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EXISTENCE OF SOLUTIONS TO A GENERALIZED BOUNDARY VALUE PROBLEM ON THE HALF-LINE

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ABSTRACT. The authors study boundary value problems of order $2k$ for $k \in \{1, 2, \dots\}$ on the half-line and prove the existence of solutions using a minimization principle and the Mountain Pass Theorem.

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1. INTRODUCTION

We consider the existence of solutions to a family of boundary value problems on the half-line of the form

$$\text{(P)} \qquad \begin{cases} (-1)^k u^{(2k)}(t) + (-1)^{k-1} u^{(2k-2)}(t) + (-1)^{k-2} u^{(2k-4)}(t) + \dots \\ \qquad\qquad + (-1)^1 u''(t) + u(t) = f(t, u(t)), & t \in [0, +\infty), \\ u^{(2i)}(0) = u^{(2i)}(+\infty) = 0, \quad i \in \{0, 1, \dots, k-1\} \end{cases}$$

where $f \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$ and $k \in \mathbb{N}^+$. While there have been numerous works for boundary value problems on finite intervals, especially in the cases $k = 1$ and $k = 2$, i.e., for second and fourth order problems, there has been considerably fewer works for such problems on the half-line. And for $k \geq 3$ there does not appear to be any such results in the literature on this type of problem.

In order to establish the setting in which to investigate our problem, we start by considering the space

$$H_0^k(0, +\infty) = \{u \in L^2(0, +\infty) : u', u'', \dots, u^{(k)} \in L^2(0, +\infty), \\ u(0) = 0, u'(0) = 0, \dots, u^{(k-1)}(0) = 0\}$$

together with its natural norm

$$\|u\| = \left(\int_0^{+\infty} (u^{(k)})^2(t) dt + \int_0^{+\infty} (u^{(k-1)})^2(t) dt + \dots + \int_0^{+\infty} u^2(t) dt \right)^{\frac{1}{2}}.$$

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In view of [5, Corollary 8.9] (or see [6, Chapter 8]) if $u \in H_0^k(0,+\infty)$, then

$$u(+\infty) = u'(+\infty) = \cdots = u^{(k-1)}(+\infty) = 0. \tag{1.1}$$

Let $p_0, p_1, \dots, p_{k-1} : [0,+\infty) \to (0,+\infty)$ be bounded and continuously differentiable functions with

$$M_i = \max(\|p_i\|_{L^2}, \|p_i'\|_{L^2}) < +\infty, \quad \text{for each} \quad i = 0, 1, \dots, k-1.$$

We also consider the spaces

$$C_{l,p_0}[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} p_0(t)u(t) \text{ exists}\}$$

endowed with the norm

$$\|u\|_{\infty,p_0} = \sup_{t\in[0,+\infty)} p_0(t)|u(t)|,$$

and

$$C_{l,p}^{k-1}[0,+\infty) = \{u \in C^{k-1}([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} p_0(t)u(t), \lim_{t\to+\infty} p_1(t)u'(t), \lim_{t\to+\infty} p_2(t)u''(t), \dots, \lim_{t\to+\infty} p_{k-1}(t)u^{(k-1)}(t) \text{ exist}\}$$

with the natural norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p_0(t)|u(t)| + \sup_{t\in[0,+\infty)} p_1(t)|u'(t)| + \sup_{t\in[0,+\infty)} p_2(t)|u''(t)| + \cdots + \sup_{t\in[0,+\infty)} p_{k-1}(t)|u^{(k-1)}(t)|.$$

Let

$$C_l[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} u(t) \text{ exists}\}$$

be endowed with the norm

$$\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|.$$

To prove that $H_0^k(0,+\infty)$ embeds compactly into $C_{l,p}^{k-1}[0,+\infty)$ (see Lemma 1.10 below), we need the following compactness criterion based on the work of Corduneanu [8, page 62].

Lemma 1.1. *Let $D \subset C_{l,p}^{k-1}[0,+\infty)$ be a bounded set. Then D is relatively compact if the following conditions hold:*

(a) *D is equicontinuous on any compact sub-interval of $[0,+\infty)$, i.e., for any compact set $J \subset [0,+\infty)$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|p_0(t_1)u(t_1) - p_0(t_2)u(t_2)| \le \varepsilon$, $|p_1(t_1)u'(t_1) - p_1(t_2)u'(t_2)| \le \varepsilon$, $\dots$, $|p_{k-1}(t_1)u^{(k-1)}(t_1) - p_{k-1}(t_2)u^{(k-1)}(t_2)| \le \varepsilon$ for all $u \in D$;*

(b) *D is equiconvergent at $+\infty$, i.e., for every $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that, $t \ge T(\varepsilon)$ implies $|p_0(t)u(t) - (p_0u)(+\infty)| \le \varepsilon$, $|p_1(t)u'(t) - (p_1u)'(\infty)| \le \varepsilon$, $\dots$, $|p_{k-1}(t)u^{(k-1)}(t) - (p_{k-1}u)^{(k-1)}(+\infty)| \le \varepsilon$ for all $u \in D$.*

We also need the following concepts from critical point theory to continue our analysis (see, for example, [1, 3, 10]).

Definition 1.2. Let X be a Banach space, $\Omega \subset X$ be an open set, and $J : \Omega \to \mathbb{R}$ be a functional. We say that J is Gâteaux differentiable at $u \in \Omega$ if there exists $A \in X'$ (dual space) such that

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = Av$$

for all $v \in X$. Now A, which is unique, is denoted by $A = J_G'(u)$.

The mapping which sends to every $u \in \Omega$ the mapping $J'_G(u)$ is called the Gâteaux differential of J and is denoted by J'_G. We say that $J \in C^1(X, \mathbb{R})$ if J is Gâteaux differentiable on Ω and J'_G is continuous at every $u \in \Omega$.

Definition 1.3. Let X be a Banach space. A functional $J : \Omega \to \mathbb{R}$ is called coercive if for every sequence $(u_k)_{k\in\mathbb{N}} \subset X$,

$$\|u_k\| \to +\infty \text{ implies } |J(u_k)| \to +\infty.$$

Definition 1.4. Let X be a Banach space. A functional $J : X \to (-\infty, +\infty]$ is said to be sequentially weakly lower semi-continuous (swlsc) if

$$J(u) \leq \liminf_{n\to+\infty} J(u_n)$$

for all sequences $u_n \rightharpoonup u$ in X as $n \to +\infty$.

The following minimization principle can be found in a number of places including the monograph of Badiale and Serra [3].

Lemma 1.5. (Minimization Principle.) *Let X be a reflexive Banach space and J be a functional defined on X such that:*

(1) $\lim\limits_{\|u\|\to+\infty} J(u) = +\infty$ *(coercivity condition),*
(2) *J is sequentially weakly lower semi-continuous.*

Then J is bounded from below on X and achieves its greatest lower bound at some point u_0.

Next, we define the well-known Palais-Smale condition.

Definition 1.6. Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(J(u_n))$ is bounded in $\mathbb{R}$, and $J'(u_n) \to 0$ as $n \to +\infty$ in X' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale (PS) condition.

We can now state the famous Mountain Pass Theorem

Lemma 1.7. (Mountain Pass Theorem.) *Let X be a Banach space, and let $J \in C^1(X, \mathbb{R})$ be such that $J(0) = 0$. Assume that J satisfies the (PS) condition and there exist positive numbers ρ and α such that:*

(1) *$J(u) \geq \alpha$ if $\|u\| = \rho$,*
(2) *there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$.*

Then there exists a critical point u^* of J satisfying

$$J'(u^*) = 0 \quad \textit{and} \quad J(u^*) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = u_0\}.$$

In order to provide the appropriate variational setting for our problem, we multiply the the equation in Problem (P) by $v \in \mathcal{S}$, integrate over $(0, +\infty)$ using integration by parts, and apply the boundary conditions and conditions (1.1), to obtain

$$\int_0^{+\infty} \Big[(-1)^k u^{(2k)}(t) + (-1)^{k-1} u^{(2k-2)}(t) + (-1)^{k-2} u^{(2k-4)}(t) \\ + \cdots + (-1)^1 u''(t) + u(t)\Big] v(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt,$$

and so from the definition of $H_0^k(0,+\infty)$ and (1.1), we see that

$$\int_0^{+\infty}\Big[u^{(k)}(t)v^{(k)}(t)+u^{(k-1)}(t)v^{(k-1)}(t)+\dots \\ +u'(t)v'(t)+u(t)v(t)\Big]\,dt=\int_0^{+\infty}f(t,u(t))v(t)dt. \tag{1.2}$$

This leads quite naturally to the notion of a weak solution to Problem (P).

Definition 1.8. By a weak solution of (P) we mean a function $u\in H_0^k(0,+\infty)$ such that (1.2) holds.

In order to study Problem (P), we consider the functional $J:H_0^k(0,+\infty)\to\mathbb{R}$ defined by

$$J(u)=\frac{1}{2}\|u\|^2-\int_0^{+\infty}F(t,u(t))dt,$$

where

$$F(t,u)=\int_0^u f(t,s)ds.$$

Next, we need to establish some embedding properties for our spaces.

Lemma 1.9. *The space $H_0^k(0,+\infty)$ embeds continuously into $C_{l,p}^{k-1}[0,+\infty)$.*

Proof. For $u\in H_0^k(0,+\infty)$ and $i=0,1,\dots,k-1$, we have

$$\begin{aligned}
|p_i(t)u^{(i)}(t)| &= |p_i(+\infty)u^{(i)}(+\infty)-p_i(t)u^{(i)}(t)|\\
&=\left|\int_t^{+\infty}(p_iu^{(i)})'(s)ds\right|\\
&\le\left|\int_t^{+\infty}p_i'(s)u^{(i)}(s)ds\right|+\left|\int_t^{+\infty}p_i(s)u^{(i+1)}(s)ds\right|\\
&\le\left(\int_t^{+\infty}(p_i'(s))^2ds\right)^{\frac{1}{2}}\left(\int_t^{+\infty}(u^{(i)}(s))^2ds\right)^{\frac{1}{2}}\\
&\quad+\left(\int_t^{+\infty}p_i^2(s)ds\right)^{\frac{1}{2}}\left(\int_t^{+\infty}(u^{(i+1)}(s))^2ds\right)^{\frac{1}{2}}\\
&\le\max\{\|p_i\|_{L^2},\|p_i'\|_{L^2}\}\,\|u\|\le M_i\|u\|,
\end{aligned}$$

where $M_i=\max\{\|p_i\|_{L^2},\|p_i'\|_{L^2}\}$. Hence, $\|u\|_{\infty,p}\le M\|u\|$, where $M=\max\{M_i:i=0,1,\dots,k-1\}$. The continuity of the embedding follows directly. □

We next show that the above embedding is in fact compact.

Lemma 1.10. *The embedding $H_0^k(0,+\infty)\hookrightarrow C_{l,p}^{k-1}[0,+\infty)$ is compact.*

Proof. Let $D\subset H_0^k(0,+\infty)$ be a bounded set. Then by Lemma 1.9, it is bounded in $C_{l,p}^{k-1}[0,+\infty)$. Let $R>0$ be such that for all $u\in D$ we have $\|u\|\le R$. We will apply Lemma 1.1.

To see that D is equicontinuous on every compact interval of $[0,+\infty)$, let $u\in D$ and $t_1,t_2\in J\subset[0,+\infty)$, where J is compact. Applying the Cauchy-Schwarz inequality for each $i=0,1,\dots,k-1$, gives

$$\begin{aligned}|p_i(t_1)u^{(i)}(t_1) - p_i(t_2)u^{(i)}(t_2)| &= \left|\int_{t_2}^{t_1} (p_i u^{(i)})'(s)ds\right| \\ &\le \left|\int_{t_2}^{t_1} p_i'(s)u^{(i)}(s) + p_i(s)u^{(i+1)}(s)ds\right| \\ &\le \left(\int_{t_2}^{t_1} (p_i'(s))^2 ds\right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} (u^{(i)}(s))^2 ds\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{t_2}^{t_1} p_i^2(s) ds\right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} (u^{(i+1)}(s))^2 ds\right)^{\frac{1}{2}} \\ &\le \max\left[\left(\int_{t_2}^{t_1} p_i^2(s)ds\right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} (p_i'(s))^2 ds\right)^{\frac{1}{2}}\right] \|u\| \\ &\le R\max\left[\left(\int_{t_2}^{t_1} p_i^2(s)ds\right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} (p_i'(s))^2 ds\right)^{\frac{1}{2}}\right] \to 0,\end{aligned}$$

as $\mid t_1 - t_2 \mid \to 0$.

Now to see that D is equiconvergent at $+\infty$, let $t \in [0, +\infty)$ and $u \in D$. Since p_0, p_1, $\ldots$, p_{k-1} are bounded, using the fact that $(p_i u)^{(i)}(+\infty) = 0$ for $i = 0, 1, \ldots, k-1$, and applying the Cauchy-Schwarz inequality, a similar calculation shows that

$$|(p_i u^{(i)})(t) - (p_i u^{(i)})(+\infty)| \le R\max\left[\left(\int_t^{+\infty} p_i^2(s)ds\right)^{\frac{1}{2}}, \left(\int_t^{+\infty} (p_i'(s))^2 ds\right)^{\frac{1}{2}}\right] \to 0$$

as $t \to +\infty$. Hence, D is relatively compact. Applying the Arzelà-Ascoli theorem completes the proof of the lemma. □

The following corollary follows directly from the above lemmas.

Corollary 1.11.

(i) $C^{k-1}_{l,p}[0, +\infty)$ *embeds continuously into* $C_{l,p_0}[0, +\infty)$.

(ii) *The embedding* $H^k_0(0, +\infty) \hookrightarrow C_{l,p_0}[0, +\infty)$ *is continuous and compact.*

Remark 1.12. As a consequence of Corollary 1.11(ii), there is a constant N such that

$$\|u\|_{\infty,p_0} \le N\|u\|.$$

2. EXISTENCE OF SOLUTIONS

In this section we prove our main existence results. Our first theorem is as follows.

Theorem 2.1. *Assume there exist a constant* $\theta \in [0, 2)$ *and a function* $a \in C([0, +\infty), \mathbb{R})$ *with* $\dfrac{a(t)}{p_0^\theta(t)} \in L^1([0, +\infty), \mathbb{R})$, *such that*

$$\limsup_{|u|\to+\infty} \frac{F(t,u)}{|u|^\theta} \le a(t) \text{ uniformly with respect to } t \in [0, +\infty). \tag{2.1}$$

Then the boundary value problem (P) has at least one weak solution.

Proof. By (2.1), there exists $R_1 > 0$ such that

$$F(t, x) \le a(t)|x|^\theta \text{ for all } t \in [0, +\infty) \text{ and all } |x| \ge R_1.$$

If we combine this with the continuity of $F(t,x)-a(t)|x|^\theta$ on $[0,+\infty)\times[-R_1,R_1]$, it is clear that there exists a function $C_1\in L^1([0,+\infty),\mathbb{R}^+)\cap C([0,+\infty),\mathbb{R}^+)$ such that

$$F(t,x)\le a(t)|x|^\theta + C_1(t) \text{ for all } t\in[0,+\infty) \text{ and all } x\in\mathbb{R}. \tag{2.2}$$

To see that J is well defined, take $u\in H_0^k(0,+\infty)$; then,

$$\begin{aligned}\int_0^{+\infty}|F(t,u(t))|dt &\le \int_0^{+\infty}\left[|a(t)||u(t)|^\theta + C_1(t)\right]dt\\ &\le \int_0^{+\infty}|a(t)||u(t)|^\theta dt + \int_0^{+\infty}C_1(t)dt\\ &\le \int_0^{+\infty}\left|\frac{a(t)}{p_0^\theta(t)}\right||p_0(t)u(t)|^\theta dt + |C_1|_{L^1}\\ &\le \int_0^{+\infty}\left|\frac{a(t)}{p_0^\theta(t)}\right|dt\,\|u\|_{\infty,p_0}^\theta + |C_1|_{L^1}\\ &\le N^\theta\left|\frac{a}{p_0^\theta}\right|_{L^1}\|u\|^\theta + |C_1|_{L^1}.\end{aligned}$$

Therefore,

$$\begin{aligned}|J(u)| &= \left|\frac{1}{2}\|u\|^2 - \int_0^{+\infty}F(t,u(t))dt\right|\\ &\le \frac{1}{2}\|u\|^2 + \int_0^{+\infty}|F(t,u(t))|dt\\ &\le \frac{1}{2}\|u\|^2 + N^\theta\left|\frac{a}{p_0^\theta}\right|_{L^1}\|u\|^\theta + |C_1|_{L^1} < +\infty,\end{aligned}$$

which is what we wanted to show.

Now,

$$J(u)=\frac{1}{2}\|u\|^2-\int_0^{+\infty}F(t,u(t))dt \ge \frac{1}{2}\|u\|^2 - N^\theta\left|\frac{a}{p_0^\theta}\right|_{L^1}\|u\|^\theta - |C_1|_{L^1}.$$

Hence, $\lim_{\|u\|\to\infty}J(u)=+\infty$ since $0<\theta<2$, and so J is coercive.

Finally, to show that J is sequentially weakly lower semi-continuous, let (u_n) be a sequence in $H_0^k(0,+\infty)$ such that $u_n \rightharpoonup u$ as $n\to+\infty$ in $H_0^k(0,+\infty)$. Then there exists a constant $A>0$ such that $\|u_n\|<A$ for all $n\ge 0$ and $\|u\|<A$. In view of Corollary 1.11(ii), $(p_0(t)u_n(t))$ converges to $(p_0(t)u(t))$ as $n\to+\infty$ for $t\in[0,+\infty)$. Since F is continuous, we have $F(t,u_n(t))\to F(t,u(t))$ as $n\to+\infty$. Also,

$$\begin{aligned}|F(t,u_n(t))| &\le |a(t)||u_n(t)|^\theta + C_1(t)\\ &\le \left|\frac{a(t)}{p_0^\theta(t)}\right||p_0(t)u_n(t)|^\theta + C_1(t)\\ &\le \left|\frac{a(t)}{p_0^\theta(t)}\right|\|u\|_{\infty,p_0}^\theta + C_1(t)\\ &\le \frac{|a(t)|}{p_0^\theta(t)}N^\theta\|u\|^\theta + C_1(t)\\ &\le \frac{|a(t)|}{p_0^\theta(t)}N^\theta A^\theta + C_1(t),\end{aligned}$$

so by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to+\infty}\int_0^{+\infty}F(t,u_n(t))dt = \int_0^{+\infty}F(t,u(t))dt.$$

Since the norm in a reflexive Banach space is sequentially weakly lower semi-continuous,

$$\liminf_{n\to+\infty} \|u_n\| \geq \|u\|.$$

Hence,

$$\begin{aligned}\liminf_{n\to+\infty} J(u_n) &= \liminf_{n\to+\infty} \left(\frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} F(t,u_n(t))dt\right)\\ &= \liminf_{n\to+\infty} \frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} F(t,u(t))dt\\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt = J(u),\end{aligned}$$

and so J is sequentially weakly lower semi-continuous.

From the Minimization Principle, Lemma 1.5, J possesses a critical point that in turn is a weak solution of Problem (P). □

Theorem 2.2. *If there exist functions b and $c \in C([0,+\infty),\mathbb{R})$ with $c, \dfrac{b}{p_0^2} \in L^1([0,+\infty),\mathbb{R})$ such that*

$$F(t,x) \leq b(t)|x|^2 + c(t) \text{ for all } t \in [0,+\infty) \text{ and all } x \in \mathbb{R},$$

with

$$N^2 \left|\frac{b}{p_0^2}\right|_{L^1} < \frac{1}{2}, \tag{2.3}$$

then the boundary value problem (P) *has at least one weak solution.*

Proof. We will prove this through a series of claims.

Claim 1: *J is well defined.* For $u \in H_0^k(0,+\infty)$, we have

$$\begin{aligned}\int_0^{+\infty} |F(t,u(t))|dt &\leq \int_0^{+\infty} |b(t)||u(t)|^2 dt + \int_0^{+\infty} |c(t)|dt\\ &\leq \int_0^{+\infty} \left|\frac{b(t)}{p_0^2(t)}\right| |p_0(t)u(t)|^2 dt + |c|_{L^1}\\ &\leq \int_0^{+\infty} \left|\frac{b(t)}{p_0^2(t)}\right| dt \|u\|^2_{\infty,p_0} + |c|_{L^1}\\ &\leq N^2 \left|\frac{b}{p_0^2}\right|_{L^1} \|u\|^2 + |c|_{L^1}.\end{aligned}$$

Thus,

$$\begin{aligned}|J(u)| &= \left|\frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt\right|\\ &\leq \frac{1}{2}\|u\|^2 + \int_0^{+\infty} |F(t,u(t))|dt\\ &\leq \frac{1}{2}\|u\|^2 + N^2 \left|\frac{b}{p_0^2}\right|_{L^1} \|u\|^2 + |c|_{L^1} < +\infty.\end{aligned}$$

Claim 2. *J is coercive.* We have

$$\begin{aligned}J(u) &= \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt\\ &\geq \frac{1}{2}\|u\|^2 - N^2 \left|\frac{b}{p_0^2}\right|_{L^1} \|u\|^2 - |c|_{L^1}\\ &\geq \left(\frac{1}{2} - N^2 \left|\frac{b}{p_0^2}\right|_{L^1}\right) \|u\|^2 - |c|_{L^1},\end{aligned}$$

so J is coercive by (2.3).

Claim 3. *J is sequentially weakly lower semi-continuous.* An argument similar to the one used in the proof of Theorem 2.1 shows this.

Thus, by Lemma 1.5, J possesses a critical point that is a weak solution of Problem (P). □

Our final existence result is contained in the following theorem.

Theorem 2.3. *Assume that:*

(F1) *There exist positive functions c_1, $c_2 \in L^1(0,+\infty)$ and $\mu \in [0, \frac{1}{2})$ such that*
 (i) $0 < F(t,x) \leq \mu x f(t,x)$ *for $t \in [0,+\infty)$ for all $x \in \mathbb{R}$, and*
 (ii) $F(t,x) \geq c_1(t)|x|^{\frac{1}{\mu}} - c_2(t)$ *for all $t \in [0,+\infty)$ and $x \in \mathbb{R}\backslash\{0\}$.*

(F2) *For any constant $R > 0$ there exists a nonnegative function $g_R \in L^1(0,+\infty)$ such that*

$$\sup_{y\in[-R,R]} \left| f\left(t, \frac{y}{p_0(t)}\right) \right| \leq g_R(t).$$

(F3) *There exists a function $\gamma \in L^\infty(0,+\infty)$ with $\gamma^* = \sup\limits_{t\in[0,+\infty)} |(p_0^2\gamma)(t)| < \frac{1}{2}$ such that*

$$\limsup_{|x|\to 0} \frac{F(t, \frac{1}{p_0(t)}x)}{|x|^2} \leq \gamma(t) \text{ uniformly with respect to } t \in [0,+\infty).$$

Then the boundary value problem (P) has at least one nontrivial weak solution.

Proof. We wish to show that J satisfies all the conditions of the Mountain Pass Theorem, Lemma 1.7 above. It is obvious from the definition of J that $J(0) = 0$. To see that J satisfies the Palais-Smale condition, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H_0^k(0,+\infty)$ such that $\lim\limits_{n\to+\infty} J'(u_n) = 0$ and $(J(u_n))$ is bounded, say $|J(u_n)| \leq K$ for some $K > 0$ and all large n. Note that

$$\langle J'(u_n), u_n\rangle = \|u_n\|^2 - \int_0^{+\infty} f(t, u_n(t))u_n(t)dt.$$

From (F1)(i), we have

$$\begin{aligned}
K \geq J(u_n) &= \frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} F(t, u_n(t))dt \\
&\geq \frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} \mu u_n(t) f(t, u_n(t))dt \\
&\geq \frac{1}{2}\|u_n\|^2 - \mu(\|u_n\|^2 - \langle J'(u_n), u_n\rangle) \\
&\geq \left(\frac{1}{2} - \mu\right)\|u_n\|^2 + \mu\langle J'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{2} - \mu\right)\|u_n\|^2 - \mu\|J'(u_n)\|\|u_n\|.
\end{aligned}$$

Since $\lim\limits_{n\to+\infty} J'(u_n) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\|J'(u_n)\| \leq 1$ for $n \geq n_0$. Therefore,

$$K \geq \left(\frac{1}{2} - \mu\right)\|u_n\|^2 + \mu\|u_n\| \text{ for } n \geq n_0,$$

which implies that (u_n) is bounded in $H_0^k(0,+\infty)$. In fact, if (u_n) is unbounded, there exists $(u_{n_k}) \subset (u_n)$ such that $\lim\limits_{k\to+\infty} \|u_{n_k}\| = +\infty$, and then we would obtain

$$K \geq \lim_{k\to+\infty} \left(\frac{1}{2} - \mu\right)\|u_{n_k}\|^2 + \mu\|u_{n_k}\| = +\infty,$$

which is a contradiction.

Since $H_0^k(0,+\infty)$ is a reflexive space, passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ in $H_0^k(0,+\infty)$ and that (u_n) is bounded in $H_0^k(0,+\infty)$. Thus, we have

$$\begin{aligned}\langle J'(u_n) - J'(u), u_n - u\rangle &= \langle J'(u_n), u_n - u\rangle - \langle J'(u), u_n - u\rangle \\ &\leq \|J'(u_n)\|\|u_n - u\| - \langle J'(u), u_n - u\rangle \to 0\end{aligned}$$

as $n \to +\infty$. Moreover, by Corollary 1.11, $\lim\limits_{n\to+\infty} \|u_n - u\|_{l,p_0} = 0$ and $\|u_n\|_{l,p_0} \leq R$ for all $n \in \mathbb{N}$. We also have $u_n \to u$ in $C_{l,p_0}[0,+\infty)$ which implies that

$$f(t, u_n(t)) \to f(t, u(t)), \text{ for all } t \in [0,+\infty).$$

From (F2) we see that

$$|f(t, u_n(t))| \leq \sup_{y\in[-R,R]} \left| f\left(t, \frac{y}{p_0(t)}\right)\right| \leq g_R(t) \in L^1(0,+\infty),$$

so by the Lebesgue Dominated Convergence Theorem, we have

$$\int_0^{+\infty} f(t, u_n(t))dt \to \int_0^{+\infty} f(t, u(t))dt \text{ as } n \to +\infty.$$

Note that for large n, we have

$$\begin{aligned}0 \geq \langle J'(u_n) - J'(u), u_n - u\rangle = \|u_n - u\|^2 \\ - \int_0^{+\infty} (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t))dt.\end{aligned}$$

Hence, $\lim\limits_{n\to+\infty} \|u_n - u\| = 0$. Thus, (u_n) converges strongly to u in $H_0^k(0,+\infty)$, and so J satisfies the (PS) condition.

Next, we show that J satisfies the first geometric condition, namely, Lemma 1.7(1). In fact, by (F3) there exist $r > 0$ and $\varepsilon > 0$ such that

$$\left| F\left(t, \frac{1}{p_0(t)}x\right)\right| \leq (\gamma(t) - \varepsilon)|x|^2 \text{ for } |x| \leq r \text{ and } t \in [0,+\infty).$$

Therefore, by using the continuous embeddings of $H_0^k(0,+\infty)$ into $L^2[0,+\infty)$ and $H_0^k(0,+\infty)$ into $C_{l,p_0}[0,+\infty)$, and the facts that $|u|_{L^2} \leq \|u\|$ and $\|u\|_{\infty,p_0} \leq N\|u\|$, then for $\|u\| = \rho$ small enough and $\alpha = (\frac{1}{2} - \gamma^* + \varepsilon \sup\limits_{t\in[0,+\infty)} |p_0(t)|^2)\rho^2 > 0$ we have $\|u\|_{\infty,p_0} \leq N\rho \leq r$, and so we obtain

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt \\
&= \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,\frac{1}{p_0(t)}(p_0u)(t))dt \\
&\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} (\gamma(t)-\varepsilon)|(p_0u)(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} (\gamma(t)-\varepsilon)|p_0(t)|^2|u(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|^2 - \sup_{t\in[0,+\infty)} (|p_0(t)|^2(\gamma(t)-\varepsilon)) \int_0^{+\infty} |u(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|^2 - \left(\sup_{t\in[0,+\infty)} |(p_0^2\gamma)(t)| - \varepsilon \sup_{t\in[0,+\infty)} |p_0(t)|^2\right) |u|_{L^2}^2 \\
&\geq \frac{1}{2}\|u\|^2 - (\gamma^* - \varepsilon \sup_{t\in[0,+\infty)} |p_0(t)|^2)\|u\|^2 \\
&= \left(\frac{1}{2} - \left(\gamma^* - \varepsilon \sup_{t\in[0,+\infty)} |p_0(t)|^2\right)\right)\|u\|^2.
\end{aligned}$$

This implies that condition (1) in Lemma 1.7 is satisfied.

Finally, we need to show that J satisfies the second geometric condition in Lemma 1.7 holds. For all $u \in H_0^k(0,+\infty)$ with $u \neq 0$, $s > 0$, the fact that $\mu \in [0, \frac{1}{2})$, and condition (F1)(ii) imply

$$\begin{aligned}
J(su) &= \frac{1}{2}s^2\|u\|^2 - \int_0^{+\infty} F(t,su(t))dt \\
&\leq \frac{1}{2}s^2\|u\|^2 - \int_0^{+\infty} (c_1(t)s^{\frac{1}{\mu}}|u(t)|^{\frac{1}{\mu}} - c_2(t))dt \\
&\leq \frac{1}{2}s^2\|u\|^2 - s^{\frac{1}{\mu}} \int_0^{+\infty} c_1(t)|u(t)|^{\frac{1}{\mu}} dt + \int_0^{+\infty} c_2(t)dt \to -\infty
\end{aligned}$$

as $s \to +\infty$. Therefore, there exists s_0 large enough so that $J(s_0u) < 0 < \alpha$. Consequently, condition (2) in Lemma 1.7 is satisfied, so J possesses a critical point that is a nontrivial weak solution of problem (P). This completes the proof of the theorem. □

As a final remark, we wish to point out that condition (F1)(i) is the famous Ambrosetti-Rabinowitz condition [2].

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