S are not weakly isotone decreasing on X. Indeed, $x = -1/2$ gives that $1/2 = Sx$ and $STx = -1$.
On the other hands, for $\lambda = \frac{1}{2}$, $S_{\frac{1}{2}}(x) = 0$, for all $x \in \mathbb{R}$ and $T_{\frac{1}{2}}(x) = x^2$, for all $x \in \mathbb{R}$. It is easy to check that the pair $T_{\frac{1}{2}}$ and $S_{\frac{1}{2}}$ are weakly isotone decreasing on X.

Motivated and inspired by the work of Berinde and Păcurar [16], Abbas et al. [4] and Gornicki and Bisht [28], we aim to enrich the class of mappings satisfying

Theorem 1.12 with respect to the weakly isotone property. Moreover, we extends and generalizes Darbo's fixed point theorem by using the concept of enriched contraction (1.6).

2. Two new class of operator on a normed space

We introduce the notions of enriched condensing contraction and enriched condensing operators as follows.

Definition 2.1. Let C be nonempty subset of normed space $(X, \|\cdot\|)$ and Ω be any an arbitrary measure of noncompactness in X. A mapping $T : C \to C$ is called:

(i) Enriched condensing contraction operator if there exists $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that for any nonempty bounded subset A of C with $\Omega(A) > 0$, $T(A)$ and $bA + T(A)$ are bounded, and the following holds:

$$\Omega(bA + T(A)) \leq \theta \Omega(A). \tag{2.1}$$

(ii) Enriched condensing operator if for any nonempty bounded subset A of C with $\Omega(A) > 0$, $T(A)$ and $bA+T(A)$ are bounded, and the following inequality holds:

$$\Omega(bA + T(A)) < (b+1)\Omega(A). \tag{2.2}$$

To indicate the measure of noncompactness Ω and constant involved in (2.1) and (2.2), we also call T a (Ω, b, θ)-set enriched contraction and (Ω, b)-enriched condensing, respectively.

Example 2.2. Any Ω-k-set-contraction (1.2) mapping T is $(0, k)$-set enriched contraction, that is, T satisfies (2.1) with $b = 0$ and $\theta = k \in [0, 1)$.

Every Ω-condensing (1.3) mapping T is $(\Omega, 0)$-enriched condensing.

We now give an example of an enriched condensing contraction operator, which is not a Ω-k-set-contraction.

Example 2.3. Let $C = [0, 1]$ be endowed with usual norm. Define $\Omega(A) = diamA$, for any closed subset of C. Let $T : C \to C$ be defined by $Tx = 1 - x$. Note that T is not a k-set-contraction but T is an $(1, 1)$-set enriched contraction. Indeed, if T would be a k-set-contraction then, by (1.2), there would exist $k \in [0, 1)$ such that

$$\begin{aligned}\Omega(T(A)) &= \sup\{\|u - v\| : u, v \in T(A)\}\\ &= \sup\{\|(1 - x) - (1 - y)\| : x, y \in A\}\\ &= \sup\{\|x - y\| : x, y \in A\}\\ &\leq k\Omega(A) = k \sup\{\|x - y\| : x, y \in A\},\end{aligned}$$

a contradiction to the fact that $k \in [0, 1)$.
On the other hand, for $b = \theta = 1$ the enriched condensing contraction operator (2.1) satisfies the following condition:

$$\Omega(A + T(A)) = \Omega\left(\left(\frac{1}{1/2} - 1\right)A + TA\right) \leq \Omega(A)$$

which can be written in an equivalent form as follows:

$$\Omega\left(T_{\frac{1}{2}}(A)\right) \leq \frac{1}{2}\Omega(A), \tag{2.3}$$

where A is any closed subset of C and $T_{\frac{1}{2}}(x) = \frac{1}{2}$, for all $x \in [0, 1]$. The above inequality (2.3) is valid because

$$\Omega\left(T_{\frac{1}{2}}(A)\right) = \sup\{\|x - y\|, x, y \in T_{\frac{1}{2}}(A)\} = 0 \leq \frac{1}{2}\Omega(A).$$

Common fixed point in ordered Banach space

Throughout this section, $(X, \|\cdot\|)$ denotes an ordered Banach space equipped with order relation $\leq$ induced by the cone C in X and E denote a nonempty closed bounded and convex subset of X. Furthermore, we use Θ as the measure of noncompactness in X in this section, as described in definition 1.8.

Now, we state one of the main results in this article, which extends and generalizes Theorem 1.12 by using the technique of enriching the existing class of operators.

Theorem 2.4. *Let $S, T : E \to E$ be two continuous operators. Assume that*

1. *S is (Ω, b_1)-enriched condensing*
2. *T is (Ω, b_2)-enriched condensing*
3. *S_{λ_1}, T_{λ_2} are weakly isotone.*

Then S and T have common fixed point, where $\lambda_i = \frac{1}{b_i+1}$, $i = 1, 2$.

Proof. Given that S and T are (Ω, b_1) and (Ω, b_2)-enriched condensing operators, respectively. Take $\lambda_i = \frac{1}{b_i+1}$, $i = 1, 2$. In this case, (2.2) becomes

$$\Theta\left(\left(\frac{1}{\lambda_2} - 1\right)A + T(A)\right) < (b_2 + 1)\Theta(A),$$

and hence

$$\Theta\left(\frac{(1 - \lambda_2)A + \lambda_2 T(A)}{\lambda_2}\right) < (b_2 + 1)\Theta(A).$$

Equivalently, we get that

$$\Theta(T_{\lambda_2}(A)) < \Theta(A). \tag{2.4}$$

Similarly, we have

$$\Theta(S_{\lambda_1}(A)) < \Theta(A). \tag{2.5}$$

Let x be an arbitrary but fixed element in X. Consider the sequence $\{x_n\}$ in E defined by

$$x_0 = x,\ x_{2n+1} = S_{\lambda_1} x_{2n},\ x_{2n+2} = T_{\lambda_2} x_{2n+1},\quad n = 0, 1, 2, .. \tag{2.6}$$

Suppose that the mappings S_{λ_1}, T_{λ_2} are weakly isotone increasing on E. Then it follows from (2.6) that

$$x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \leq \ldots \tag{2.7}$$

Let $A = \{x_1, x_2, \ldots, x_n, \ldots\}$. Then,

$$\begin{aligned} A &= \{x_1\} \cup \{x_3, x_5, \ldots, x_{2n+1}, \ldots\} \cup \{x_2, x_4, \ldots, x_{2n}, \ldots\} \\ &= \{x_1\} \cup S_{\lambda_1}(A_1) \cup T_{\lambda_2}(A_2), \end{aligned}$$

where $A_1 = \{x_2, x_4, \ldots, x_{2n}, \ldots\} \subset A$ and $A_2 = \{x_3, x_5, \ldots, x_{2n+1}, \ldots\} \subset A$. Clearly $A \subset E$ and hence A is bounded.

We now prove that A is precompact. Assume on the contrary that A is not precompact, then by the definition of Θ and (2.5) and (2.4), we get

$$\begin{aligned} \Theta(A) &= \Theta\big(\{x_1\} \cup S_{\lambda_1}(A_1) \cup T_{\lambda_2}(A_2)\big) \\ &= \max\{S_{\lambda_1}(A_1), T_{\lambda_2}(A_2)\} \\ &< \Theta(A), \end{aligned}$$

a contradiction, and hence A is precompact and $\bar{A}$ is compact. In view of (2.7), the sequence $\{x_n\}$ is monotone increasing in $\bar{A}$. Therefore, there is a unique limit x^* in $\bar{A}$ such that $\lim_{n\to\infty} x_n = x^*$. Again every subsequence of the sequence $\{x_n\}$ converges to the same limit point $x^* \in E$. Thus we have

$$\lim_{n\to\infty} x_{2n+1} = x^* \quad \textit{and} \quad \lim_{n\to\infty} x_{2n+2} = x^*.$$

By the continuity of S_{λ_1} and T_{λ_2}, we obtain

$$x^* = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} S_{\lambda_1} x_{2n} = S_{\lambda_1}(\lim_{n\to\infty} x_{2n}) = S_{\lambda_1} x^* = S x^*,$$

and

$$x^* = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T_{\lambda_2} x_{2n+1} = T_{\lambda_2}(\lim_{n\to\infty} x_{2n+1}) = T_{\lambda_1} x^* = T x^*.$$

Similarly, if we assume that S_{λ_1} and T_{λ_2} are weakly isotone decreasing on E, then it can be proved that the sequence $\{x_n\}$ is monotone decreasing and converges to the unique limit point $y^* \in E$, which is a common fixed point of S and T. □

If we put $b_1 = b_2 = 0$ in Theorem 2.4, we obtain the Theorem 1.12.

Corollary 2.5. [25] *Let $S, T : E \to E$ be two continuous and Ω-condensing operators. Further if S and T are weakly isotone, then they have a common fixed point.*

Theorem 2.6. *Let $S, T : E \to E$ be two continuous operators. Assume that*

1. *S is (Ω, b_1)-enriched condensing*
2. *T is (Ω, b_2)-enriched condensing*
3. *S_{λ_1}, T_{λ_2} are isotone increasing*
4. *S_{λ_1} and T_{λ_2} are commutative, that is,*

$$S_{\lambda_1}(T_{\lambda_2}(x)) = T_{\lambda_2}(S_{\lambda_1}(x)) \ \forall \ x \in E,$$

5. *$x \leq S_{\lambda_1}(x)$ and $x \leq T_{\lambda_2}(x)$ for some $x \in E$.*

Then S and T have common fixed point, where $\lambda_i = \frac{1}{b_i+1}$, $i = 1, 2$.

Proof. Following arguments similar to those given in the proof of Theorem 2.4, define a sequence $\{x_n\}$ in E by (2.6). Then in view of the hypothesis (3)-(4), it follows that

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$$

The rest of the proof follows using similar arguments given in the proof of Theorem 2.4. □

If we put $b_1 = b_2 = 0$ in Theorem 2.6, we obtain Theorem 2.2 of [25].

Corollary 2.7. [25] *Let $S, T : E \to E$ be two continuous and Ω-condensing operators. If*

1. *S and T are isotone mappings*
2. *$S(T(x)) = T(S(x))$ for all $x \in E$*
3. *$x \leq Sx$ and $x \leq Tx$ for some $x \in E$.*

Then S and T have common fixed point.

By applying Theorem 2.4 to a pair of mappings on Banach spaces, we obtain some results which guarantees the existence of the unique common fixed point of two operators. It is worth mentioning that these results do not require the compactness type conditions, however the mappings under consideration satisfy certain contraction type conditions.
For this, we start with the following proposition.

Lemma 2.8. *Every (b, θ)-enriched contraction $T : E \to E$ is a (Ω, b, θ)-set contraction with respect to the Kuratowski measure of noncompactness and hence (Ω, b)-enriched nonexpansive condensing.*

Proof. Let us denote $\lambda = \frac{1}{b+1}$. Obviously $\lambda \in (0, 1]$ and the enriched contractive conditions (1.6) becomes

$$\|T_\lambda x - T_\lambda y\| \leq d\,\|x - y\| \quad \forall \ x, y \in E, \tag{2.8}$$

where $d = \theta\lambda$. As $\theta \in [0, b + 1)$, $d \in [0, 1)$ and hence by (2.8) T_λ is $(0, d)$-enriched contraction.

On the other hand, (Θ, b, θ)-set contraction condition (1.2) is equivalent to the following inequality for $\lambda = \frac{1}{b+1}$.

$$\Theta(T_\lambda(A)) \leq d\Theta(A). \tag{2.9}$$

By (2.9), it follows that T_λ is (Ω, b, θ)-set contraction.
It follows from [25] that every $(0, d)$-enriched contraction is a $(\Omega, 0, d)$-set contraction with respect to the Kuratowski measure of noncompactness. Due to equivalences of (2.8) and (2.9) with (1.6) and (1.2), respectively, we note that every (Ω, b, θ)-enriched contraction is a (Ω, b, θ)-set contraction with respect to the Kuratowski measure of noncompactness. Similarly, it can be shown that T is (Ω, b)-enriched nonexpansive condensing operator. □

Theorem 2.9. *If $S, T : E \to E$ satisfy the following conditions;*

1. *T is (b_1, θ)-enriched contraction,*
2. *S is continuous and (Ω, b_2)-enriched condensing,*
3. *T_{λ_1}, S_{λ_2} are weakly isotone.*

Then S and T have unique common fixed point which is the unique fixed point of T, where $\lambda_i = \frac{1}{b_i+1}$, $i = 1, 2$.

Proof. It follows from Lemma 2.8 that T is a (Ω, b_1)-enriched condensing with respect to the Kuratowski measure of noncompactness Θ. Since T is (b_1, θ)-enriched contraction, it is continuous on E.
Thus all the conditions of Theorem 2.4 are fulfilled and hence S and T have a common fixed point. It follows from [17] that T cannot have more than one fixed points. This completes the proof. □

Corollary 2.10. *Let $S, T : E \to E$ be two mappings satisfying*

1. *T is (b_1, θ_1)-enriched contraction,*
2. *S is continuous and (Ω, b_2, θ_2)-set contraction,*
3. *T_{λ_1}, S_{λ_2} are weakly isotone.*

Then S and T have unique common fixed point which is the unique fixed point of T, where $\lambda_i = \frac{1}{b_i+1}$, $i = 1, 2$.

Proof. The proof follows using the arguments similar to those given in the proof of Theorem 2.9. □

3. A new generalization of Darbo's theorem in Banach space

Now we state one of the main result in this article which extends and generalizes Darbo's fixed point theorem by using the concept of enriched contraction (1.6). Throughout this section, X denotes a real Banach space equipped with the norm $\|\cdot\|$.

Theorem 3.1. *Let C be a nonempty closed bounded and convex subset of a Banach space X and $T : C \to C$ a continuous and (μ, b, θ)-set enriched contraction. Then T has at least one fixed point.*

Proof. Let us denote $\lambda = \frac{1}{b+1}$. Clearly $0 < \lambda < 1$. Note that for any $x, y \in C$, (2.1) becomes:

$$\mu(T_\lambda(A)) \leq k\mu(T(A)), \tag{3.1}$$

where $k = \lambda\theta$. Clearly, $k \in [0, 1)$.
We define a sequence C_n inductively by letting $C_0 = C$ and $C_n = co(T_\lambda C_{n-1})$, $n \geq 1$. Note that

$$T_\lambda C_0 = T_\lambda C \subseteq C = C_0,$$
$$C_1 = co(T_\lambda C_0) \subseteq C = C_0.$$

Continuing this process, we obtain that

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots .$$

If there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$, then C_N is relatively compact. Also, we have

$$T_\lambda C_N \subseteq co(T_\lambda C_N) = C_{N+1} \subseteq C_N.$$

By Theorem 1.6 , T has a fixed point.
Assume that $\mu(C_n) \neq 0$ for $n \geq 0$. By (3.1), we have

$$\mu(C_{N+1}) = \mu(co(T_\lambda C_N)) = \mu(T_\lambda C_N) \leq k\mu(T_\lambda C_N) < \mu(T_\lambda C_N), \tag{3.2}$$

which implies that $\mu(C_N)$ is a positive decreasing sequence of real numbers. Thus, there is an $s \geq 0$ so that $\mu(C_N) \to s$ as $n \to \infty$. We now show that $s = 0$. Assume on contrary that $s \neq 0$. From (3.2), we have

$$\frac{\mu(C_{N+1})}{\mu(C_N)} \leq k < 1,$$

which yields, $k = 1$ as $n \to \infty$ and hence $s = 0$. Since $C_{n+1} \subseteq C_n$ and $T_\lambda C_n \subseteq C_n$ for all $n \geq 1$. Then from (1.1), each C_n is a nonempty closed convex set, invariant under T_λ and belongs to $Ker\mu$. The result now follows from Theorem 1.6 . □

As a corollary of our result, we can obtain Darbo fixed point theorem, in the setting of a Banach space.

Corollary 3.2. *(Darbo [24]) Let C be a nonempty close bounded and convex subset of a Banach space X. If $T : C \to C$ is continuous and μ-k-set-contraction mapping, then T has at least one fixed point in C.*

By Theorem 3.1 we obtain Theorem 2.4 of [17].

Corollary 3.3. [17] *Let C be a nonempty closed bounded and convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ a (b, θ)-enriched contraction. Then T has a unique fixed point.*

Proof. Define $\mu(A) = diam(A)$ for any closed and convex subset of C. Take $\lambda = \frac{1}{b+1}$. Clearly, $0 < \lambda < 1$. Note that for any $x, y \in C$, (1.6) becomes:

$$\|T_\lambda x - T_\lambda y\| \leq \theta\lambda \|x - y\| . \tag{3.3}$$

By using (3.3), we have

$$\begin{aligned}\Omega(T_\lambda(A)) &= \sup\{\|T_\lambda x - T_\lambda y\| : x, y \in A\} \\ &\leq \theta\lambda \sup\{\|x - y\| : x, y \in A\} \\ &= \theta\lambda\Omega(A).\end{aligned}$$

This implies that

$$\Omega(T_\lambda(A)) \leq \theta\lambda\Omega(A)$$

Equivalently, we have

$$\Omega(bA + T(A)) \leq \theta\Omega(A).$$

The result now follows from Theorem 3.1. □

4. Application to differential equations

As an application of the result obtained in the above section, we prove the existence of common solutions of nonlinear differential equations under certain appropriate conditions.

First, we recall the following concept.

Suppose that $I = [t_0, t_0 + 1]$ is interval for some $t_0 \in \mathbb{R}$. Let X be the real Banach space equipped with the norm $\|\cdot\|_X$ and an order relation $\leq$ induced by the cone K in X. Denote by Θ, the Kurastowski measure of noncompactness in X.

Consider the system of nonlinear differential equations with the same initial condition

$$\begin{cases} x' = f(t, x), \ t \in I, \\ x(t_0) = x_0 \in X \end{cases} \tag{4.1}$$

and

$$\begin{cases} x' = g(t, x), \ t \in I, \\ x(t_0) = x_0 \in X, \end{cases} \tag{4.2}$$

where $f, g : I \times X \to X$ are continuous functions. Denote by $Y = C(I, X)$, the set of all continuous functions on the interval endowed with the norm

$$\|x\|_Y = \sup_{t \in I} \|x(t)\|_X . \tag{4.3}$$

We define an order relation $\leq$ in Y by the cone $\overline{K}$ in $C(I,X)$ given by

$$\overline{K} = \{x \in C(I,X) : x(t) \in K \text{ for all } t \in I\}. \tag{4.4}$$

Clearly, $C(I,X)$ with the norm $\|\cdot\|_Y$ and order relation $\leq$ is an ordered Banach space. Let $B \subset C(I,X)$, then

$$B(t) = \{z(t) : z \in B\} \subset X \text{ and } B(I) = \cup_{t\in I} B(t).$$

We need the following technical lemma.

Lemma 4.1. [13] *For any bounded and equicontinuous set B in $C(I,X)$, we have*

1. $\Theta\big(\int_{t_0}^t B(s)ds\big) \leq \int_{t_0}^t \Theta\big(B(s)\big),\ t \in I,$ *and*
2. $\Theta(B) = \max_{t\in I} \Theta\big(B(t)\big)$.

Let us assume that

(A1) The function f and g are bounded on $I \times X$ with bound 1.
(A2) The function f and g are uniformly continuous on $I \times X$.
(A3) For $t \in I$,

$$\Theta\big(f(t,B)\big) < 4\Theta(B) \tag{4.5}$$

and

$$\Theta\big(g(t,B)\big) < 4\Theta(B), \tag{4.6}$$

for any bounded set $B \subset X$.
(A4) The function $f(t,\cdot)$ and $g(t,\cdot)$ are nondecreasing on X for each $t \in I$.
(A5) The function $f(t,x) \leq g(t,f(t,x))$ and $g(t,x) \leq f(t,g(t,x))$ for all $(t,x) \in I\times X$.
(A6) $f(t,x(t)) \leq x_0 + \int_{t_0}^t f(\tau,x(\tau))d\tau$ and $g(t,x(t)) \leq x_0 + \int_{t_0}^t g(\tau,x(\tau))d\tau$ for all $(t,x) \in I \times C(I,X)$ and for a fixed element $x_0 \in X$ given in (4.1) and (4.2).

Theorem 4.2. *Assume that (A1)-(A6) hold, then the differential equations (4.1) and (4.2) have a common solution.*

Proof. Define a subset E of the ordered Banach space $C(I,X)$ by

$$E = \{x \in C(I,X) :\ |x(t) - x(s)| \leq |t-s| \text{ and } x(t_0) = x_0\}. \tag{4.7}$$

Clearly, E is closed bounded, convex and equi-continuous set in $C(I,X)$. Note that the differential equations (4.1) and (4.2) are equivalent to the integral equations

$$x(t) = x_0 + \int_{t_0}^t f(s,x(s))ds, \quad t \in I, \tag{4.8}$$

and

$$x(t) = x_0 + \int_{t_0}^t g(s,x(s))ds, \quad t \in I, \tag{4.9}$$

respectively.
Define the operators S and T on E by

$$Sx(t) = 2\big(x_0 + \int_{t_0}^{t} f(s, x(s))ds\big) - x(t),\ t \in I, \tag{4.10}$$

$$Tx(t) = 2\big(x_0 + \int_{t_0}^{t} g(s, x(s))ds\big) - x(t),\ t \in I. \tag{4.11}$$

If $b_1 = b_2 = 1$, then $\lambda_1 = \lambda_2 = \frac{1}{2}$ gives that

$$S_{\frac{1}{2}}x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,\ t \in I, \tag{4.12}$$

and

$$T_{\frac{1}{2}}x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds,\ t \in I. \tag{4.13}$$

It is straightforward to check that $S_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are weakly isotone increasing on E. Note that for any $x \in E$,

$$\begin{aligned} S_{\frac{1}{2}}x(t) &= x_0 + \int_{t_0}^{t} f(s, x(s))ds \\ &\leq x_0 + \int_{t_0}^{t} g\big(s, f(s, x(s))\big)ds \\ &\leq x_0 + \int_{t_0}^{t} g\left(s, x_0 + \int_{t_0}^{s} f(\tau, x(\tau))d\tau\right)ds \\ &= x_0 + \int_{t_0}^{t} g\big(s, S_{\frac{1}{2}}x(s)\big)ds \\ &= T_{\frac{1}{2}}S_{\frac{1}{2}}x(t) \end{aligned}$$

for all $t \in I$, that is, $S_{\frac{1}{2}}x \leq T_{\frac{1}{2}}S_{\frac{1}{2}}x$ for all $x \in E$. Similarly, we have $T_{\frac{1}{2}}x \leq S_{\frac{1}{2}}T_{\frac{1}{2}}x$ for all $x \in E$.

Note that S and T are 1-enriched condensing operators on E with respect to the measure of noncompactness $\overline{\Theta}$ in Y defined by

$$\overline{\Theta}(B) = \max_{t \in I} \Theta(B(t)),$$

where B is a bounded set in Y.

Now for any bounded set B in E, from Lemma 4.1, we have

$$\begin{aligned}\Theta\left(S_{\frac{1}{2}}\big(B(t)\big)\right) &\leq \int_{t_0}^{t} \Theta\big(f(s,B(s))\big)ds + \Theta\big(\{x_0\}\big) \\ &\qquad \big(\text{Since } \Theta(A+B) \leq \Theta(A)+\Theta(B)\big) \\ &\leq \int_{t_0}^{t} \Theta\big(f(s,B(s))\big)ds \\ &< 4\int_{t_0}^{t} \Theta(B(s))ds \\ &\leq 4\overline{\Theta}(B). \qquad (4.14)\end{aligned}$$

Taking maximum over t in (4.14), we obtain that

$$\overline{\Theta}\left(S_{\frac{1}{2}}B\right) < 4\overline{\Theta}(B) \quad \text{if } \overline{\Theta}(B) > 0, \tag{4.15}$$

equivalently,

$$\overline{\Theta}\big(B+S(B)\big) < 2\overline{\Theta}(B) \quad \text{if } \overline{\Theta}(B) > 0. \tag{4.16}$$

This shows that S is $(\overline{\Theta},1)$-enriched condensing operator on E. Also, T is a $(\overline{\Theta},1)$-enriched condensing operator on E. Obviously, S and T are continuous. Thus all the condition of Theorem 2.4 are satisfied and hence the differential equations (4.1) and (4.2) have a common solution in E. This complete the proof. □

5. **Conclusions**

1. We introduced the classes of enriched condensing contraction operators and enriched condensing operators that include k-set contractions as well as condensing operators as particular cases.
2. We present examples to show that the class of enriched condensing contraction operators strictly includes the k-set contractions.
3. We obtain Theorem 2.4 which enrich the class of mappings satisfying Theorem 1.12 with respect to the weakly isotone property.
4. We obtain Theorem 3.1, which extends the Darbo's fixed point theorem and (Theorem 2.4, [17]).
5. As an application of our result (Theorem 2.4), the existence of common the solution to the problem of differential equation (Theorem 4.2) is presented.

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