Dynamic Systems and Applications 32 (2023) 351-368

ADDITIVE DECOMPOSITION OF VECTOR-VALUED CONTINUOUS FUNCTIONS WITH SOME DIMENSIONAL CONSIDERATIONS OF THE SUMMANDS

K. K. PANDEY, S. VERMA*, AND P. VISWANATHAN

Department of Mathematics, IIT Delhi, New Delhi 110016, India.
*Department of Applied Sciences, IIIT Allahabad, Prayagraj 211015, India

ABSTRACT. Additive decomposition of a continuous real-valued function on the unit interval in the framework of fractal dimensions of the graphs of the summands has received significant attention recently. This is intimately connected with a rather old problem concerning the fractal dimensions of the graph of a generic continuous function. The primary objective of the current note is to revisit the aforementioned results on the decomposition of a continuous real-valued function and provide certain aspects of suitable vector-valued analogues. We show, for instance, that a continuous function $\mathbf{f} : [0,1] \to \mathbb{R}^n$ with a suitable choice of $\beta \in [1, n+1]$ can be decomposed as the sum of two continuous functions such that their graphs have the Hausdorff dimension β. Similar results regarding decompositions of a continuous vector-valued function in the light of packing dimension and box dimension of the graphs of the summands are indicated. Along the way, some elementary properties of the set-valued maps that arise in connection with the additive decomposition are also provided.

AMS (MOS) Subject Classification. 28A78, 26A30, 46E15.

Key Words and Phrases. Prevalent, Generic, Hausdorff dimension, Decomposition, Multivalued map.

1. INTRODUCTION

Estimation of the fractal dimensions such as the Hausdorff dimension, packing dimension and box dimension of the graphs of real-valued continuous functions from different standpoints has received great research attention, see, for instance, [1, 4, 5, 6, 9, 10, 11, 12, 13, 14, 16, 18, 19, 20, 21, 24, 25, 26, 27, 31, 33, 34, 35]. For a compact set A in $\mathbb{R}^m$ and a function $\mathbf{f} : A \to \mathbb{R}^n$ we shall denote the graph of $\mathbf{f}$ by $G_{\mathbf{f}}(A)$ or simply by $G_{\mathbf{f}}$. That is,

$$G_{\mathbf{f}}(A) = \big\{(\mathbf{x}, \mathbf{f}(\mathbf{x})) : \mathbf{x} \in A\big\} \subset \mathbb{R}^m \times \mathbb{R}^n.$$

The Hausdorff dimension of the graph of $\mathbf{f} : A \subset \mathbb{R}^m \to \mathbb{R}^n$ will be denoted by $\dim_H \big(G_{\mathbf{f}}(A)\big)$.

Received March 2, 2023 ISSN 1056-2176(Print); ISSN 2693-5295 (online)
$15.00 © Dynamic Publishers, Inc. https://doi.org/10.46719/dsa2023.32.19
www.dynamicpublishers.org Special Issue - ICNAAO 2022-IIT.

The relationship between the analytical properties (such as the Hölder continuity) of a function and the Hausdorff dimension of its graph is of particular interest [13]. In the first part of this note, we shall record some elementary results on the Hausdorff dimension of the graph of a continuous vector-valued function defined on $[0, 1]$.

The second part of this note provides a modest contribution to the dimensional aspects of the decomposition of a vector-valued continuous function. As is customary, let us denote the space of all continuous real-valued functions defined on a compact metric space X by $\mathcal{C}(X, \mathbb{R})$, and endow it with the supnorm. Let us first recall that the Hausdorff dimension of the graph of a function $f \in \mathcal{C}([0, 1], \mathbb{R})$ can be any number in $[1, 2]$. On the other hand, Mauldin and Williams [27] showed that the set

$$\big\{f \in \mathcal{C}([0, 1], \mathbb{R}) : \dim_H \big(G_f([0, 1])\big) = 1\big\}$$

is a dense G_δ-set in $\mathcal{C}([0, 1], \mathbb{R})$, and hence, in particular, this set is *residual* or *comeager* in $\mathcal{C}([0, 1], \mathbb{R})$. Recall that a property of a function in $\mathcal{C}(X, \mathbb{R})$ is treated to be *typical*, if the set of functions satisfying this property is a residual subset of $\mathcal{C}(X, \mathbb{R})$. For the convenience of the reader, we shall collect these notions in the next section. Typicality is one of the important approaches to describe the *generic* behavior of elements in a Banach space. Therefore, the aforementioned result by Mauldin and Williams can be paraphrased as follows. The graph of a generic function in $\mathcal{C}([0, 1], \mathbb{R})$ has the Hausdorff dimension 1. Furthermore, using a Baire category argument, Mauldin and Williams [27] deduced an engaging additive decomposition of a continuous function with the Hausdorff dimension of the graphs of the summands being one (vide infra).

Theorem 1.1. [27, Theorem 2] *For any $f \in \mathcal{C}([0, 1], \mathbb{R})$, there exist two functions g and h in $\mathcal{C}([0, 1], \mathbb{R})$ such that $f = g + h$ and $\dim_H \big(G_g([0, 1])\big) = \dim_H \big(G_h([0, 1])\big) = 1$.*

Following the work of Mauldin and Williams [27], Humke and Petruska [18] showed that the graph of a generic function in $\mathcal{C}([0, 1], \mathbb{R})$ has the packing dimension 2. In fact, over the last two decades, several authors investigated the fractal dimension (such as the Haudorff dimension, Box dimension, Packing dimension and topological Hausdorff dimension) of the graph of a generic continuous function, where the notion of being *generic* is interpreted in different ways (for instance, from a topological point view using the Baire categorical notions and a measure theoretic point of view by appealing to the notion of prevalence). In particular, Fraser and Hyde proved the following result.

Theorem 1.2. [16, Theorem 2.1] *The set $\big\{f \in \mathcal{C}([0, 1], \mathbb{R}) : \dim_H(G_f([0, 1])) = 2\big\}$ is a prevalent subset of $\mathcal{C}([0, 1], \mathbb{R})$. Consequently, any $f \in \mathcal{C}([0, 1], \mathbb{R})$ can be written as*

$f = g + h$, where g and h are functions in $\mathcal{C}([0,1],\mathbb{R})$ such that $\dim_H\big(G_g([0,1])\big) = \dim_H\big(G_h([0,1])\big) = 2$.

Note that Theorem 1.1 and Theorem 1.2 deal with the additive decomposition of a continuous function with the Hausdorff dimension of the graph of each summand taking the possible extreme values 1 and 2. Filling this "gap", Liu and Wu [24] proved the following.

Theorem 1.3. [24, Theorem 1.2] *Let $\beta \in [1,2]$ and $f \in \mathcal{C}([0,1],\mathbb{R})$. There exist functions g, h in $\mathcal{C}([0,1],\mathbb{R})$ such that $f = g+h$ and* $\dim_H\big(G_g([0,1])\big) = \dim_H\big(G_h([0,1])\big) = \beta$.

It is worth to note that Priyadarshi and his student [33] recently prove that for any $\beta \in [1,2]$, a given strictly positive real-valued continuous function can be decomposed as a product of two real-valued continuous functions whose graphs have upper box-counting dimension equal to β under certain conditions. A natural question that translates the previous theorem to the vector-valued setting is the following:

Let $1 \le \beta \le n+1$ and $\mathbf{f} : [0,1] \to \mathbb{R}^n$ be a continuous function. Does there exist a pair of vector-valued continuous functions $\mathbf{g}, \mathbf{h}$ such that $\mathbf{f} = \mathbf{g} + \mathbf{h}$ and $\dim_H\big(G_{\mathbf{g}}([0,1])\big) = \dim_H\big(G_{\mathbf{h}}([0,1])\big) = \beta$?

In the second part of this note, we provide a partial affirmative answer to the above question. To be precise, we prove:

Theorem 1.4 (Additive decomposition of a continuous vector-valued function)**.** *Let $\boldsymbol{f} : [0,1] \to \mathbb{R}^n$ be a continuous function and $\beta \in [1, n+1]$ satisfies one of the following conditions:*

(1) $\beta \le \dim_H\big(G_{\boldsymbol{f}}([0,1])\big)$
(2) $\dim_H\big(G_{\boldsymbol{f}}([0,1])\big) \le \beta$ *and* $n \le \beta \le n+1$.

Then there exist two vector-valued continuous functions $\boldsymbol{g}, \boldsymbol{h}$ such that $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ and $\dim_H\big(G_{\boldsymbol{g}}([0,1])\big) = \dim_H\big(G_{\boldsymbol{h}}([0,1])\big) = \beta$.

We shall hint at similar additive decomposition of a continuous function with the packing dimension and box dimension considerations of the graph of the summands. Thus, this part of the current note can be viewed as a sequel to [24, 25, 26]. This note is interspersed with examples and remarks which serve to supplement the general results. We also present some interesting sidelights.

2. Notation and Preliminaries

2.1. **Genericity.** As hinted in the introductory section, generic behaviour of a class of mathematical objects can be approached in different ways. However, extensively used in the study of dimensional properties of the graphs of continuous functions are *typicality* and *prevalence*, which, respectively, are topological and measure-theoretic notions of genericity. We shall collect the basic definitions here; the reader may refer [29, 31] for more details.

Definition 2.1. Let (X, d) be a complete metric space. A set $S \subseteq X$ is said to be of the *first category*, or, *meager*, if it can be written as a countable union of nowhere dense sets and a set $T \subseteq X$ is *residual*, or, *co-meager*, if $X \setminus T$ is meager. A property is called *typical* if the set of points which have the property is residual.

The Baires's category theorem, a version of which is stated below, can be used to test for typicality.

Theorem 2.2. *Let (X, d) be a complete metric space. A set $T \subseteq X$ is residual if and only if T contains a countable intersection of open dense sets or, equivalently, T contains a dense G_δ subset of X.*

Definition 2.3. Let X be a separable abelian topological group endowed with a compatible complete metric. A set $M \subseteq X$ is called *shy* if there exists a Borel set $B \subseteq X$ and a Borel probability measure μ on X such that $M \subseteq B$ and $\mu(M + x) = 0$ for all $x \in X$. The complement of a shy set is called a *prevalent* set.

2.2. **Fractal Dimensions.** We shall use the notation $\underline{\dim}_B$, $\overline{\dim}_B$, $\dim_H$ and $\dim_P$ for the lower box dimension, upper box dimension, Hausdorff dimension and packing dimension of a bounded set in $\mathbb{R}^d$. The s-dimensional Hausdorff measure of a set A is denoted by $\mathcal{H}^s(A)$. We shall provide a brief exposition of the aforementioned notions of fractal dimension here; we refer the reader to [13] for further details.

Definition 2.4. For a non-empty subset U of $\mathbb{R}^n$, the diameter of U is defined as

$$|U| = \sup\Big\{\|x - y\|_2 : x, y \in U\Big\},$$

where $\|x - y\|_2$ denotes the usual Euclidean distance between x, y in $\mathbb{R}^n$. A δ-cover of $F \subseteq \mathbb{R}^n$ is a countable collection of sets $\{U_i\}$ that cover F such that each U_i is of diameter at most δ. Suppose F is a subset of $\mathbb{R}^n$ and s is a non-negative real number. For any $\delta > 0$, we define

$$H^s_\delta(F) = \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta - \text{cover of } F\Big\}.$$

We define the $s-$dimensional Hausdorff measure of F by $H^s(F) = \lim_{\delta\to 0} H^s_\delta(F)$.

Definition 2.5. Let $F \subseteq \mathbb{R}^n$ and $s \geq 0$. The Hausdorff dimension of F is

$$\dim_H(F) = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

Remark 2.6. For $s = \dim_H(F)$, the $s-$dimensional Hausdorff measure $H^s(F)$ may be zero, infinite, or may satisfy $0 < H^s(F) < \infty$.

The following result points to a fundamental property of the Hausdorff dimension.

Theorem 2.7. *Let $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$.*

(i) *If $f : A \to \mathbb{R}^n$ is a Lipschitz map, then* $\dim_H\big(f(A)\big) \leq \dim_H(A)$.
(ii) *If $f : A \to \mathbb{R}^n$ is a bi-Lipschitz map, i.e.*

$$c_1\|x-y\|_2 \leq \|f(x)-f(y)\|_2 \leq c_2\|x-y\|_2$$

for all $x, y \in A$ and some $0 < c_1 \leq c_2 < \infty$, then $\dim_H\big(f(A)\big) = \dim_H(A)$.

Definition 2.8. Let $F \neq \emptyset$ be a bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower box dimension and upper box dimension of F respectively are defined as

$$\underline{\dim}_B(F) = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta},$$

and

$$\overline{\dim}_B(F) = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}.$$

If the above two are equal, we define the box dimension of F as the common value, that is,

$$\dim_B(F) = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}.$$

Definition 2.9. For a set $F \subseteq \mathbb{R}^n$ and $\delta > 0$, a δ-packing of F is defined as a collection of at most countable number of disjoint balls of radii at most δ with centers in F. Fix $s \geq 0$. The s-dimensional packing measure of F is defined by

$$\mathcal{P}^s(F) = \inf\Big\{\sum_i \mathcal{P}^s_0(F_i) : F \subseteq \cup_{i=1}^\infty F_i\Big\},$$

where $\mathcal{P}^s_0(F_i) = \lim_{\delta\to 0^+}\sup\sum_j |G_{j,i}|^s$ and supremum is taken over all δ-packing $\{G_{j,i}\}$ of F_i. The packing dimension of F is defined by

$$\dim_P(F) = \inf\{s : \mathcal{P}^s(F) = 0\} = \sup\{s : \mathcal{P}^s(F) = \infty\}.$$

As mentioned in the introductory section, it is a well-known result by Mauldin and Williams [27] that the graph of the generic $f \in \mathcal{C}\big([0,1],\mathbb{R}\big)$ is of Hausdorff dimension one. Let us note the following generalization to the generic function (in the sense of Baire category) on an arbitrary compact metric space.

Theorem 2.10. [5, Theorem 5.2] *Let A be a compact metric space and $n \in \mathbb{N}$. For a generic (typical) $\boldsymbol{f} \in \mathcal{C}(A, \mathbb{R}^n)$, we have* $\dim_H(G_{\boldsymbol{f}}(A)) = \dim_H(A)$.

If prevalence is used as a notion of genericity, then we have the following result.

Theorem 2.11. [6, Theorem 7.5] *Let A be an uncountable compact metric space and $n \in \mathbb{N}$. For a prevalent $\boldsymbol{f} \in \mathcal{C}(A, \mathbb{R}^n)$, we have* $\dim_H(G_{\boldsymbol{f}}(A)) = \dim_H(A) + n$.

It is worth to recall the following pair of analogous results for other notions of dimension.

Theorem 2.12. [4, Theorem 1.14] *Let A be an uncountable compact metric space and $n \in \mathbb{N}$. For a prevalent $\boldsymbol{f} \in \mathcal{C}(A, \mathbb{R}^n)$, we have* $\dim_P(G_{\boldsymbol{f}}(A)) = \dim_P(A) + n$.

Theorem 2.13. [4, Theorem 1.10] *Let A be an uncountable compact metric space with at most finitely many isolated points and $n \in \mathbb{N}$. For a prevalent $\boldsymbol{f} \in \mathcal{C}(A, \mathbb{R}^n)$, we have* $\underline{\dim}_B(G_{\boldsymbol{f}}(A)) = \underline{\dim}_B(A) + n$ *and* $\overline{\dim}_B(G_{\boldsymbol{f}}(A)) = \overline{\dim}_B(A) + n$.

The following result is fundamental.

Theorem 2.14. [13, Theroem 4.10] *Let $A \subseteq \mathbb{R}^n$ be a Borel set such that* $0 < \mathcal{H}^s(A) \leq \infty$. *Then we have a compact set $K \subseteq A$ such that* $0 < \mathcal{H}^s(K) < \infty$.

Using the previous theorem, Liu and Wu [24] noted the upcoming lemma.

Lemma 2.15. *Let A be a compact subset of $\mathbb{R}^n$ and $s \leq \dim_H(A)$. Then there exists a compact set $B \subseteq A$ such that* $\dim_H(B) = s$.

3. Some Basic Results on the Dimension of the Graph of a Vector-valued Function

Here we prove a few basic results on the Hausdorff dimension of the graph of a vector-valued continuous function. Similar results are well-recorded for real-valued functions. These vector-valued analogues appear to be folklore - but we have been unable to track down complete proofs.

For a vector valued function $\mathbf{f} : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$, let us denote by $f_i : A \to \mathbb{R}$ the coordinate functions of $\mathbf{f}$. That is, for each $\mathbf{x} \in A$,

$$\mathbf{f}(\mathbf{x}) = \big(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})\big).$$

Furthermore,

$$G_{\mathbf{f}}(A) = \big\{(\mathbf{x}, \mathbf{f}(\mathbf{x})) : \mathbf{x} \in A\big\} \subseteq \mathbb{R}^m \times \mathbb{R}^n.$$

We may indentify $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$. For $k \in \mathbb{N}$, we shall endow $\mathbb{R}^k$ with the usual Euclidean norm $\|.\|_2$. The proofs of the next two lemmas are simple, hence, are avoided.

Lemma 3.1. *For a continuous function* $\boldsymbol{f}\colon A \subseteq \mathbb{R}^m \to \mathbb{R}^n$, *we have* $\dim_H\big(G_{\boldsymbol{f}}(A)\big) \geq \max\{\dim_H(G_{f_i}(A)) : 1 \leq i \leq n\}$.

Remark 3.2. Let us note that a space filling curve is a continuous and surjective function $f : [0,1] \to [0,1] \times [0,1]$. In [23] the author proved that both the coordinate functions of the Peano space filling curve $f : [0,1] \to [0,1]\times[0,1]$ are generated by self-affine iterated function systems and futher that they have positive finite $\frac{3}{2}$-dimensional Hausdorff measure. Consequently, $\dim_H\big(G_{f_1}([0,1])\big) = \dim_H\big(G_{f_2}([0,1])\big) = 1.5$. On the other hand, being a continuous surjective function, the Peano space filling curve $f : [0,1] \to [0,1] \times [0,1]$ satisfies $\dim_H\big(G_f([0,1])\big) \geq 2$. This illustrates that the inequality in the above lemma can be strict. Let us further remark that the coordinate functions of the Hilbert space filling curve also have graphs of Hausdorff dimension 1.5. For more details the reader can consult [12, 28, 30].

Lemma 3.3. *Let* $\boldsymbol{f}\colon A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ *be continuous on* A, *where* $f_1, f_2, \dots, f_n : A \to \mathbb{R}$ *are the co-ordinate functions of* $\boldsymbol{f}$. *Suppose that there exists a natural number* i, $1 \leq i \leq n$, *such that the coordinate maps* $f_1, f_2, \dots, f_{i-1}, f_{i+1}, \dots, f_n$ *are all Lipschitz. Then,* $\dim_H(G_{\boldsymbol{f}}(A)) = \dim_H(G_{f_i}(A))$.

Lemma 3.4. *Let* $\boldsymbol{f}\colon A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ *be a Lipschitz continuous function and* $\boldsymbol{g} : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ *be a continuous function. Then,* $\dim_H\big(G_{\boldsymbol{f}+\boldsymbol{g}}(A)\big) = \dim_H(G_{\boldsymbol{g}}(A))$.

Proof. We define a map $T : G_{\mathbf{g}} \to G_{\mathbf{f}+\mathbf{g}}$ by $T(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \big(\mathbf{x}, \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\big)$. The assertion is an easy consequence of the fact that T is a surjective bi-Lispchitz map, and the details are omitted. □

Remark 3.5. Since the lower (upper) box dimension (if exists) is Lipschitz invariant, the above lemma holds for the lower (upper) box dimension as well.

As usual, let us define the multiplication of two functions $\mathbf{f}, \mathbf{g} : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ componentwise $(\mathbf{fg})(\mathbf{x}) = \mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x}) = \big(f_1(\mathbf{x})g_1(\mathbf{x}), f_2(\mathbf{x})g_2(\mathbf{x}), \dots, f_n(\mathbf{x})g_n(\mathbf{x})\big)$.

Lemma 3.6. *Let* $A \subset \mathbb{R}^m$ *be a compact set,* $\boldsymbol{f}\colon A \subset \mathbb{R}^m \to \mathbb{R}^n$ *be a Lipschitz map and* $\boldsymbol{g} : A \subset \mathbb{R}^m \to \mathbb{R}^n$ *be continuous. Then* $\dim_H\big(G_{\boldsymbol{fg}}(A)\big) \leq \dim_H\big(G_{\boldsymbol{g}}\big)$. *Furthermore, we have* $\dim_H(A) \leq \dim_H(G_{\boldsymbol{g}}(A))$ *for any continuous function* $g : A \subset \mathbb{R}^m \to \mathbb{R}^n$.

Proof. Here let us define a map $T : G_{\mathbf{g}} \to G_{\mathbf{fg}}$ by $T(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \big(\mathbf{x}, \mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})\big)$. It is plain to see that T is onto. Let $L_{\mathbf{f}}$ be the Lipschitz constant of $\mathbf{f}$, $M_{\mathbf{f}} = \|\mathbf{f}\|_\infty$, $M_{\mathbf{g}} = \|\mathbf{g}\|_\infty$ and $M = \max\{\sqrt{1 + 2M_{\mathbf{g}}L_{\mathbf{f}}^2}, \sqrt{2}M_{\mathbf{f}}\}$. By some simple calculations one can show that

$$\Big\|T(\mathbf{x}, g(\mathbf{x})) - T(\mathbf{y}, g(\mathbf{y}))\Big\|_2 \leq M\Big\|(\mathbf{x}, \mathbf{g}(\mathbf{x})) - (\mathbf{y}, \mathbf{g}(\mathbf{y}))\Big\|_2.$$

Therefore, T is a Lipschitz map and $\dim_H\big(G_{\mathbf{fg}}(A)\big) \leq \dim_H\big(G_{\mathbf{g}}(A)\big)$. □

Remark 3.7. Note that conclusion of the above lemma holds if we replace compactness assumption on A by the boundedness of the functions $\mathbf{f}$ and $\mathbf{g}$.

Remark 3.8. Since the lower (upper) box dimension is Lipschitz invariant, the above lemma is also true for the lower (upper) box dimension.

Remark 3.9. In the previous lemma, we may not have the equality, in general. To this end, define a vector-valued function $\mathbf{g} : [0,1] \to \mathbb{R}^n$, that is, $\mathbf{g} = (g_1, g_2, \dots, g_n)$ such that each $g_i : [0,1] \to \mathbb{R}$ is a Weierstrass type function with the graph having the Hausdorff dimension strictly greater than one (see, for instance, [32]). Further, let $\mathbf{f}$ be the zero function. In this case, we have $1 = \dim_H\big(G_{\mathbf{fg}}([0,1])\big) < \dim_H\big(G_{\mathbf{g}}[0,1]\big)$.

Proposition 3.10. *If $\boldsymbol{f} : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a Hölder continuous function with exponent s, then each coordinate function is also Hölder continuous function with exponent s. Conversely, if all the coordinate functions of $\boldsymbol{f} : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ are Hölder continuous with the same exponent s, then so is $\boldsymbol{f}$.*

Proof. Using simple mathematical inequality, the proof follows. □

The relationship between dimensions of the sets and the graphs of real-valued continuous functions with some prescribed smoothness defined on those sets has received considerable attention in the literature. One such result as reported in [22] is worth recalling here. Let $A \subset \mathbb{R}^n$ be a compact set and $\dim_H(A) = d$. Let us recall that the Hausdorff dimension of the graph of a Hölder continuous function with the Hölder exponent $s \in (0,1)$ defined on A is less than or equal to $\min\{d+1-s, \frac{d}{s}\}$. The previous proposition, taken in conjunction with the aforementioned result, provides:

1. If $\mathbf{f} : [0,1] \to \mathbb{R}^n$ is a Hölder continuous function with the exponent s, then $\dim_H\big(G_{f_i}([0,1])\big) \le 2 - s$ for each $1 \le i \le n$.
2. If $\mathbf{f} : [0,1] \times [0,1] \to \mathbb{R}^n$ is a Hölder continuous function with the exponent s, then $\dim_H\big(G_{f_i}([0,1])\big) \le 3 - s$ for each $1 \le i \le n$.
3. Let Δ denote the Sierpinski triangle and $\mathbf{f} : \triangle \to \mathbb{R}^n$ be a Hölder continuous function with the exponent s, then $\dim_H\big(G_{f_i}(\Delta)\big) \le 1 + \frac{\log 3}{\log 2} - s$ for each $1 \le i \le n$.

Remark 3.11. Note that the Peano space filling curve $\mathbf{f} : [0,1] \to [0,1] \times [0,1]$ is Hölder continuous with the Hölder exponent $\frac{1}{2}$. As mentioned in Remark 3.2, the component functions satisfy $\dim_H\big(G_{f_1}([0,1])\big) = \dim_H\big(G_{f_2}([0,1])\big) = 1.5$. On the other hand, we have $\dim_H\big(G_{\mathbf{f}}([0,1])\big) \ge 2$. This example should convince the reader that, in contrast to the case of a real-valued function, in general, one may not provide an upperbound for the Hausdorff dimension of the graph of a vector-valued function in terms of its Hölder exponent.

4. Decomposition of Continuous Vector-Valued Functions and Some Remarks

4.1. **Dimensional aspects of additive decomposition of continuous functions.** In this subsection we shall consider vector-valued analogues of some additive decompositions of a continuous functions with dimensional aspects of the summands, which are scattered in the literature [24, 25, 26]. Let us note that while a few of these results can be extended verbatim to the vector-valued setting, the arguments in many others do not extend per se to the vector-valued situation. However, we should admit that our proofs, to a considerable extent, depend on the machinery and methods developed for the corresponding results for a real-valued function.

Proposition 4.1. *Let K be a compact subset of $\mathbb{R}$ and $\boldsymbol{f}\colon K \to \mathbb{R}^n$ be a continuous function. Then there exist two continuous $\mathbb{R}^n$-valued functions $\boldsymbol{g}, \boldsymbol{h}$ on K such that $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ and $\dim_H\big(G_{\boldsymbol{g}}(K)\big) = \dim_H\big(G_{\boldsymbol{h}}(K)\big) = 1$.*

Proof. From Theorem 2.10 and the definition of a generic (typical) set, it follows that the set $S := \big\{\mathbf{g} \in \mathcal{C}(K, \mathbb{R}^n) : \dim_H(G_{\mathbf{g}}(K)\big) = \dim_H(K)\big\}$ is a dense G_δ-set. By a property of the generic set, its translate $\mathbf{f} + S$ will also be a dense G_δ-set. Therefore, $S \cap (\mathbf{f} + S) \neq \emptyset$, and consequently, $\mathbf{f} = \mathbf{g} - \mathbf{g}_1$, where the functions $\mathbf{g}$ and $\mathbf{g}_1$ are in S, providing the proof. □

Proposition 4.2. *Let K be a compact subset of $\mathbb{R}$ and $\boldsymbol{f} : K \to \mathbb{R}^n$ be a continuous function. Then there exist two continuous functions $\boldsymbol{g}, \boldsymbol{h} : K \to \mathbb{R}^n$ such that $\dim_H\big(G_{\boldsymbol{g}}(K)\big) = \dim_H(G_{\boldsymbol{h}}(K)) = n + \dim_H(K)$.*

Proof. Theorem 2.11 asserts that the set $S := \big\{\mathbf{g} \in \mathcal{C}(K, \mathbb{R}^n) : \dim_H(G_{\mathbf{g}}(K)) = n + \dim_H(K)\big\}$ is a prevalent set, which further implies that its translate $\mathbf{f} + S$ will be a prevalent set as well. We must have $S \cap (\mathbf{f} + S) \neq \emptyset$, from which the assertion can be easily deduced. □

On similar lines, via Theorem 2.12, we can prove the following proposition.

Proposition 4.3. *Let K be a compact subset of $\mathbb{R}$ and $\boldsymbol{f}\colon K \to \mathbb{R}^n$ be a continuous function. Then there exist two continuous functions $\boldsymbol{g}, \boldsymbol{h} : K \to \mathbb{R}^n$ such that $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ and $\dim_P\big(G_{\boldsymbol{g}}(K)\big) = \dim_P(G_{\boldsymbol{h}}(K)) = n + \dim_P(K)$.*

Now we shall revisit the following question posed in the introductory section:

Given a real number $\beta \in [1, n+1]$ and a continuous function $\mathbf{f}\colon [0,1] \to \mathbb{R}^n$, can we find an additive decomposition of $\mathbf{f}$, namely, $\mathbf{f} = \mathbf{g} + \mathbf{h}$, where $\mathbf{g}, \mathbf{h}$ are continuous and $\dim_H(G_{\mathbf{g}}([0,1])) = \dim_H\big(G_{\mathbf{h}}([0,1])\big) = \beta$?

Note that Propositions 4.1 and 4.2 together provide an affirmative answer to the above question for the extreme values of β, namely, $\beta = 1$ and $\beta = n+1$. In what follows, we shall attempt to answer the question for some values of $\beta \in (1, n+1)$. First we need a preparatory lemma, which can be viewed as a vector-valued analogue of Proposition 2.3 appeared in [24]. For the sake of completeness of the exposition, we supply a detailed proof.

Lemma 4.4. *Let X be a compact subset of $[0,1]$. Then a continuous function $\boldsymbol{f}: X \to \mathbb{R}^n$ can be extended to a continuous function $\boldsymbol{g}$ on $[0,1]$ which satisfies $\dim_H \big(G_{\boldsymbol{g}}([0,1])\big) = \max\big\{ \dim_H(G_{\boldsymbol{f}}(X)), 1\big\}$.*

Proof. Let X be a compact subset of $[0,1]$ and $\mathbf{f}: X \to \mathbb{R}^n$ be a continuous function with $\mathbf{f} = (f_1, f_2, \dots, f_n)$. If $X = [0,1]$, then there is nothing to prove. For a proper subset X of $[0,1]$, we have the following possibilities.

(a) $0, 1 \in X$,
(b) $0 \in X$ and $1 \notin X$,
(c) $1 \in X$ and $0 \notin X$,
(d) $0, 1 \notin X$.

Corresponding to each of the above cases, we obtain

(a) $[0,1]\backslash X = \cup_{i=1}^{\infty}(a_i, b_i)$,
(b) $[0,1]\backslash X = \cup_{i=1}^{\infty}(a_i, b_i) \cup \{1\}$,
(c) $[0,1]\backslash X = \cup_{i=1}^{\infty}(a_i, b_i) \cup \{0\}$,
(d) $[0,1]\backslash X = \cup_{i=1}^{\infty}(a_i, b_i) \cup \{0, 1\}$,

where $a_i, b_i \in X$ for each $i \in \mathbb{N}$ and the open intervals (a_i, b_i) are pairwise disjoint. By the countable stability of the Hausdorff dimension, it is enough to deal with the first case. We extend the coordinate functions f_j of $\mathbf{f}$ as follows.

$$g_j(x) = \begin{cases} f_j(x), & x \in X, \\ \frac{f_j(b_i) - f_j(a_i)}{b_i - a_i}(x - a_i) + f_j(a_i), & x \in (a_i, b_i) \text{ for some } i \in \mathbb{N}. \end{cases}$$

Clearly g_j is continuous for each $j = 1, 2, \dots, n$. Consider the continuous function $\mathbf{g} : [0,1] \to \mathbb{R}^n$ defined by $\mathbf{g} = (g_1, g_2, \dots, g_n)$. Using the countable stability of the Hausdorff dimension and Lemma 3.6 we see that

$$\begin{aligned} \dim_H \big(G_{\mathbf{g}}([0,1]\backslash X)\big) &= \sup_{i \in \mathbb{N}} \big\{ \dim_H \big(G_{\mathbf{g}}((a_i, b_i))\big)\big\} \\ &= \sup_{i \in \mathbb{N}} \{\dim_H((a_i, b_i))\} \\ &= \dim_H([0,1]\backslash X). \end{aligned}$$

Furthermore,

$$\begin{aligned} \dim_H(G_{\mathbf{g}}\big([0,1]\big)) &= \max\big\{ \dim_H(G_{\mathbf{g}}(X)), \dim_H(G_{\mathbf{g}}([0,1]\backslash X))\big\} \\ &= \max\big\{ \dim_H(G_{\mathbf{f}}(X)), \dim_H([0,1]\backslash X)\big\} \\ &\le \max\big\{ \dim_H(G_{\mathbf{f}}(X)), \dim_H([0,1])\big\}. \end{aligned}$$

Since $1 \leq \dim_H \big(G_{\mathbf{g}}([0,1])\big)$ and $\dim_H \big(G_{\mathbf{f}}(X)\big) \leq \dim_H \big(G_{\mathbf{g}}(X)\big) \leq \dim_H \big(G_{\mathbf{g}}([0,1])\big)$ we have the result. $\square$

We are now ready to answer the question on the additive decomposition of a continuous vector-valued function with a consideration to the Hausdorff dimension of the graphs of the summands posed in the above question, for some appropriate values of $\beta \in (1, n+1)$. This is a vector-valued analogue of a similar decomposition researched in [26]. While the proofs share a natural kinship, the reader will also notice a reasonable degree of difference between them.

Theorem 4.5. *Let $\boldsymbol{f} \colon [0,1] \to \mathbb{R}^n$ be a continuous function and $1 \leq \beta \leq n+1$ be a real number so that one of the following holds:*

(1) $\beta \leq \dim_H \big(G_{\boldsymbol{f}}([0,1])\big)$,
(2) $\dim_H \big(G_{\boldsymbol{f}}([0,1])\big) \leq \beta$ *and* $n \leq \beta \leq n+1$.

Then there exist two vector-valued continuous functions $\boldsymbol{g}, \boldsymbol{h}$ such that $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ and $\dim_H \big(G_{\boldsymbol{g}}[0,1]\big) = \dim_H(G_{\boldsymbol{h}}([0,1])) = \beta$.

Proof. Propositions 4.1 and 4.2 settle the proof for $\beta = 1$ and $\beta = n+1$ respectively. For $1 < \beta < n+1$, the proof proceeds in two cases.

Case 1: Let $\dim_H \big(G_{\mathbf{f}}([0,1])\big) \geq \beta$. Proposition 4.1 produces two functions $\mathbf{f}_1, \mathbf{f}_2$ in $\mathcal{C}([0,1], \mathbb{R}^n)$ such that $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ and

$$\dim_H \big(G_{\mathbf{f}_1}([0,1])\big) = \dim_H \big(G_{\mathbf{f}_2}([0,1])\big) = 1.$$

In view of Lemma 2.15 we can find a compact set $A \subseteq [0,1]$ such that $\dim_H(G_{\mathbf{f}}(A)) = \beta$. We consider the restriction maps $\mathbf{g}_1 = \mathbf{f}_1|_A$ and $\mathbf{g}_2 = \mathbf{f}_2|_A$. With the help of Lemma 4.4, we extend $\mathbf{g}_1$ and $\mathbf{g}_2$ to $[0,1]$, and denote the respective extensions by $\tilde{\mathbf{g}}_1$ and $\tilde{\mathbf{g}}_2$. Finally we consider the functions $\mathbf{g}$ and $\mathbf{h}$ as follows.

$$\mathbf{g} = \mathbf{f}_2 + \frac{1}{2}\tilde{\mathbf{g}}_1 - \frac{1}{2}\tilde{\mathbf{g}}_2, \quad \mathbf{h} = \mathbf{f}_1 + \frac{1}{2}\tilde{\mathbf{g}}_2 - \frac{1}{2}\tilde{\mathbf{g}}_1.$$

Note that $\mathbf{g} = \frac{1}{2}\mathbf{f}$ on A and

$$\dim_H(G_{\mathbf{g}}(A)) = \dim_H(G_{\frac{1}{2}\mathbf{f}}(A)) = \dim_H(G_{\mathbf{f}}(A)) = \beta.$$

We see that the function $\frac{1}{2}\tilde{\mathbf{g}}_1 - \frac{1}{2}\tilde{\mathbf{g}}_2$ is piecewise linear on $[0,1]\backslash A$. From the countable stability of the Hausdorff dimension and Lemma 3.4 we obtain

$$\begin{aligned}
\dim_H \big(G_{\mathbf{g}}([0,1]\backslash A)\big) &= \sup_{i \in \mathbb{N}} \big\{ \dim_H \big(G_{\mathbf{f}_2 + \frac{1}{2}\tilde{\mathbf{g}}_1 - \frac{1}{2}\tilde{\mathbf{g}}_2}((a_i, b_i))\big)\big\} \\
&= \sup_{i \in \mathbb{N}} \big\{ \dim_H(G_{\mathbf{f}_2}((a_i, b_i)))\big\} \\
&= \dim_H(G_{\mathbf{f}_2}([0,1]\backslash A)) \\
&\leq \dim_H \big(G_{\mathbf{f}_2}([0,1])\big) \\
&= 1.
\end{aligned}$$

Thus we have

$$\begin{aligned}\beta \leq\ & \dim_H\big(G_{\mathbf{g}}([0,1])\big)\\ =\ & \max\big\{\dim_H(G_{\mathbf{g}}([0,1]\backslash A)), \dim_H(G_{\mathbf{g}}(A))\big\}\\ \leq\ & \max\{1,\beta\}\\ =\ & \beta.\end{aligned}$$

On similar lines one can prove that $\dim_H(G_{\mathbf{h}}([0,1]) = \beta$.

Case 2: Let $\dim_H\big(G_{\mathbf{f}}([0,1])\big) \leq \beta$ and $n \leq \beta \leq n+1$. Since $\beta - n < 1$ and $\dim_H([0,1]) \geq \beta - n$, using Lemma 2.15 we obtain a compact set $A \subset [0,1]$ with $\dim_H(A) = \beta - n$. Proposition 4.2 ensures the existence of functions $\mathbf{g}_1, \mathbf{g}_2$ in $\mathcal{C}(A, \mathbb{R}^n)$ such that $\mathbf{f}|_A = \mathbf{g}_1 + \mathbf{g}_2$ and $\dim_H(G_{\mathbf{g}_1}(A)) = \dim_H(G_{\mathbf{g}_2}(A)) = \beta$. As in Case 1, the functions $\mathbf{g}_1$ and $\mathbf{g}_2$ can be extended to $[0,1]$, and we denote the extensions of these functions by $\tilde{\mathbf{g}}_1$ and $\tilde{\mathbf{g}_2}$ respectively. Define functions $\mathbf{g}$ and $\mathbf{h}$ by $\mathbf{g} = \frac{1}{2}(\mathbf{f} - \tilde{\mathbf{g}}_1 + \tilde{\mathbf{g}_2})$ and $\mathbf{h} = \frac{1}{2}(\mathbf{f} - \tilde{\mathbf{g}_2} + \tilde{\mathbf{g}_1})$. Following the similar arguments as in Case 1, bearing in mind that $\dim_H\big(G_{\mathbf{f}}([0,1])\big) \leq \beta$, we obtain the required decomposition of $\mathbf{f}$. That is, $\mathbf{f} = \mathbf{g} + \mathbf{h}$ and $\dim_H\big(G_{\mathbf{g}}([0,1])\big) = \dim_H\big(G_{\mathbf{h}}([0,1])\big) = \beta$.

This completes the proof. □

Remark 4.6. For $n = 1$, the above theorem reduces to Theorem 1.3. The case $1 < \beta < n$ with $\dim_H\big(G_{\mathbf{f}}([0,1])\big) \leq \beta$ still eludes us. This case leads to an open problem that:
for functions $\mathbf{f}, \mathbf{g} : [0,1] \to \mathbb{R}^n$ and any $\beta \leq \min\{\dim_H\big(G_{\mathbf{f}}([0,1])\big), \dim_H\big(G_{\mathbf{g}}([0,1])\big)\}$, can we have a set $A \subset [0,1]$ such that

$$\dim_H\big(G_{\mathbf{f}}(A)\big) = \dim_H\big(G_{\mathbf{g}}(A))\big) = \beta?$$

The problem may require a new tool from dimension theory. We also could not find any counterexample to this.

Next we state a packing dimension counterpart of the previous theorem. This can also be viewed as an extension of [25, Theorem 1.7] to the vector-valued setting. Invoking Proposition 4.3, the proof follows on lines similar to the proof of Case 2 in the previous theorem and hence omitted.

Theorem 4.7. *Let $\boldsymbol{f} \in \mathcal{C}([0,1], \mathbb{R}^n)$ and β be a real number such that $\beta \in [n, n+1]$ and $\dim_P\big(G_{\boldsymbol{f}}([0,1])\big) \leq \beta$. Then there exist $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{C}([0,1], \mathbb{R}^n)$ such that $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ and $\dim_P\big(G_{\boldsymbol{g}}([0,1])\big) = \dim_P(G_{\boldsymbol{h}}\big([0,1])\big) = \beta$.*

4.2. **Some remarks and loose ends.** Here we note that a few more results available on the decomposition of a real-valued function and dimensions of the graphs of the summands cannot be extended to the vector-valued case. First let us gather some results on the dimensional aspects of the additive decomposition that concern us.

Theorem 4.8. [25, Proposition 3.2] *Assume that* $f : [0,1] \to \mathbb{R}$ *is a continuous function that satisfies* $\dim_P(G_f(U)) = \dim_P(G_f([0,1]))$ *for any nonempty open set* $U \subset [0,1]$. *Then for any functions* g, h *such that* $f = g + h$, *we have*

$$\dim_P\big(G_{\boldsymbol{f}}([0,1])\big) \le \max\big\{\dim_P(G_{\boldsymbol{g}}([0,1])), \dim_P(G_{\boldsymbol{h}}([0,1]))\big\}.$$

Theorem 4.9. [14, Lemma 2.1] *Let* $f, g : [0,1]^d \to \mathbb{R}$ *be continuous functions. Then*

$$\overline{\dim}_B(G_{f+g}\big([0,1]^d)\big) \le \max\big\{\overline{\dim}_B(G_f([0,1]^d)), \overline{\dim}_B(G_g([0,1]^d))\big\}.$$

Theorem 4.10. [25, Corollary 1.6] *Let* $\beta \in [1,2]$ *and* $f \in \mathcal{C}([0,1],\mathbb{R})$. *Then there exist* $g, h \in \mathcal{C}([0,1],\mathbb{R})$ *with the properties that* $f = g+h$ *and* $\overline{\dim}_B(G_g) = \overline{\dim}_B(G_h) = \beta$ *if and only if* $\overline{\dim}_B(G_f) \le \beta$.

Theorem 4.11. [24, Lemma 3.1] *For* $f, g \in \mathcal{C}([0,1],\mathbb{R})$ *we have*

$$\underline{\dim}_B\big(G_{f+g}([0,1])\big) \le \max\Big\{\underline{\dim}_B\big(G_f([0,1])\big), \overline{\dim}_B(G_g([0,1]))\Big\}.$$

Turning now to the vector-valued case, we provide the following example.

Example 4.12. Let $\mathbf{f} : [0,1] \to [0,1] \times [0,1]$ be the Lebesgue space filling curve. Let us take $\mathbf{g} = (f_1, 0)$ and $\mathbf{h} = (0, f_2)$, where f_1 and f_2 are coordinate functions of $\mathbf{f}$. Clearly $\mathbf{f} = \mathbf{g} + \mathbf{h}$. Since $\mathbf{f}$ is a surjective continuous function defined on a compact set $[0,1]$, it follows that $\mathbf{f}$ is an open map. It is known [2] that the graphs of the coordinate functions of $\mathbf{f}$ have the same box dimension, Hausdorff dimension and packing dimension which are all equal to $1 + \log_9 2$. However, $\dim_P\big(G_{\mathbf{f}}([0,1])\big) \ge 2$ and $\dim_B\big(G_{\mathbf{f}}([0,1])\big) \ge 2$. This example illustrates the following.

1. The inequality $\dim_P\big(G_{\mathbf{f}}([0,1])\big) \le \max\big\{\dim_P(G_{\mathbf{g}}([0,1])), \dim_P(G_{\mathbf{h}}([0,1]))\big\}$ does not hold. This is in contrast to the case of real valued functions (cf. Theorem 4.8).
2. The inequality $\dim_B\big(G_{\mathbf{f}}([0,1])\big) \le \max\big\{\dim_B(G_{\mathbf{g}}([0,1])), \dim_B(G_{\mathbf{h}}([0,1]))\big\}$ does not hold. This should be compared with the result available for continuous real-valued functions (cf. Theorem 4.9).
3. There exists an additive decomposition $\mathbf{f} = \mathbf{g} + \mathbf{h}$ with the property that $\dim_B(G_{\mathbf{g}}\big([0,1])\big) = \dim_B\big(G_{\mathbf{h}}([0,1])\big) = 1 + \log_9 2$, which lies in $[1,3]$. However, $\dim_B(G_{\mathbf{f}}) > 1 + \log_9 2$. The reader may compare this with Theorem 4.10 above.
4. The result stated in Theorem 4.11 does not hold for a continuous vector-valued function.

The reader, if so inclined, will no doubt be able to carry out similar analysis as in the proof of [26, Theorem 1.6] to prove the following vector-valued analogue.

Theorem 4.13. *Let* $\alpha, \beta \in [1, n+1]$ *with* $\alpha < \beta$. *Then following hold:*

(a) *Let* $\boldsymbol{f} \in \mathcal{C}([0,1],\mathbb{R}^n)$ *such that* $\dim_H \big(G_{\boldsymbol{f}}([0,1])\big) \geq \beta$. *Then there exist functions* $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{C}([0,1],\mathbb{R}^n)$ *such that* $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$, $\dim_H \big(G_{\boldsymbol{g}}([0,1])\big) = \alpha$ *and* $\dim_H(G_{\boldsymbol{h}}\big([0,1])\big) = \beta$.

(b) *There exists a function* $\boldsymbol{f} \in \mathcal{C}([0,1],\mathbb{R}^n)$ *satisfying* $n < \dim_H \big(G_{\boldsymbol{f}}([0,1])\big) < \beta$ *for which the decomposition as given above does not exist.*

Remark 4.14. With help of an example we observed that Theorem 4.11 is not true in general for a continuous vector-valued map. That is, for $\mathbf{f}, \mathbf{g} \in \mathcal{C}([0,1],\mathbb{R}^n)$ the inequality

$$\underline{\dim}_B\big(G_{\mathbf{f}+\mathbf{g}}([0,1])\big) \leq \max\big\{\underline{\dim}_B(G_{\mathbf{f}}([0,1])), \overline{\dim}_B(G_{\mathbf{g}}([0,1]))\big\}$$

does not hold in general. Let us note that Proposition 1.7, Theorems 1.4 and 1.5 appeared in [24] are proved using the result stated in Theorem 4.11. Therefore appropriate extensions of the aforementioned results in [24] to the vector-valued setting still remain open.

Remark 4.15. Let $f \in \mathcal{C}([0,1],\mathbb{R})$ be such that $f(x) > 0$ for every $x \in [0,1]$. The function $h : [0,1] \to \mathbb{R}$ by $h(x) = \log(f(x))$ is continuous. Hence, by Theorem 4.5, for each $\beta \in [1,2]$ there exist $\phi, \xi \in \mathcal{C}([0,1],\mathbb{R})$ such that $h = \phi + \xi$ and $\dim_H \big(G_\phi([0,1])\big) = \dim_H \big(G_\xi([0,1])\big) = \beta$. It is easy to check that $f = g.p$, where $g(x) := \exp(\phi(x))$ and $p(x) := \exp(\xi(x))$ satisfy $\dim_H \big(G_g([0,1])\big) = \dim_H \big(G_p([0,1])\big) = \beta$. If $f(x) < 0$ for all $x \in [0,1]$ then we can define $h(x) = \log(-f(x))$, and in this case also we get a multiplicative decomposition for f. This can also be compared with [33].

5. Elementary Properties of set-valued Maps Associated with the Additive Decomposition

First let us recall a few fundamental concepts connected with the set-valued maps. For details, the reader can refer [3].

Definition 5.1. Let (X,d) and (Y,ρ) be metric spaces. We say that T is a set-valued map from X to Y, denoted by $T : (X,d) \rightsquigarrow (Y,\rho)$, if for every $x \in X$, $T(x)$ is a subset of Y.

Definition 5.2. A set-valued map $T : (X,d) \rightsquigarrow (Y,\rho)$ is called lower semicontinuous at $x \in X$ if for any open set U in Y such that $U \cap T(x) \neq \emptyset$ there exists $\delta > 0$ satisfying $U \cap T(x') \neq \emptyset$ whenever $d(x,x') < \delta$. The map T is called lower semicontinuous if it is lower semicontinuous at every $x \in X$.

Definition 5.3. A set-valued map $T : (X,d) \rightsquigarrow (Y,\rho)$ is said to be closed if the graph of T defined and denoted by $G_T = \{(x,y) : y \in T(x)\}$ is a closed subset of $X \times Y$.

Definition 5.4. A set-valued map $T : X \rightsquigarrow Y$ between two normed linear spaces X and Y is said to be Lipschitz at $x \in \mathrm{Dom}(T)$ if there exist a neighborhood $U \subset \mathrm{Dom}(T)$ of x and a constant $l > 0$ such that

$$T(x_1) \subset T(x_2) + l\|x_1 - x_2\|\, \overline{B}_1, \quad \forall\, x_1, x_2 \in U,$$

where $\mathrm{Dom}(T) = \{x \in X : T(x) \neq \emptyset\}$ and $\overline{B}_1$ is the closed unit ball centered at 0 in Y. The set-valued map T is said to be Lipschitz if it is Lipschitz at every $x \in \mathrm{Dom}(T)$.

Proposition 5.5. *Let $1 \leq \beta \leq n+1$. The set S_β defined by*

$$S_\beta = \big\{\boldsymbol{f} \in \mathcal{C}([0,1], \mathbb{R}^n) : \dim(G_{\boldsymbol{f}}([0,1])) = \beta\big\}$$

is dense in $\mathcal{C}([0,1], \mathbb{R}^n)$. Here the notation dim *is used to represent* $\dim_H$, $\dim_P$ *or* $\dim_B$.

Proof. Let $\mathbf{f} \in \mathcal{C}([0,1], \mathbb{R}^n)$ and $\epsilon > 0$. Recall that the set of all vector-valued Lipschitz functions denoted by $\mathcal{L}ip\big([0,1], \mathbb{R}^n\big)$ is a dense subset of $\mathcal{C}\big([0,1], \mathbb{R}^n\big)$. Therefore there exists $\mathbf{g} \in \mathcal{L}ip([0,1], \mathbb{R}^n)$ such that

$$\|\mathbf{f} - \mathbf{g}\|_\infty < \frac{\epsilon}{2}.$$

Choose a nonzero function $\mathbf{h} \in S_\beta$. Define $\tilde{\mathbf{h}} = \mathbf{g} + \frac{\epsilon}{2\|\mathbf{h}\|_\infty}\mathbf{h}$. Since $\mathbf{g}$ is a Lipschitz function, Lemma 3.4 and Remark 3.5 imply that $\dim\big(G_{\tilde{\mathbf{h}}}([0,1])\big) = \dim\big(G_{\mathbf{h}}([0,1])\big) = \beta$. Thus we have $\tilde{\mathbf{h}} \in S_\beta$ and

$$\|\mathbf{f} - \tilde{\mathbf{h}}\|_\infty \leq \|\mathbf{f} - \mathbf{g}\|_\infty + \|\mathbf{g} - \tilde{\mathbf{h}}\|_\infty < \epsilon,$$

providing the claim. □

Proposition 5.6. *The set-valued function $\Phi : [1, n+1] \rightsquigarrow \mathcal{C}([0,1], \mathbb{R}^n)$ defined by*

$$\Phi(\beta) = S_\beta := \big\{\boldsymbol{f} \in \mathcal{C}([0,1], \mathbb{R}^n) : \dim_H(G_{\boldsymbol{f}}([0,1])) = \beta\big\}$$

is lower semicontinuous.

Proof. Let $\beta \in [1, n+1]$ be fixed and U be an open set such that $\Phi(\beta) \cap U \neq \emptyset$. Since S_α is dense in $\mathcal{C}([0,1], \mathbb{R}^n)$ for each $\alpha \in [1, n+1]$, we obtain $\Phi(\alpha) \cap U \neq \emptyset$ for all $\alpha \in [1, n+1]$. This tells that δ required in the definition of a lower semicontinuous function (cf. Definition 5.2) can be taken to be arbitrary. □

Proposition 5.7. *The set-valued map $\Phi : [1,2] \rightsquigarrow \mathcal{C}([0,1], \mathbb{R})$ defined above is not closed.*

Proof. Let $f : [0,1] \to \mathbb{R}$ be a Weierstrass type nowhere differentiable function such that $\dim_H(G_f([0,1])) > 1$, see, for instance, [8]. Choose a sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ converging uniformly to f. For each $n \in \mathbb{N}$, we have $\dim_H(p_n([0,1])) = 1$ and hence $p_n \in \Phi(1)$. That is, $(1, p_n) \in G_\Phi$. Note that $(1, p_n) \to (1, f)$ as $n \to$

∞. However, since $\dim_H(G_f([0,1])) > 1$ we have $(1, f) \notin G_\Phi$. This proves the assertion. □

Proposition 5.8. *Fix a* $\beta \in [1, n+1]$ *and define a map* $\Psi_\beta : \mathcal{L}ip\big([0,1], \mathbb{R}^n\big) \rightsquigarrow \mathcal{C}([0,1], \mathbb{R}^n) \times \mathcal{C}([0,1], \mathbb{R}^n)$ *by*

$$\Psi_\beta(\boldsymbol{f}) = \Big\{(\boldsymbol{g}, \boldsymbol{h}) \in \big(\mathcal{C}([0,1], \mathbb{R}^n)\big)^2 : \boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h},\ \dim_H(G_{\boldsymbol{g}}([0,1])) = \dim_H(G_{\boldsymbol{h}}([0,1])) = \beta\Big\}$$

is Lipschitz.

Proof. Let $\mathbf{f} \in \mathrm{Dom}(\Psi_\beta) \subset \mathcal{L}ip\big([0,1], \mathbb{R}^n\big)$. Take U as the open unit ball centered at $\mathbf{f}$ in $\mathrm{Dom}(\Psi_\beta)$ and $l = 2$. We claim that $\Psi_\beta(\mathbf{f}_1) \subset \Psi_\beta(\mathbf{f}_2) + l\|\mathbf{f}_1 - \mathbf{f}_2\|_\infty \overline{B}_1$ for all $\mathbf{f}_1, \mathbf{f}_2 \in U$.

To establish the claim, let $(\mathbf{g}_1, \mathbf{h}_1) \in \Psi_\beta(\mathbf{f}_1)$ and $\epsilon = 2\|\mathbf{f}_1 - \mathbf{f}_2\|_\infty$. Since S_β is dense in $\mathcal{C}([0,1], \mathbb{R}^n)$, for $\mathbf{g}_1 \in \mathcal{C}([0,1], \mathbb{R}^n)$ there exists a function $\mathbf{g}_2 \in S_\beta$ such that

$$\|\mathbf{g}_1 - \mathbf{g}_2\|_\infty < \frac{\epsilon}{2}.$$

Define $\mathbf{h}_2 := \mathbf{f}_2 - \mathbf{g}_2$. It is easy to see that

$$\mathbf{f}_2 = \mathbf{g}_2 + \mathbf{h}_2, \quad \dim_H(G_{\mathbf{h}_2}([0,1])) = \dim_H(G_{\mathbf{g}_2}([0,1])) = \beta.$$

Consequently,

$$\|\mathbf{h}_1 - \mathbf{h}_2\|_\infty \leq \|\mathbf{g}_1 - \mathbf{g}_2\|_\infty + \|\mathbf{f}_1 - \mathbf{f}_2\|_\infty < \epsilon.$$

Define $\mathbf{g}_3 = \frac{1}{\epsilon}(\mathbf{g}_1 - \mathbf{g}_2)$ and $\mathbf{h}_3 = \frac{1}{\epsilon}(\mathbf{h}_1 - \mathbf{h}_2)$ so that $(\mathbf{g}_3, \mathbf{h}_3) \in \overline{B}_1$. Furthermore,

$$(\mathbf{g}_1, \mathbf{h}_1) = (\mathbf{g}_2, \mathbf{h}_2) + 2\|\mathbf{f}_1 - \mathbf{f}_2\|_\infty(\mathbf{g}_3, \mathbf{h}_3) \in \Psi_\beta(\mathbf{f}_2) + l\ \overline{B}_1,$$

from which the proof follows. □

Statements and Declarations

Data availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Funding: During the preparation of this manuscript, the first author was financially supported by the University Grants Commission (UGC), India in the form of a Senior Research Fellowship.

Conflict of interest: We declare that we do not have any conflict of interest.

Author Contributions: All authors contributed equally to this manuscript.

REFERENCES

[1] V. Agrawal, T. Som, L^p approximation using fractal functions on the Sierpiński gasket, *Results Math*, 77(2):74, 2022.

[2] P.C. Allaart, K. Kawamura, Dimensions of the coordinate functions of space-filling curves, *J. Math. Anal. Appl.*, 335:1161–1176, 2007.

[3] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäauser, Boston, 1990.

[4] R. Balka, Dimensions of graphs of prevalent continuous maps, *arxiv preprint*, https://doi.org/10.48550/arXiv.1503.00865, 2015.

[5] R. Balka, Z. Buczolich, M. Elekes, Topological Hausdorff dimension and level sets of generic continuous functions on fractals, *Chaos, Solitons & Fractals*, 45:1579–1589, 2012.

[6] R. Balka, U.B. Darji and M. Elekes, Hausdorff and packing dimension of fibers and graphs of prevalent continuous maps, *Adv. Math.*, 293:221–274, 2016.

[7] R. Balka, Á. Farkas, M. J. Fraser, J.T. Hyde Dimension and measure for generic continuous images, *Ann. Acad. Sci. Fenn. Math.*, 38(1):389–404, 2013.

[8] K. Baranski. Dimension of the graphs of the Weierstrass-type functions, *Fractal Geometry and Stochastics V., Springer International Publishing*, 77–91, 2015.

[9] F. Bayart, Y. Heurteaux, On the Hausdorff dimension of graphs of prevalent continuous functions on compact sets, *Further Developments in Fractals and Related Fields*, Birkhäuser, Boston, 25–34, 2013.

[10] S. Chandra, S. Abbas, On fractal dimensions of fractal functions using functions spaces, *Bull. Aust. Math. Soc.*, 1–11, 2022.

[11] S. Chandra, S. Abbas, S. Verma, Bernstein Super Fractal Interpolation Function for Countable Data Systems, *Numerical Algorithms*, 92(4):2457–2481, 2023.

[12] G.A. Edgar, Kiesswetter's fractal has Hausdorff dimension $\frac{3}{2}$, *Real Anal. Exchange*, 14:215–223, 1988/89.

[13] K.J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley Sons Inc., New York, 1999.

[14] K.J. Falconer, J.M. Fraser, The horizon problem for prevalent surfaces, *Math. Proc. Cambridge Philos. Soc.*, 151:355–372, 2011.

[15] K.J. Falconer, J.D. Howroyd, Projection theorem for box and packing dimensions, *Math. Proc. Cambridge Philos. Soc.*, 119:287–295, 1996.

[16] J.M. Fraser, J.T. Hyde, The Hausdorff dimension of graphs of prevalent continuous functions, *Real Analysis Exchange*, 37(2):333—352, 2011/2012.

[17] V. Gruslys, J. Jonušas, V. Mijović, O. Ng, L. Olsen, I. Petrykiewicz, Dimensions of prevalent continuous functions, *Monatsh. Math.*, 166:153–180, 2012.

[18] P.D. Humke, G. Petruska, The packing dimension of a typical continuous function is 2, *Real Anal. Exchange*, 14:345–358, 1988/1989.

[19] J. Hyde, V. Laschos, L. Olsen, I. Petrykiewicz, A. Shaw, On the box dimension of graphs of typical continuous functions, *J. Math. Anal. Appl.*, 391:567–581, 2012.

[20] S. Jha, S. Verma, Dimensional Analysis of α-Fractal Functions, *Results Math*, 76(4):1–24, 2021.

[21] S. Jha, S. Verma, A.K.B. Chand, Non-stationary zipper α-fractal functions and associated fractal operator, *Fract. Calc. Appl. Anal.*, 25:1527—1552, 2022.

[22] J.-P. Kahane. *Some random series of functions*, Cambridge Univ. Press, 2nd edition, 1985.

[23] N. Kono, On self-affine functions, *Japan J. Appl. Math.*, 3:259–269, 1986.

[24] J. Liu, D. Liu, On the decomposition of continuous functions and dimensions, *Fractals*, doi: 10.1142/S0218348X20500073.

[25] J. Liu, B. Tan, J. Wu, Graphs of continuous functions and packing dimension, *J. Math. Anal. Appl.*, 435:1099–1106, 2016.

[26] J. Liu, J. Wu, A remark on decomposition of continuous functions, *J. Math. Anal. Appl.*, 401:404–406, 2013.

[27] R.D. Mauldin, S.C. Williams, On the Hausdorff dimension of some graphs, *Trans. Amer. Math. Soc.*, 298:793–803, 1986.

[28] M. Mcclure, The Hausdorff dimension of Hilbert's coordinate functions, *Real Anal. Exchange*, 24:875–884, 1998/99.

[29] J. C. Oxtoby, *Measure and Category*, Springer, 2nd ed., 1996.

[30] H. Sagan, *Space-filling curves*, Springer-Verlag, New York, 1994.

[31] A. Shaw, *Prevalence*, M. Math Dissertation, University of St. Andrews, 2010.

[32] W. Shen, Hausdorff dimension of the graphs of the classical Weierstrass functions, *Math. Z.*, 289:223–266, 2018.

[33] M. Verma, A. Priyadarshi, Graphs of continuous functions and fractal dimension, *Chaos, Solitons and Fractals*, 172:113513, 2023.

[34] M. Verma, A. Priyadarshi, S. Verma, Fractal dimensions of fractal transformations and Quantization dimensions for bi-Lipschitz mappings, *arxiv preprint*, https://doi.org/10.48550/arXiv.2212.09669, 2022.

[35] S. Verma, P.R. Massopust, Dimension preserving approximation, *Aequationes Mathematicae*, 96(6):1233–1247, 2022.

[36] P. Wingren, Dimensions of graphs of functions and lacunary decompositions of spline approximations, *Real Anal. Exchange*, 26:17–26, 2000/2001.