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SOME PROPERTIES OF FRACTAL OPERATOR ASSOCIATED WITH COMPLEX-VALUED FRACTAL FUNCTIONS ON THE SIERPIŃSKI GASKET

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ABSTRACT. In this paper, we demonstrate several properties, such as Fredholm, non-compactness of the complex-valued fractal operator associated with the complex-valued fractal functions, which is constructed using a germ function, base function, and scaling functions defined on the Sierpiński gasket. We show the existence of a non-trivial closed subspace of a complex-valued fractal operator. We prove that a complex-valued fractal function has finite energy under certain conditions on the parameters involved. Further, the existence of Schauder basis consisting of a complex-valued fractal function is shown.

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Key Words and Phrases. Complex-valued fractal function; Sierpiński gasket, Schauder basis, Energy, Fractal operator.

1. INTRODUCTION

One out of various tools for fitting and analysing scientific data is a fractal interpolation. A fractal set is formed by the union of several smaller copies of itself. Barnsley [4] invented the technique that produces the self-referential function, namely the fractal interpolation functions (FIFs), which is defined by iterated function system (IFS). Furthermore, Navascués [26] has defined FIFs on a compact interval of $\mathbb{R}$. Fractal interpolation was developed as an interpolation approach for collecting data with intrinsic fractal structure. Unlike classical interpolation, which is based on elementary functions such as polynomials, the fractal interpolation is based on the theory of iterated function system and it produces an interpolants suitable for fitting physical or experimental data. By extending Barnsley's notion of FIFs to the domain of Sierpiński Gasket (SG in short), Celik et al. [8] have made significant progress in the field of FIFs. On a post critically finite (p.c.f.) self-similar set, Ruan [31] has developed FIFs and linear FIFs. Further, Ri and Ruan [30] have studied some basic properties of uniform FIFs on SG. Readers are referred to see [23] for understanding the fractal functions, fractal surfaces, and wavelets. In [16], Jha and Verma have studied the dimensions of FIFs defined on a compact interval of $\mathbb{R}$. Further, they have provided an accurate estimate

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of the box dimension of fractal functions. In [41], Verma and Sahu have estimated the upper and lower box dimensions of the graph of a function defined on SG. Further, they have also computed an upper bound for Hausdorff dimension and box dimension of the graph of a function, which has a finite energy. In [40], dimension preserving approximation for continuous functions defined on compact interval of $\mathbb{R}$ has been investigated by Verma and Massopust. Kigami [19] has explored fractal analysis on a post critically finite (p. c. f.) self-similar sets. In [10], Chandra and Abbas have studied the integral transforms and fractional order integral transforms of the bivariate FIFs. In [11], Chandra and Abbas have investigated the properties such as continuity, bounded variation and boundedness of the mixed Weyl-Marchaud fractional derivative of a function. In [28], Prasad constructed the space of Coalescence Hidden-variable FIFs. Jha et al. [17] have studied non-stationary zipper α-fractal functions and associated fractal operator. To understand the fractal dimension, we refer the reader to [12, 13, 20, 21, 22]. The authors of [24] have studied the non-stationary α-fractal surfaces. The authors of [33] have discussed the quantization for uniform distributions on stretched SG. In [25, 42], Verma and his group have studied the dimensional results for vector-valued FIFs. In [14], Chandra and Abbas have obtained dimension results for fractal functions. With the help of oscillation spaces, they also established certain bounds for the fractal dimension of fractal functions. Recently, Chandra et al. [9] have studied the properties of Bernstein super FIFs.

In [35], Sahu and Priyadarshi have determined the bounds for the box dimension of the graph of a harmonic function on SG. Further, they have also obtained the upper and lower bounds for the box dimension of graph of a function having a finite energy on SG. Further, they have studied the solutions of Laplacian and p-Laplacian in [34, 36]. The authors of [32] have generalized existing results about the fractal dimensions of many other IFSs.

Author of [29] has demonstrated that the graphs of FIFs formed on SG using nonconstant harmonic functions of the fractal analysis are attractors of iterated function system. Agrawal and Som have determined the fractal dimension of α-fractal function on SG in [1]. Furthermore, Agrawal and Som [2] have studied the L^p approximation using fractal functions on SG, whereas very recently Prasad and Verma [27] have constructed FIFs on the product of two SGs and in [15], Jha and Verma have developed the concept of a non-stationary fractal operator as well as several approximations and convergence properties. Furthermore, they have also examined the approximation properties of non-stationary fractal polynomials towards a continuous function.

The present paper is organized as follows: In Section 2, we give technical introduction to create attractor for defined IFS. In Section 3, we discuss the construction of SG and recall the theorem to construct the α-fractal fractal function and the graph of α-fractal fractal function defined on SG. In Section 4, via graphical representation of the graph of an α-fractal function on SG, we observe that the α-fractal function is continuously dependent on parameters. Furthermore, we derive that a complex-valued fractal function has finite energy under certain conditions on the parameters involved. In the last section, we establish several properties such as Fredholm, non-compactness of the complex-valued fractal operator associated with the complex-valued fractal functions which is constructed from a germ function, base function and scaling functions defined on SG. Furthermore, we show the existence of a non-trivial closed subspace of a complex-valued fractal operator, then we establish the existence of Schauder bases consisting of complex-valued fractal functions defined on SG.

2. TECHNICAL INTRODUCTION

To obtain an attractor, we consider the following iterated function system (IFS)

$$\{X; W_j, j = 1, 2, \ldots, k\},$$

where (X, d) is a complete metric space and $W_j : X \to X$ are contractive mappings with contraction ratio α_j respectively. Using the maps of this IFS, we define another mapping W from $S(X)$ into $S(X)$ as follows:

$$W(F) = \bigcup_{j=1}^{k} W_j(F),$$

where $S(X)$ is the class of all non-empty compact subsets of X. The Hutchinson-Barnsley map W acting on $S(X)$ endowed with Hausdorff metric h_d is a contraction mapping. The contraction ratio α of W is taken to be $\max\{\alpha_j : 1 \le \alpha_j \le k\}$, where α_j be the contraction ratio of each W_j .Then, by the Banach contraction principle, we get a unique nonempty compact subset H_*, which satisfies $H_* = \cup_{j=1}^{k} W_j(H_*)$, the set H_* is called an attractor of the IFS. For further details, we refer [5].

3. FRACTAL INTERPOLATION FUNCTION ON SG

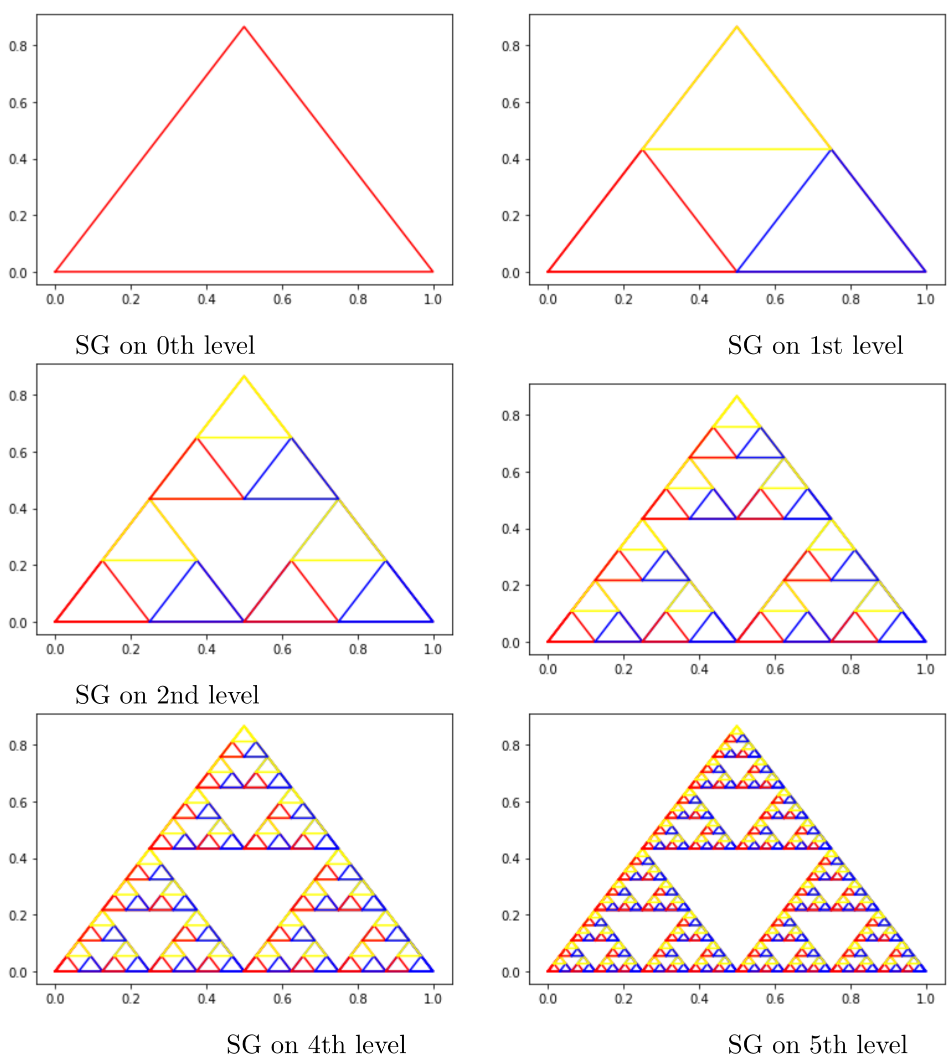


FIGURE 1. SG on various levels

The collection of vertices of the equilateral triangle is defined by

$$V_0 = \{p_1, p_2, p_3\}.$$

For $i \in \{1, 2, 3\}$, let us define the contraction maps $\chi_i : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$\chi_i(t) = \frac{1}{2}(t + p_i), \text{where } p_i \in V_0.$$

The aforementioned three contraction mappings form an IFS, which is an attractor for SG, that is,

$$SG = \chi_1(SG) \cup \chi_2(SG) \cup \chi_3(SG).$$

Let $N \in \mathbb{N} \cup \{0\}$, the set I^N be a cartesian product of the set I up to N times, where $I = \{1, 2, 3\}$ and it is the notation of all words having a length of N. If $\boldsymbol{i} \in I^N$, then

$$\boldsymbol{i} = i_1 i_2, \ldots, i_N = (i_1, i_2, \ldots, i_N), \text{where } i_j \in I.$$

To obtain SG up to the Nth level, we use N composition of contraction maps. Let $\chi_{\boldsymbol{i}}$ be the iteration and defined as follows:

$$\chi_{\boldsymbol{i}} = \chi_{i_1} \circ \chi_{i_2} \circ \cdots \circ \chi_{i_N}, \text{ where } \boldsymbol{i} \in I^N.$$

For $N \in \mathbb{N}$, consider the set V_N defined by

$$V_N = \{p_1, p_2, p_3, \chi_{\boldsymbol{i}}(p_2), \chi_{\boldsymbol{i}}(p_3), \chi_{\boldsymbol{i}}(p_1) : \boldsymbol{i} \in I^N\}.$$

The set V_N is Nth level of vertices and consists of all the images of V_0 with respect to iteration $\chi_{\boldsymbol{i}}$. For $N \in \mathbb{N}$, we further define $V_* = \cup_{N=1}^{\infty} V_N$. Let $f : SG \to \mathbb{R}$ be a continuous function on SG. The following IFS arises as an attractor for the graph of a continuous function defined on SG denoted by f^α, which satisfies $f^\alpha|_{V_N} = f|_{V_N}$. Let us assume $Y = SG \times \mathbb{R}$ and define the maps $W_{\boldsymbol{i}} : Y \to Y$ by

$$W_{\boldsymbol{i}}(t, x) = \Big(\chi_{\boldsymbol{i}}(t), E_{\boldsymbol{i}}(t, x)\Big), \ \boldsymbol{i} \in I^N,$$

where $E_{\boldsymbol{i}}(t, x) : SG \times \mathbb{R} \to \mathbb{R}$ is a contraction map in the second variable, where $\boldsymbol{i} \in I^N$ with $E_{\boldsymbol{i}}(p_j, f(p_j)) = f(\chi_{\boldsymbol{i}}(p_j))$. More precisely, we define

$$E_{\boldsymbol{i}}(t, x) = \alpha_{\boldsymbol{i}}(t)x + f(\chi_{\boldsymbol{i}}(t)) - \alpha_{\boldsymbol{i}}(t)b(t),$$

where the function $b : SG \to \mathbb{R}$ is a continuous base function, which satisfies $b|_{V_0} = f|_{V_0}$, and for any $\boldsymbol{i} \in \{1, 2, 3\}^N$, $\alpha_{\boldsymbol{i}} : SG \to \mathbb{R}$ is a continuous function satisfying $\|\alpha_{\boldsymbol{i}}\|_\infty < 1$. We now have an IFS $\{Y; W_{\boldsymbol{i}}, \boldsymbol{i} \in \{1, 2, 3\}^N\}$. Further, we take $\|\alpha\|_\infty = \max\{\|\alpha_{\boldsymbol{i}}\|_\infty : \boldsymbol{i} \in I^N\}$.

Theorem 3.1. [1] *Let $f : SG \to \mathbb{R}$ be a continuous function on SG. The IFS $\{Y; W_{\boldsymbol{i}}, \boldsymbol{i} \in I^N\}$ defined as above, has a unique attractor graph(f^α). The set graph(f^α) is the graph of a continuous function $f^\alpha : SG \to \mathbb{R}$, which satisfies $f^\alpha|_{V_N} = f|_{V_N}$. Furthermore, f^α satisfies the following functional equation*

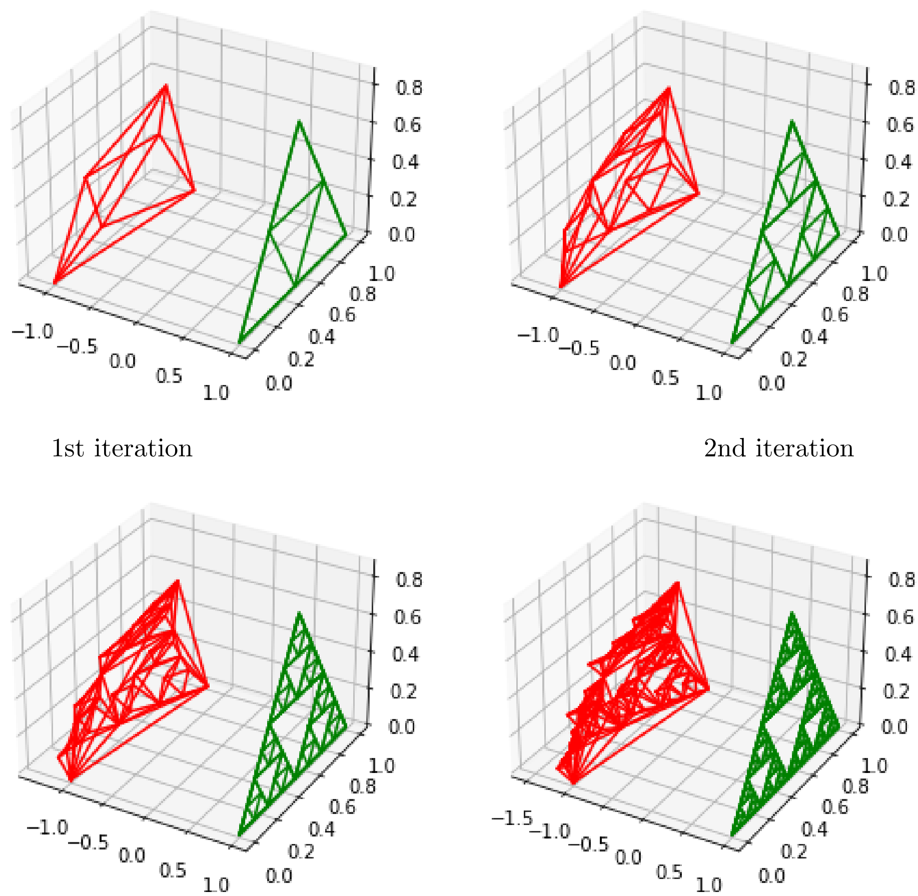
$$f^\alpha(t) = f(t) + \alpha_i(\chi_{\boldsymbol{i}}^{-1}(t))(f^\alpha - b)\big(\chi_{\boldsymbol{i}}^{-1}(t)\big) \ \forall \ t \in \chi_{\boldsymbol{i}}(SG), \ \boldsymbol{i} \in I^N. \tag{3.1}$$

For $f^\alpha(t) = E_{\boldsymbol{i}}\big(\chi_{\boldsymbol{i}}^{-1}(t), f^\alpha(\chi_{\boldsymbol{i}}^{-1}(t))\big) \ \forall t \in \chi_{\boldsymbol{i}}(SG), \boldsymbol{i} \in I^N$. This is further represented as $f^\alpha(\chi_{\boldsymbol{i}}(t)) = E_{\boldsymbol{i}}(t, f^\alpha(t))$ for all $t \in SG$ and $\boldsymbol{i} \in I^N$. It can be verified that the graph of f^α is an attractor of the IFS and hence,

$$\bigcup_{\boldsymbol{i} \in I^N} W_{\boldsymbol{i}}(graph(f^\alpha)) = graph(f^\alpha).$$

We denote the space of all the complex-valued continuous functions defined on SG by $\mathcal{C}(SG, \mathbb{C})$. For any complex-valued continuous function $g : SG \to \mathbb{C}$, that is, $g \in \mathcal{C}(SG, \mathbb{C})$, we define a complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha} : \mathcal{C}(SG, \mathbb{C}) \to \mathcal{C}(SG, \mathbb{C})$ by $\mathcal{F}_{\mathbb{C}}^{\alpha}(g) = g_{\mathbb{C}}^{\alpha}$, where $g_{\mathbb{C}}^{\alpha}$ is the fractal version of a complex-valued function $g \in \mathcal{C}(SG, \mathbb{C})$. Further, we define complex-valued bounded linear operator $L : \mathcal{C}(SG, \mathbb{C}) \to \mathcal{C}(SG, \mathbb{C})$ by $(Lg)|_{V_0} = g|_{V_0}$ and operator $Id : \mathcal{C}(SG, \mathbb{C}) \to \mathcal{C}(SG, \mathbb{C})$ is complex-valued identity operator.

The following figures [1, 2] represent the variations in the graph of f^{α}, one can observe that the graph of f^{α} is continuously dependent on parameters, that is, germ function $f(x, y)$, base function $b(x, y)$ and scaling factor α.



1st iteration 2nd iteration

3rd iteration 4th iteration

FIGURE 2. $f(x, y) = 2x^2 + y^2 - e^x$, $b(x, y) = 2x^2 + y^2 - e^x - x^2(x - 1)^2(4y^2 - 3)$ and $\alpha = 0.9$.

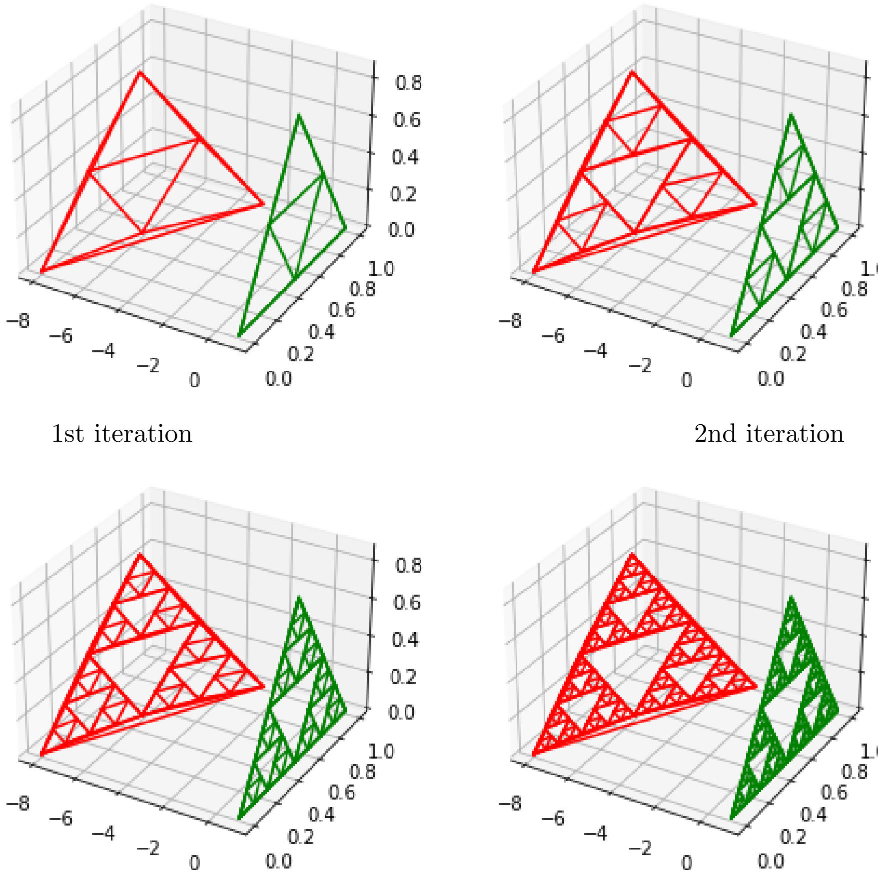
Theorem 3.2. *Consider a complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha} : \mathcal{C}(SG, \mathbb{C}) \to \mathcal{C}(SG, \mathbb{C})$, the perturbation error satisfies:*

$$\|\mathcal{F}_{\mathbb{C}}^{\alpha}(g) - g\|_{\infty} \leq \|\alpha\|_{\infty} \|\mathcal{F}_{\mathbb{C}}^{\alpha}(g) - Lg\|_{\infty}.$$

Consequently,

$$\|\mathcal{F}_{\mathbb{C}}^{\alpha}(g) - g\|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \|g - Lg\|_{\infty}.$$

Hence, a complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha}$ is bounded. Furthermore, if $\|\alpha\|_{\infty} = 0$ or $L = Id$, then $\mathcal{F}_{\mathbb{C}}^{\alpha} = Id$.



1st iteration 2nd iteration

3rd iteration 4th iteration

FIGURE 3. $f(x,y) = 3e^x - 11$, $b(x,y) = 3e^x - 11 - x^3(x-1)^2(4y^2-3)$ and $\alpha = 0.1$.

Lemma 3.3 ([7], Lemma 1)**.** *Let $M : X \to X$ be a linear operator, where $(X, \|.\|)$ be a Banach space. If the constants $c_1, c_2 \in [0,1)$ are so choosen such that*

$$\|Mx - x\| \leq c_1\|x\| + c_2\|Mx\|, \quad \textit{for all } x \in X.$$

Then M is topological isomorphism.

Theorem 3.4. *Consider a bounded linear operator with $\|\alpha\|_\infty < \|L\|^{-1}$. Then the complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}} : \mathcal{C}(SG, \mathbb{C}) \to \mathcal{C}(SG, \mathbb{C})$ is topological isomorphism.*

Proof. Using Theorem 3.2, we get

$$\|\mathcal{F}^\alpha_{\mathbb{C}}(f) - f\|_\infty \leq \|\alpha\|_\infty \|\mathcal{F}^\alpha_{\mathbb{C}}(f) - Lf\|_\infty \tag{3.2}$$

$$\leq \|\alpha\|_\infty \Big[\|\mathcal{F}^\alpha_{\mathbb{C}}(f)\|_\infty + \|L\|\|f\|_\infty \Big]. \tag{3.3}$$

Note that $\|\alpha\|_\infty < 1$ and $\|\alpha\|_\infty < \|L\|^{-1}$, then using the Lemma 3.3, one can prove that a complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}}$ is a topological isomorphism.

□

4. ENERGY OF COMPLEX-VALUED FRACTAL FUNCTIONS

There are many approaches available to define the Laplacian. Energy is used to define Laplacian on SG. Energy also computes the solutions of PDEs, see, for instance, [39] and so the notion of energy has attracted the attention of many researchers in the fractal community.

We first form a complete graph Υ_0 on the set V_0. Then, we form the complete graph on each level of SG. For $k \in \mathbb{N} \cup \{0\}$. Let Υ_k be the complete graph on the kth level of vertices of SG. Using V_k, we define the edge relation on kth level of SG. For any $a, b \in V_k$, the edge relation $a \sim_k b$ exists if and only if $a \in \chi_i(a'), b \in \chi_i(b')$ with $a' \sim_{k-1} b'$ and $i \in I$. Equivalently, $a \sim_k b$ if and only if one can choose $\boldsymbol{i} \in I^k$ such that $a, b \in \chi_k(V_0)$.

Definition 4.1. For $k \in \mathbb{N} \cup \{0\}$, the graph energy E_k on Υ_k is given by

$$E_k(g) = \left(\frac{5}{3}\right)^k \sum_{a \sim_k b} (g(a) - g(b))^2.$$

The sequence of graph energy, that is, $\big(E_k\big)_{k=0}^{\infty}$ admits $E_{k-1}(g) = \min E_k(g^*)$, where the minimum is taken over all g^* admits $g^*|_{V_{k-1}} = g$ for every $g : V_* \to \mathbb{R}$ and for every $k \in \mathbb{N}$. Note that the sequence $\big(E_k(g)\big)_{k=0}^{\infty}$ is increasing for every g on V_*. We refer the following limit as the energy of g on V_*

$$E(g) := \lim_{k \to \infty} E_k(g),$$

if $E(g) < \infty$, then g has a finite energy.

Recall that if a function g has a finite energy, that is, $E(g) < \infty$, then g is uniformly continuous.

We now determine the required conditions on the given parameters which ensure that the complex-valued fractal version $g^{\alpha}_{\mathbb{C}}$ has finite energy whenever the germ function $g : SG \to \mathbb{C}$ has finite energy. A complex-valued function $g : SG \to \mathbb{C}$ can be written as $g = g_{re} + ig_{im} = (g_{re}, g_{im})$, where g_{re} and g_{im} are real-valued functions defined by $g_{re} : SG \to \mathbb{R}$ and $g_{im} : SG \to \mathbb{R}$, respectively. The energy of a complex-valued is defined by

$$E(g, \mathbb{C}) = \max\{E(g_{re}), E(g_{im})\}.$$

For $\alpha \in \mathcal{C}(SG, \mathbb{C})$, where $\alpha = \alpha_{re} + \alpha_{im} = (\alpha_{re}, \alpha_{im})$, the energy $E(\alpha_{\mathbb{C}})$ is defined by

$$E(\alpha, \mathbb{C}) = \max\{E(\alpha_{re}), E(\alpha_{im})\}.$$

Similarly, for a complex-valued base function $b \in \mathcal{C}(SG, \mathbb{C})$, where $b = b_{re} + \alpha_{im} = (b_{re}, b_{im})$, we define the energy $E(b, \mathbb{C})$ as follows:

$$E(b, \mathbb{C}) = \max\{E(b_{re}), E(b_{im})\}.$$

Remark 4.2. The energy of a complex-valued function $g : SG \to \mathbb{C}$ can be defined. Now, define the graph energy $E^*_k(g, \mathbb{C})$ on Υ_k as follows:

$$E^*_k(g, \mathbb{C}) = \left(\frac{5}{3}\right)^k \sum_{a \sim_k b} \|(g(a) - g(b))\|_2^2.$$

Using the above, we define the energy of a complex-valued function g, that is, $E^*(g, \mathbb{C})$ as follows:

$$E^*(g, \mathbb{C}) = \lim_{k \to \infty} E^*_k(g, \mathbb{C}).$$

Using $\|z\|_\infty \le \|z\|_2 \le \sqrt{2}\|z\|_\infty$, we can deduce that

$$E^*(g, \mathbb{C}) < \infty \text{ if and only if } E(g, \mathbb{C}) < \infty.$$

We now define $dom(E,\mathbb{C}) = \{g \in \mathcal{C}(SG,\mathbb{C}) : E(g,\mathbb{C}) < \infty\}$.

Theorem 4.3. *Consider a germ complex-valued function $g \in dom(E,\mathbb{C})$ and let the parameter $b \in dom(E,\mathbb{C})$ admit $b|_{V_0} = g|_{V_0}$. For $N \in \mathbb{N}$, if $\alpha \in dom(E,\mathbb{C})$ with $\|\alpha\|_\infty < \frac{1}{2\sqrt{5^N}}$, then the corresponding complex-valued α-fractal function $g^\alpha_{\mathbb{C}} \in dom(E,\mathbb{C})$. Furthermore, it gives*

$$E(g^\alpha_{\mathbb{C}}) \leq \frac{4\ E(g,\mathbb{C}) + 8\ 5^N\|\alpha\|^2_\infty E(b,\mathbb{C}) + 4\ 5^N(\|g^\alpha_{\mathbb{C}}\|^2_\infty + 2\|b\|^2_\infty)E(\alpha,\mathbb{C})}{1 - 4\ 5^N\|\alpha\|^2_\infty}.$$

Proof. We use the functional equation to obtain the desired result.

$$\begin{aligned}
|g^\alpha_{re}(u) - g^\alpha_{re}(v)|^2 =& |g_{re}(u) - g_{re}(v) + \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u))g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(v))\\
&+ \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))b_{re}(\chi_{\boldsymbol{i}}^{-1}(v)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u))b_{re}(\chi_{\boldsymbol{i}}^{-1}(u))|^2\\
\leq& 4|g_{re}(a) - g_{re}(b)|^2\\
&+ 4\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u))|^2\ |g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 4\ |g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 8\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\ |b_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - b_{re}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 8\ |b_{re}(\chi_{\boldsymbol{i}}^{-1}(u))|^2\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
\leq& 4\ |g_{re}(a) - g_{re}(b)|^2\\
&+ 4\ \|\alpha\|^2_\infty\ |g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 4\ \|g^\alpha\|^2_\infty\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 8\ \|\alpha\|^2_\infty\ |b_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - b_{re}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 8\ \|b\|^2_\infty\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
\leq& 4\ |g_{re}(a) - g_{re}(b)|^2\\
&+ 4\ \|\alpha\|^2_\infty\ |g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - g^\alpha_{re}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 4\ (\|g^\alpha_{\mathbb{C}}\|^2_\infty + 2\|b\|^2_\infty)\ |\alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(u)) - \alpha_{\boldsymbol{i}}(\chi_{\boldsymbol{i}}^{-1}(v))|^2\\
&+ 8\ \|\alpha\|^2_\infty\ |b_{re}(\chi_{\boldsymbol{i}}^{-1}(u)) - b_{re}(\chi_{\boldsymbol{i}}^{-1}(v)|^2.
\end{aligned}$$

We compute the k-th level of energy

$$\begin{aligned}
E_k(g^\alpha_{re}) \leq& 4\ E_k(g_{re}) + 4\ 3^N\left(\frac{5}{3}\right)^N\ \|\alpha\|^2_\infty\ E_{k-N}(g^\alpha_{re})\\
&+ 8\ 3^N\left(\frac{5}{3}\right)^N\ \|\alpha\|^2_\infty\ E_{k-N}(b_{re})\\
&+ 4\ 3^N\left(\frac{5}{3}\right)^N\ (\|g^\alpha_{\mathbb{C}}\|^2_\infty + 2\|b\|^2_\infty)\ E_{k-N}(\alpha_{\boldsymbol{i}}).
\end{aligned} \tag{4.1}$$

Taking limit as $k \to \infty$, the aforementioned inequality gives

$$\begin{aligned}
E(g^\alpha_{re}) - 4\ 5^N\|\alpha\|^2_\infty E(g^\alpha_{re}) \leq& 4\ E(g_{re}) + 8\ 5^N\|\alpha\|^2_\infty E(b_{re})\\
&+ 4\ 5^N(\|g^\alpha_{\mathbb{C}}\|^2_\infty + 2\|b\|^2_\infty)E(\alpha_{\boldsymbol{i}}).
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
(1 - 4\ 5^N\|\alpha\|^2_\infty)E(g^\alpha_{re}) \leq& 4\ E(g_{re}) + 8\ 5^N\|\alpha\|^2_\infty E(b_{re})\\
&+ 4\ 5^N(\|g^\alpha_{\mathbb{C}}\|^2_\infty + 2\|b\|^2_\infty)E(\alpha_{\boldsymbol{i}}).
\end{aligned} \tag{4.3}$$

Similarly, we can obtain the similar inequalities for $E(g_{im})$, $E(b_{im})$ and $E(g^\alpha_{im})$. If $\|\alpha\|_\infty < \frac{1}{2\sqrt{5^N}}$, then from the definitions of $E(g,\mathbb{C})$, $E(\alpha,\mathbb{C})$ and $E(b,\mathbb{C})$, we get

$$(1 - 4\ 5^N \|\alpha\|_\infty^2) E(g_{\mathbb{C}}^\alpha) \leq 4\ E(g, \mathbb{C}) + 8\ 5^N \|\alpha\|_\infty^2 E(b, \mathbb{C}) \tag{4.4}$$
$$+ 4\ 5^N (\|g_{\mathbb{C}}^\alpha\|_\infty^2 + 2\|b\|_\infty^2) E(\alpha, \mathbb{C}). \tag{4.5}$$

Finally, we have

$$E(g_{\mathbb{C}}^\alpha) \leq \frac{4\ E(g, \mathbb{C}) + 8\ 5^N \|\alpha\|_\infty^2 E(b, \mathbb{C}) + 4\ 5^N (\|g_{\mathbb{C}}^\alpha\|_\infty^2 + 2\|b\|_\infty^2) E(\alpha, \mathbb{C})}{1 - 4\ 5^N \|\alpha\|_\infty^2}.$$

This proves the result.

□

We know that energy of constant function is always 0. On SG, we find the energy of a few continuous complex-valued function. Consider a continuous complex-valued functions. $f(z) = x + iy = (x, y)$, where $f_1(z) = x$ and $f_2(z) = y$ are continuous real-valued functions on SG. Note that for $k \in \mathbb{N} \cup \{0\}$, the graph energy of $E_k(f)$ on Υ_k is given by

$$E_k(f) = \left(\frac{5}{3}\right)^k \sum_{a \sim_k b} (f(a) - f(b))^2.$$

On $0th$ level vertices, that is, $V_0 = \left\{(0,0), (1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}$, we get

$$E_0(f_1) = \frac{5}{3} \sum_{a \sim_0 b} (f_1(a) - f_1(b))^2, \text{ where } a, b \in V_0.$$

By simple calculation, we obtain

$$E_0(f_1) = \frac{5}{3}\left[1^2 + 2\left(\frac{1}{2}\right)^2\right].$$

For the $1st$ level vertices $V_1 = \{(0,0), \left(\frac{1}{2}, 0\right), (1,0), \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\}$, we now find the $1st$ level of energy for f_1 as

$$E_1(f_1) = \left(\frac{5}{3}\right)^2 \sum_{a \sim_1 b} (f_1(a) - f_1(b))^2, \text{ where } a, b \in V_1.$$

A simple computation yields

$$E_1(f_1) = 3\left(\frac{5}{3}\right)^2 \left[\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{4}\right)^2\right].$$

For the $2nd$ level vertices V_2, we now find the $2nd$ level of energy for f_1, that is,

$$E_2(f_1) = \left(\frac{5}{3}\right)^3 \sum_{a \sim_2 b} (f_1(a) - f_1(b))^2, \text{ where } a, b \in V_2.$$

By simple calculation, we get

$$E_2(f_1) = 3^2\left(\frac{5}{3}\right)^3 \left[\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{8}\right)^2\right].$$

Similarly, for nth level vertices, we get the nth level of energy E_n as

$$E_n(f_1) = 3^n\left(\frac{5}{3}\right)^{n+1} \left[\left(\frac{1}{2^n}\right)^2 + 2\left(\frac{1}{2^{n+1}}\right)^2\right].$$

Hence,

$$E_n(f_1) = \frac{5^{n+1}}{2^{2n+1}}.$$

We finally get the graph energy of $f_1(z) = x$, that is,

$$E(f_1) = \lim_{k\to\infty} E_n(f_1) \tag{4.6}$$

$$= \lim_{k\to\infty} \frac{5^{n+1}}{2^{2n+1}} \tag{4.7}$$

$$= \infty. \tag{4.8}$$

Note that $E(f) = \max\{E(f_1), E(f_2)\}$. Thus,

$$E(f) = \infty.$$

Example 4.4. We calculate the graph energy of $g(z) = sin(x) + icos(y) = (sin(x), cos(y))$ on SG, where $g_1(z) = sin(x)$ and $g_2 = cos(y)$. We first find the graph energy of $g_1(z) = sin(x)$ on SG.

By the Mean Value Theorem, we have

$$|\sin(x) - \sin(y)| \geq \cos(1)\,|x-y| \;\; \forall\, x, y\, \in (0,1)\,.$$

Combining the above inequality with the nth level of graph energy of the above function $f(z) = x$, we can find the bounds of the graph energy for each level of the function $g_1(z) = sin(x)$. In particular, the bounds for the nth level of energy is given by

$$E_n(g_1) \geq \frac{5^{n+1}}{2^{2n+1}} cos(1)^2.$$

Using

$$\lim_{n\to\infty} \frac{5^{n+1}}{2^{2n+1}} cos(1)^2 = \infty.$$

We get

$$\lim_{n\to\infty} E_n(g_1) = \infty.$$

Hence,

$$E(g_1) = \infty.$$

Note that $E(g) = \max\left(E(g_1), E(g_2)\right)$. Hence, we get

$$E(g) = \infty.$$

In particular, from the above functions, we get $E(g) = E(\overline{g})$.

5. SOME PROPERTIES OF COMPLEX-VALUED FRACTAL OPERATOR

Theorem 5.1. *For $\|\alpha\|_\infty < \|L\|^{-1}$, the complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}}$ is not a compact operator.*

Proof. For $\|\alpha\|_\infty < \|L\|^{-1}$, one can deduce that $\mathcal{F}^\alpha_{\mathbb{C}} : \mathcal{C}(SG,\mathbb{C}) \to \mathcal{C}(SG,\mathbb{C})$ is one-one. Recall that range space of complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}}$ is infinite dimensional. Let us now define the inverse complex-valued map $(\mathcal{F}^\alpha_{\mathbb{C}})^{-1} : \mathcal{F}^\alpha_{\mathbb{C}}\big(\mathcal{C}(SG,\mathbb{C})\big) \to \mathcal{C}(SG,\mathbb{C})$. A complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}}$ is bounded below with respect to selected α, and this implies that $(\mathcal{F}^\alpha_{\mathbb{C}})^{-1}$ is bounded linear operator.

Now suppose that complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}}$ is a compact operator. Using the fact that the composition of a compact operator and a bounded operator is again a compact operator, one can establish that the complex-valued operator $\mathcal{F}^\alpha_{\mathbb{C}}(\mathcal{F}^\alpha_{\mathbb{C}})^{-1} : \mathcal{F}^\alpha_{\mathbb{C}}\big(\mathcal{C}(SG,\mathbb{C})\big) \to \mathcal{C}(SG,\mathbb{C})$ is a compact operator, which is a contradiction to the infinite dimensionality of the space $\mathcal{F}^\alpha_{\mathbb{C}}\big(\mathcal{C}(SG,\mathbb{C})\big)$. Thus, complex-valued fractal operator $\mathcal{F}^\alpha_{\mathbb{C}}$ is not a compact operator. ∎

Theorem 5.2. *If* $\|\alpha\|_\infty < (1+\|Id-L\|)^{-1})^{-1}$, *then complex-valued fractal operator* $\mathcal{F}_{\mathbb{C}}^{\alpha}$ *is Freadholm and its index is zero.*

Proof. Under the hypothesis, range space of complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha}$ is closed. Furthermore, complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha}$ is invertible. Keep in mind if $T : X \to Y$ in invertible, then T^* is invertible [6], this implies that complex-valued fractal operator $(\mathcal{F}_{\mathbb{C}}^{\alpha})^*$ is invertible. Hence, complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha}$ is a Fredholm. Note that we define the index of a Fredholm operator as follows:

$$\text{index}(\mathcal{F}_{\mathbb{C}}^{\alpha}) = \dim\big(\ker(\mathcal{F}_{\mathbb{C}}^{\alpha})\big) - \dim\big(\ker(\mathcal{F}_{\mathbb{C}}^{\alpha})^*\big),$$

thus, the index is 0. This completes the proof. □

In the next theorem, we prove the existence of a complex-valued non-trivial closed invariant subspace of the bounded linear complex-valued fractal operator.

The invariant subspace problem is an important aspect of operator theory that must be taken into consideration. For researchers it is an interesting problem to find the existence of non-trivial invariant subspaces for many distinct operators. Consequently, finding the existence of non-trivial invariant subspaces of an operator is extremely intriguing. Various researchers have constructed an operator without a non-trivial closed invariant subspace; see, for instance, [38].

Theorem 5.3. *There exists a complex-valued non-trivial closed invariant subspace for the bounded linear complex-valued fractal operator* $\mathcal{F}_{\mathbb{C}}^{\alpha} : \mathcal{C}(SG, \mathbb{C}) \to (SG, \mathbb{C})$.

Proof. Select a non-zero complex-valued function $h \in \mathcal{C}(SG, \mathbb{C})$ satisfying $h(t) = 0$ for all $t \in V_N$. We denote the composition of $\mathcal{F}_{\mathbb{C}}^{\alpha}$ with itself $k - times$ by $(\mathcal{F}_{\mathbb{C}}^{\alpha})^k$ and $(\mathcal{F}_{\mathbb{C}}^{\alpha})^0(h) = h$. Let $\mathcal{V}_h$ be the non-zero subspace

$$\mathcal{V}_h = \text{ span}\{h, \mathcal{F}_{\mathbb{C}}^{\alpha}(h), (\mathcal{F}_{\mathbb{C}}^{\alpha})^2(h), \dots\}.$$

It can be easily shown that $\mathcal{V}_h$ is an invariant subspace of complex-valued fractal operator $\mathcal{F}_{\mathbb{C}}^{\alpha}$, that is, $\mathcal{F}_{\mathbb{C}}^{\alpha}(\mathcal{V}_h) \subseteq \mathcal{V}_h$. Note that if $f \in \mathcal{V}_h$, then $f(t) = 0 \ \forall \ t \in V_N$. Using the definition of $\mathcal{V}_h$, one can get a constants $A_i \in \mathbb{C}$ such that

$$f = \sum_{i=1}^{i=m} A_i (\mathcal{F}_{\mathbb{C}}^{\alpha})^{k_i}(h),$$

where $k_i \in \mathbb{N} \cup \{0\}$. Using the interpolatory property of the fractal operator, one gets

$$(\mathcal{F}_{\mathbb{C}}^{\alpha}(h))(t) = h(t), \text{for all } t \in V_N.$$

Consequently, for all $t \in V_N$, one have $f(t) = 0$.

Consider $\mathcal{V} = \overline{\mathcal{V}_h}$. It can be easily verified that $\mathcal{F}_{\mathbb{C}}^{\alpha}(\mathcal{V}) \subseteq \mathcal{V}$, we now prove that $\mathcal{V} \neq \mathcal{C}(SG, \mathbb{C})$. For $g \in \mathcal{V}$, one can get a complex-valued sequence $(g_m)_{m\in\mathbb{N}} \subset \mathcal{V}_h$ such that $g_m \to g$ uniformly. Since $g_m(t) = 0$ for all m and $t \in V_N$. Consequently, we have $g(t) = 0$ for all $t \in V_N$. Thus, a complex-valued continuous function g is non-vanishing at some points of V_N can not be an element in $\mathcal{V}$. In particular, $\mathcal{V} \neq \mathcal{C}(SG, \mathbb{C})$, this completes the proof.

□

In the upcoming theorem, we show the existence of a complex-valued Schauder bases consisting of fractal functions for the complex-valued space.

Finding the Schauder bases for different domains and spaces such as the interval, square or rectangle is itself a huge challenge. For $k \in \mathbb{N}$, we know the Schauder bases for the space of k times continuously differentiable spaces on the compact interval I of $\mathbb{R}$, that is $\mathcal{C}^K(I)$.

Schonefeld [37] constructed the Schauder bases for the space of one time continuously differentiable on $[0\ 1] \times [0\ 1]$ in 1968, however, numerous spaces exist that don't have a Schauder basis, see, for instance, the space l^{∞} does not have a Schauder basis.

Theorem 5.4. *For $\mathcal{C}(SG, \mathbb{C})$, there exist a complex-valued Schauder basis consisting of fractal functions.*

Proof. We consider a Schauder basis (z_n) of the complex-valued space $\mathcal{C}(SG, \mathbb{C})$, where $z_n = x_n + iy_n$. Select α satisfying $\|\alpha\|_{\infty} < \|L\|^{-1}$, then using Theorem 3.4, one can see that a complex-valued fractal operator $\mathcal{F}^{\alpha}_{\mathbb{C}}$ is topological isomorphism. If $h \in \mathcal{C}(SG, \mathbb{C})$ then $(\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h) \in \mathcal{C}(SG, \mathbb{C})$, then we can write

$$(\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h) = \sum_{n=1}^{\infty} a_n \Big((\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h) \Big) z_n.$$

Using the continuity of the fractal operator, one can obtain

$$h = \mathcal{F}^{\alpha}_{\mathbb{C}} (\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h) = \sum_{n=1}^{\infty} a_n \Big((\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h) \Big) z_n^{\alpha},$$

where $z_n^{\alpha} = \mathcal{F}^{\alpha}_{\mathbb{C}}(z_n)$. We can also write h as $h = \sum_{n=1}^{\infty} d_n z_n^{\alpha}$. Using the continuity of $(\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}$, we obtain the following inequality

$$(\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h) = \sum_{n=1}^{\infty} d_n z_n,$$

this gives $d_n = a_n\big((\mathcal{F}^{\alpha}_{\mathbb{C}})^{-1}(h)\big)$ for every n. Hence, we obtain the Schauder basis (z_n^{α}) for $\mathcal{C}(\mathrm{SG}, \mathbb{C})$. This completes the proof.

□

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