

COMPARISON THEOREMS FOR SECOND ORDER LINEAR AND HALF-LINEAR DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, discrete comparison theorems of Sturm’s type is developed for a pair of second order linear difference equations. In particular, we present here Sturm-Picone type and Leighton type comparison theorems on second order linear difference equations. Our research is motivated by famous book [15] dealing with a similar problem for second order linear differential equations. Furthermore we extend and investigate the more general half-linear difference equation by referring the famous paper [10]. Examples are given to illustrate the relevance of the results.
AMS (MOS) Subject Classification. 34C10, 34C15, 34K40.

Key Words and Phrases. Oscillation, comparison theorem, difference equation.

1. INTRODUCTION

Difference equations is discrete analogue of differential equations. Therefore, second order differential equations discrete counterparts are analyzed paralelly. For example, Sturm-Liouville problems is one of these examples. Sturm-Liouville problem has an important role because of a lot of applications in mathematical physics. This paper is devoted to the existence and location of the zeros of the solutions of second order linear or half-linear difference equations. At first, we will consider second order linear difference equations of the form

$$(E_1) \quad l[u_n] \equiv \Delta(a_n \Delta u_n) + c_n u_{n+1} = 0,$$

$$(E_2) \quad L[v_n] \equiv \Delta(A_n \Delta v_n) + C_n v_{n+1} = 0,$$

where Δ is the forward difference operator, that is, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, and $\alpha, \beta \in \mathbb{Z}$, $\alpha < \beta$. The sequences c_n, C_n are real-valued sequences on $[\alpha, \beta]$, and a_n, C_n are positive real-valued sequences on $n \in \mathbb{Z}$.

The second purpose of this paper we shall discuss about the second order half-linear difference equations

$$(E_3) \quad l_r[u_n] \equiv \Delta(a_n \Psi_r(\Delta u_n)) + c_n \Psi_r(u_{n+1}) = 0,$$

$$(E_4) \quad L_r[v_n] \equiv \Delta(A_n \Psi_r(\Delta v_n)) + C_n \Psi_r(v_{n+1}) = 0,$$

where $\Psi_r = |s|^{r-1}s$ and $r > 0$ is a constant.

By a *solution* of (E_i) ($i = 1, 2, 3, 4$), we mean a nontrivial real sequence x_n which is defined for all positive integer $n \geq n_0$ and satisfies (E_i) ($i = 1, 2, 3, 4$) for $n \geq n_0$. A solution x_n of (E_i) ($i = 1, 2, 3, 4$) is said to be *oscillatory* if for every positive integer $N \geq n_0$, there exists $n \geq N$ such that $x_n = 0$ otherwise x_n is said to be *nonoscillatory*. Indeed, this definition says that nontrivial solutions of equations (E_i) ($i = 1, 2, 3, 4$) can have only simple zeros, where a solution x_n of (E_i) ($i = 1, 2, 3, 4$) is said to have a *simple zero* at $n_0 \geq 0$ if $x_{n_0} = 0$.

The second order linear difference equation

$$(1.1) \quad \Delta(a_n \Delta x_n) + c_n x_{n+1} = 0$$

have been investigated many authors [1-9,11-14]. The authors [4,7] proved Reid's roundabout theorem, Sturm-type separation theorem and Sturm-type comparison theorem by using the concept of disconjugacy. A solution x_n of (1.1) is said to have a *generalized zero* at m provided $x_m = 0$ if $m = a$, and if $m > a$ either $x_m = 0$ or $x_m x_{m+1} < 0$. Equation (1.1) is said to be *disconjugate* provided no nontrivial solution of equation (1.1) has two or more generalized zeros.

Reid's roundabout theorem (see [2].) *All of the following statements are equivalent*

1. Equation (1.1) is disconjugate on $[\alpha, \beta]$.
2. Equation (1.1) has a solution x_n without generalized zeros in the interval $[\alpha, \beta + 1]$.
3. The Riccati difference associated with (1.1), namely,

$$\mathcal{R}[w_n] = \Delta w_n + \frac{w_n^2}{w_n + a_n} + c_n = 0,$$

where $w_n = a_n \Delta x_n / x_n$, has a solution w_n on $[\alpha, \beta]$ satisfying $a_n + w_n > 0$ on $[\alpha, \beta]$.

4. The functional \mathcal{F} is positive definite on $U(\alpha, \beta)$, where

$$\mathcal{F}(\xi; \alpha, \beta) = \sum_{i=\alpha}^{\beta} [a_i |\Delta \xi_i|^2 - c_i |\xi_{i+1}|^2]$$

and

$$U(\alpha, \beta) = \{\xi : [\alpha, \beta + 2] \rightarrow \mathbb{R} : \xi_\alpha = \xi_{\beta+1} = 0\},$$

where \mathcal{F} is positive definite on U provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U$, and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

Let x_n and y_n be solutions of equation (1.1) in $[\alpha, \beta]$. Let the *Wronskian* of x_n and y_n by

$$W[x_n, y_n] = \det \begin{pmatrix} x_n & y_n \\ x_{n+1} & y_{n+1} \end{pmatrix} = \det \begin{pmatrix} x_n & y_n \\ \Delta x_n & \Delta y_n \end{pmatrix}.$$

If x_n and y_n are *linearly independent* solution of (1.1), then $W[x_n, y_n] \neq 0$ for $[\alpha, \beta]$.

Sturm separation theorem (discrete) (see [2].) *Two linearly independent solutions of equation (1.1) cannot have common zero. If a nontrivial solution of equation (1.1) has a zero at t_1 and a generalized zero at $t_2 > t_1$, then any second linearly independent solution (1.1) has a generalized zero in $(t_1, t_2]$. If a nontrivial solution of equation (1.1) has a generalized zero at t_1 and a generalized zero at $t_2 > t_1$, then any second linearly independent solution has a generalized zero in $[t_1, t_2]$.*

Sturm's comparison theorem (discrete) (see [2].) *Suppose $a_n \geq A_n$ and $c_n \leq C_n$ for n in the interval $[\alpha, \beta]$. If (E_1) is disconjugate on $[\alpha, \beta]$, then (E_2) is also disconjugate on $(\alpha, \beta + 1)$.*

In [3], authors proved Sturm-type comparison theorem on finite or infinite intervals by using the concept of recessive solution. They defined that iff there exist two linearly independent solution x_n and y_n of (1.1) such that $\frac{y_n}{x_n} \rightarrow 0$ as $n \rightarrow \infty$, then x_n is called recessive or sub-dominant and y_n is called dominant. However, it seems that the discrete analogue of comparison theorem is not yet studied. For this reason, our research is motivated by continuous case. It is a famous book [15] dealing with the comparison theorems for self-adjoint equations

$$l[u] \equiv (a(x)u'(x))' + c(x)u(x) = 0,$$

$$L[v] \equiv (A(x)v'(x))' + C(x)v(x) = 0$$

on a bounded open interval $\tilde{\alpha} < x < \tilde{\beta}$ and $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$, where a, c, A and C are real-valued continuous functions and $a(x) > 0, A(x) > 0$ on $[\tilde{\alpha}, \tilde{\beta}]$. In this book he present following theorems:

Sturm-Picone Comparison Theorem (continuous) (see [15].) *Suppose $a(x) > A(x)$ and $c(x) < C(x)$ in the interval $\alpha < x < \beta$. If there exists a nontrivial real solution u of $l[u] = 0$ such that $u(\alpha) = u(\beta) = 0$, then every real solution of $L[v] = 0$ has at least one zero in (α, β) .*

Sturm separation theorem (continuous) (see [15].) *The zeros of linearly independent solutions of $l[u] = 0$ separate each other.*

From the above theorem we see that if u and v are two independent solutions, then zeros of u and v appear alternately.

In the third section the foregoing results are generalized for linear equations (E_i) ($i = 3, 4$). The following equation is the discrete version of the second order half-linear

difference equation

$$(1.2) \quad \Delta(a_n \Psi_r(\Delta x_n)) + c_i \Psi_r(x_{n+1}) = 0,$$

By defining the concept of disconjugacy, generalized zero, positive definite for (E_i) ($i = 3, 4$) as former (see [2]), these generalizations will be derive a following Reid's roundabout theorem, the results was proved in [13, 14].

Reid's roundabout theorem (see [13].) *All of the following statements are equivalent*

1. Equation (1.2) is disconjugate on $[\alpha, \beta]$.
2. Equation (1.2) has a solution x_n without generalized zeros in the interval $[\alpha, \beta + 1]$.
3. The Riccati difference associated with (1.2), namely,

$$\mathcal{R}[w_n] = \Delta w_n + w_n \left(1 - \frac{\Psi_r(w_n)}{\Psi_r(w_{n+1})} \right) + c_n = 0,$$

where $w_n = a_n \Psi_r(\Delta x_n) / \Psi_r(x_n)$, has a solution w_n on $[\alpha, \beta]$ satisfying $a_n + w_n > 0$ on $[\alpha, \beta]$.

4. The functional \mathcal{F} is positive definite on $U(\alpha, \beta)$, where

$$\mathcal{F}(\xi; \alpha, \beta) = \sum_{i=\alpha}^{\beta} [a_i |\Delta \xi_i|^r - c_i |\xi_{i+1}|^r]$$

and

$$U(\alpha, \beta) = \{\xi : [\alpha, \beta + 2] \rightarrow \mathbb{R} : \xi_\alpha = \xi_{\beta+1} = 0\},$$

where \mathcal{F} is positive definite on U provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U$, and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

Sturm separation theorem (discrete) (see [13].) *Two linearly independent solutions of equation (1.2) cannot have common zero. If a nontrivial solution of equation (1.2) has a zero at t_1 and a generalized zero at $t_2 > t_1$, then any second linearly independent solution (1.2) has a generalized zero in $(t_1, t_2]$. If a nontrivial solution of equation (1.2) has a generalized zero at t_1 and a generalized zero at $t_2 > t_1$, then any second linearly independent solution has a generalized zero in $[t_1, t_2]$.*

Sturm's comparison theorem (discrete) (see [2].) *Suppose $a_n \geq A_n$ and $c_n \leq C_n$ in the interval $n \in [\alpha, \beta]$. If (E_3) is disconjugate on $[\alpha, \beta]$, then (E_4) is also disconjugate on $(\alpha, \beta + 1)$.*

Picone identity concerning the second order half-linear differential equation

$$l_r[u] \equiv (a(x) \Psi_r(u'(x)))' + c(x) \Psi_r(u(x)) = 0,$$

$$L_r[v] \equiv (A(x) \Psi_r(v'(x)))' + C(x) \Psi_r(v(x)) = 0$$

can be found in [9] and there was given following comparison theorem.

Sturm-Picone comparison theorem (continuous) (see [10].) *If $a(x) \geq A(x)$ and $C(x) \geq c(x)$ on a given interval I and there exists an u such that $L_r[u] = 0$, then any solution of $L_r[v] = 0$ either has a zero in (α, β) or it is a constant multiple of u .*

In this paper our purpose is to derive the discrete version of comparison theorems similar to continuous version of comparison theorems for second order linear or half-linear difference equations.

2. COMPARISON THEOREMS FOR LINEAR DIFFERENCE EQUATIONS

Based on the book [15], we establish the following comparison theorems.

Theorem 2.1 (Sturm). *Suppose $a_n = A_n$ and $c_n < C_n$ in the interval $n \in [\alpha, \beta + 1]$. If there exists a nontrivial solution u_n of $l[u_n] = 0$ such that $u_\alpha = u_{\beta+1} = 0$, then every solution of $L[v_n] = 0$ has at least one zero in $(\alpha, \beta + 1)$.*

Proof. Suppose to the contrary that v_n does not vanish in $(\alpha, \beta + 1)$. It may be supposed without loss of generality that $v_n > 0$ and also $u_n > 0$ in $(\alpha, \beta + 1)$. Multiplication of (E₁) by v_{n+1} , (E₂) by u_{n+1} , subtraction of the resulting equations, and summing yields

$$(2.1) \quad \sum_{i=\alpha}^{\beta} \left\{ \Delta(a_i \Delta u_i) \cdot v_{i+1} - \Delta(a_i \Delta v_i) \cdot u_{i+1} \right\} = \sum_{i=\alpha}^{\beta} (C_i - c_i) u_{i+1} v_{i+1}.$$

On the other hand, the left part of (2.1) is equal to

$$\sum_{i=\alpha}^{\beta} \left\{ \Delta(a_i \Delta u_i) \cdot v_{i+1} - \Delta(a_i \Delta v_i) \cdot u_{i+1} \right\} = \sum_{i=\alpha}^{\beta} \Delta \left(a_i \Delta u_i \cdot v_i - a_i u_i \cdot \Delta v_i \right).$$

Since the sum on the left side is the forward difference of $a_i(\Delta u_i v_i - u_i \Delta v_i)$ and $C_i - c_i > 0$ by hypothesis, it follows that

$$(2.2) \quad a_i(\Delta u_i \cdot v_i - u_i \cdot \Delta v_i) \Big|_{i=\alpha}^{\beta+1} > 0.$$

However, $u_\alpha = u_{\beta+1} = 0$ by hypothesis, and since $u_i > 0$ in $(\alpha, \beta + 1)$, $\Delta u_\alpha > 0$ and $\Delta u_{\beta+1} < 0$. This inequality $\Delta u_{\beta+1} < 0$ holds because the equation (E₁) is rewritten as

$$a_{i+1} \Delta u_{i+1} - a_i \Delta u_i + c_i u_{i+1} = 0,$$

and so, substituting the above into $i = \beta$ yields

$$a_{\beta+1} \Delta u_{\beta+1} = a_\beta \Delta u_\beta < 0.$$

Thus the left inequality of (2.2) is negative, which is a contradiction. □

Theorem 2.2 (Sturm-Picone). *Suppose $a_n > A_n$ and $c_n < C_n$ in the interval $n \in [\alpha, \beta + 1]$. Then the conclusion of Theorem 2.1 is valid.*

The proof will be deferred since this is a special case of a theorem of Leighton (Theorem 2.5) to be proved later. The following lemma will be stated in terms of the quadratic functional defined by the equation

$$(2.3) \quad J[u_n] = \sum_{i=\alpha}^{\beta} (A_i \Delta u_i^2 - C_i u_{i+1}^2).$$

The domain D of J is defined to be the set of all real-valued functions $u_n \in C[\alpha, \beta + 1]$ such that $u_\alpha = u_{\beta+1} = 0$.

Lemma 2.3. *If there exists a function $u_n \in D$, not identically zero, such that $J[u_n] \leq 0$, then every real solution of $L[v_n] = 0$ except a constant multiple of u_n vanishes at some point of $(\alpha, \beta + 1)$.*

Proof. Suppose to the contrary that there exists a solution $v_n \neq 0$ in $(\alpha, \beta + 1)$. Now, since

$$(2.4) \quad A_n v_n v_{n+1} \left[\Delta \left(\frac{u_n}{v_n} \right) \right]^2 = A_n \left(\frac{v_n}{v_{n+1}} \right) (\Delta u_n)^2 - \frac{2A_n u_n \Delta u_n \Delta v_n}{v_{n+1}} + A_n \frac{(u_n \Delta v_n)^2}{v_n v_{n+1}}$$

and

$$(2.5) \quad \Delta \left(\frac{A_n u_n^2 \Delta v_n}{v_n} \right) = \frac{u_{n+1}^2}{v_{n+1}} \Delta (A_n \Delta v_n) + \frac{A_n \Delta v_n (u_{n+1} + u_n) \Delta u_n}{v_{n+1}} - A_n \frac{(u_n \Delta v_n)^2}{v_n v_{n+1}},$$

so it is obvious that

$$\begin{aligned} & A_n v_n v_{n+1} \left[\Delta \left(\frac{u_n}{v_n} \right) \right]^2 + \Delta \left(\frac{A_n u_n^2 \Delta v_n}{v_n} \right) \\ &= A_n \left(\frac{v_n}{v_{n+1}} \right) (\Delta u_n)^2 + \frac{A_n \Delta v_n (u_{n+1} - u_n) \Delta u_n}{v_{n+1}} + \frac{u_{n+1}^2}{v_{n+1}} \Delta (A_n \Delta v_n) \\ &= A_n \left(\frac{v_n + \Delta v_n}{v_{n+1}} \right) (\Delta u_n)^2 + \frac{u_{n+1}^2}{v_{n+1}} \Delta (A_n \Delta v_n) \\ &= A_n (\Delta u_n)^2 + \frac{u_{n+1}^2}{v_{n+1}} (L[v_n] - C_{n+1} v_{n+1}). \end{aligned}$$

Hence, for all v_n and $u_n \in D$, the following identity is valid in $(\alpha, \beta + 1)$:

$$(2.6) \quad \begin{aligned} & A_n v_n v_{n+1} \left[\Delta \left(\frac{u_n}{v_n} \right) \right]^2 + \Delta \left(\frac{A_n u_n^2 \Delta v_n}{v_n} \right) \\ &= A_n (\Delta u_n)^2 - C_{n+1} (u_{n+1})^2 + \frac{u_{n+1}^2}{v_{n+1}} L[v_n]. \end{aligned}$$

Since $L[v_n] = 0$ in $n \in (\alpha, \beta + 1)$, sum of (2.6) has

$$(2.7) \quad \sum_{i=y}^z (A_i \Delta u_i^2 - C_i u_{i+1}^2) = \sum_{i=y}^z A_i v_i v_{i+1} \left[\Delta \left(\frac{u_i}{v_i} \right) \right]^2 + \left(\frac{A_i u_i^2 \Delta v_i}{v_i} \right) \Big|_{i=y}^z$$

for arbitrary y and z satisfying $\alpha < y < z < \beta$.

1) If $v_\alpha \neq 0$ and $v_{\beta+1} \neq 0$. It follows from (2.3) that and hypotheses $u_\alpha = u_{\beta+1} = 0$ that

$$(2.8) \quad J[u_n] = \sum_{i=\alpha}^{\beta} A_i v_i v_{i+1} \left[\Delta \left(\frac{u_i}{v_i} \right) \right]^2.$$

Since $A_i > 0$, $J[u_n] \geq 0$, equality if and only if $\Delta \left(\frac{u_i}{v_i} \right)$ is identically zero, i.e., u_i is a constant multiple of v_i . The latter cannot occur since $u_\alpha = 0$ and $v_\alpha \neq 0$, and hence $J[u_n] > 0$. The contradiction shows that v_n must have a zero in $(\alpha, \beta + 1)$.

2) If $v_\alpha = v_{\beta+1} = 0$. According to the result of [4. Chap 1] and [12], we consider solutions $u_{\alpha[\varepsilon]}$ satisfying

$$u_{\alpha[\varepsilon]} = v_{\alpha[\varepsilon]} = \varepsilon$$

for $\varepsilon > 0$ is sufficiently small. Then it show that $\alpha^{[\varepsilon]} \rightarrow \alpha$ and

$$u_{\alpha^{[\varepsilon]}} \rightarrow u_\alpha \quad \text{and} \quad v_{\alpha^{[\varepsilon]}} \rightarrow v_\alpha \quad \text{as} \quad \varepsilon \rightarrow 0.$$

So we obtain

$$\frac{A_{\alpha^{[\varepsilon]}} u_{\alpha^{[\varepsilon]}}^2 \Delta v_{\alpha^{[\varepsilon]}}}{v_{\alpha^{[\varepsilon]}}} = \frac{A_{\alpha^{[\varepsilon]}} u_{\alpha^{[\varepsilon]}} \Delta v_{\alpha^{[\varepsilon]}}}{\left(\frac{e^{u_{\alpha^{[\varepsilon]}} \log(1+v_{\alpha^{[\varepsilon]}})} u_{\alpha^{[\varepsilon]}}^{-1} - 1}{u_{\alpha^{[\varepsilon]}}} \right)},$$

which leads to

$$\lim_{\varepsilon \rightarrow 0} \frac{A_{\alpha^{[\varepsilon]}} u_{\alpha^{[\varepsilon]}}^2 \Delta v_{\alpha^{[\varepsilon]}}}{v_{\alpha^{[\varepsilon]}}} = \lim_{\varepsilon \rightarrow 0} \frac{A_{\alpha^{[\varepsilon]}} u_{\alpha^{[\varepsilon]}} \Delta v_{\alpha^{[\varepsilon]}}}{\left(\frac{e^{u_{\alpha^{[\varepsilon]}} \log(1+v_{\alpha^{[\varepsilon]}})} u_{\alpha^{[\varepsilon]}}^{-1} - 1}{u_{\alpha^{[\varepsilon]}}} \right)} = 0$$

by applying $\lim_{\xi \rightarrow 0} (1 + \xi)^{\frac{1}{\xi}} = e$ and $\lim_{\xi \rightarrow 0} \frac{e^\xi - 1}{\xi} = 1$. Similarly, we choose $\varepsilon > 0$ is sufficiently small, then $u_{\beta^{[\varepsilon]}+1}$ and $v_{\beta^{[\varepsilon]}+1}$ satisfies

$$u_{\beta^{[\varepsilon]}+1} = v_{\beta^{[\varepsilon]}+1} = \varepsilon,$$

which lead to

$$u_{\beta^{[\varepsilon]}+1} \rightarrow u_{\beta+1} \quad \text{and} \quad v_{\beta^{[\varepsilon]}+1} \rightarrow v_{\beta+1} \quad \text{as} \quad \varepsilon \rightarrow 0.$$

And so, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{A_{\beta^{[\varepsilon]}+1} u_{\beta^{[\varepsilon]}+1}^2 \Delta v_{\beta^{[\varepsilon]}+1}}{v_{\beta^{[\varepsilon]}+1}} = 0.$$

It then follows from (2.7) in the limit $y, z \rightarrow \alpha, \beta + 1$ that (2.8) is still valid, and $J[u_n] > 0$. Hence we obtain the contradiction $J[u_n] > 0$ unless v_n is a constant multiple of u_n .

3) If $v_\alpha = 0, v_{\beta+1} \neq 0$ or $v_\alpha \neq 0, v_{\beta+1} = 0$. It is clear from the foregoing proof that (2.8) still holds and accordingly that v_n has a zero in $(\alpha, \beta + 1)$.

This complete the proof of Lemma 2.3. \square

Remark 2.4. It is obviously that $J[u_n] = 0$ if and only if

$$\Delta \left(\frac{u_i}{v_i} \right) = \frac{\Delta u_i \cdot v_i - u_i \cdot \Delta v_i}{v_i v_{i+1}} = 0.$$

Hence, it notes that this case occurs if and only if

$$\Delta u_i = \frac{u_i \cdot \Delta v_i}{v_i},$$

or equivalently,

$$(2.9) \quad \frac{u_{i+1}}{u_i} = \frac{v_{i+1}}{v_i}$$

Product (2.9) $i = 1$ to $n - 1$ as follows:

$$\prod_{i=0}^{n-1} \left(\frac{u_{i+1}}{u_i} \right) = \prod_{i=0}^{n-1} \left(\frac{v_{i+1}}{v_i} \right),$$

which implies

$$u_n = \left(\frac{u_0}{v_0} \right) v_n.$$

Consequently, $J[u_n] = 0$ means that corresponds to the case where $\Delta u_i \equiv u_i \cdot \Delta v_i / v_i$, i.e. u_n is a constant multiple of v_i .

Lemma 2.3 extends Leighton's result slightly by weakening the hypothesis $J[u_n] < 0$ to $J[u_n] \leq 0$. In addition to (2.3) consider the quadratic functional defined by

$$(2.10) \quad j[u_n] = \sum_{i=\alpha}^{\beta} (a_i \Delta u_i^2 - c_i u_{i+1}^2)$$

for $u_n \in D$. The variation of $j[u_n]$ is defined as $V[u_n] = j[u_n] - J[u_n]$, that is

$$(2.11) \quad V[u_n] = \sum_{i=\alpha}^{\beta} \left\{ (a_i - A_i) \Delta u_i^2 + (C_i - c_i) u_{i+1}^2 \right\}$$

with domain D .

Theorem 2.5 (Leighton). *If there exists a nontrivial real solution u_n of $l[u_n] = 0$ in $[\alpha, \beta + 1]$ such that $u_\alpha = u_{\beta+1} = 0$ and $V[u_n] > 0$, then every real solution of $L[v_n] = 0$ has at least one zero in $(\alpha, \beta + 1)$.*

In the following result the hypothesis $V[u_n] > 0$ is weakened to $V[u_n] \geq 0$.

Theorem 2.6. *If there exists a nontrivial real solution u_n of $l[u_n] = 0$ in $[\alpha, \beta + 1]$ such that $u_\alpha = u_{\beta+1} = 0$ and $V[u_n] \geq 0$, then every real solution of $L[v_n] = 0$ has one of the following properties:*

- (i) v_n has at least one zero in $(\alpha, \beta + 1)$,
- (ii) v_n is a constant multiple of u_n .

Proof. Since $u_\alpha = u_{\beta+1} = 0$ and $l[u_n] = 0$, it is clear that

$$\begin{aligned} \sum_{i=\alpha}^{\beta} u_{i+1} \cdot l[u_i] &= \sum_{i=\alpha}^{\beta} u_{i+1} \left(\Delta(a_i \Delta u_i) + c_i u_{i+1} \right) \\ &= \sum_{i=\alpha}^{\beta} \left\{ \Delta(u_i \cdot a_i \Delta u_i) - a_i (\Delta u_i)^2 + c_i u_{i+1}^2 \right\}, \end{aligned}$$

which implies that

$$\sum_{i=\alpha}^{\beta} u_{i+1} \cdot l[u_i] + j[u_n] = \sum_{i=\alpha}^{\beta} \Delta(u_i \cdot a_i \Delta u_i) = 0.$$

So it is obvious that $j[u_n] = 0$. The hypothesis $V[u_n] \geq 0$ of Theorem 2.6 is equivalent to $J[u_n] \leq j[u_n]$. Hence the hypothesis $J[u_n] \leq 0$ of Lemma 2.3 is fulfilled, and v_n vanishes at least one in $(\alpha, \beta + 1)$ unless v_n is a constant multiple of u_n .

Under the hypothesis $V[u_n] > 0$ of Theorem 2.5 alternative (ii) of Theorem 2.6 implies $J[u_n] = 0$ by (2.3), and hence $V[u_n] = 0$. The contradiction establishes Theorem 2.5. □

Since 1910, when the original paper of Picone was published, this identity has been extended to various equations (not only to ODE's but also to PDE's and to difference equations). Now, we will formulate the discrete version of the so-called Picone identity.

Lemma 2.7 (Picone's identity). *Let u_n and v_n be defined on $[\alpha, \beta + 1]$ and assume $v_n \neq 0$ for $n \in (\alpha, \beta + 1)$. Then for $n \in [\alpha, \beta + 1]$, the following equality holds:*

$$\begin{aligned} &\Delta \left\{ \frac{u_n}{v_n} \left[v_n a_n \Delta u_n - u_n A_n \Delta v_n \right] \right\} \\ &= (a_n - A_n) \Delta u_n^2 + (C_n - c_n) u_{n+1}^2 \\ &\quad + \left(\frac{u_{n+1}}{v_{n+1}} \right) \left\{ v_{n+1} \cdot l[u_n] - u_{n+1} \cdot L[v_n] \right\} + A_n v_n v_{n+1} \left[\Delta \left(\frac{u_n}{v_n} \right) \right]^2 \end{aligned}$$

holds for $n \in [\alpha, \beta]$.

The Sturm-Picone theorem 2.2 follows immediately from Theorem 2.5 since the hypotheses $a_n > A_n$ and $c_n < C_n$ of the former imply that $V[u_n] > 0$. Likewise the following improvement of Theorem 2.2 is an immediate consequence of Theorem 2.6 and Lemma 2.7.

Theorem 2.8. *Suppose $a_n \geq A_n$ and $c_n \leq C_n$ in $n \in (\alpha, \beta + 1)$. If there exists a nontrivial real solution u_n of $l[u_n] = 0$ such that $u_\alpha = u_{\beta+1} = 0$, then every real solution of $L[v_n] = 0$ except a constant multiple of u_n has at least one zero in $(\alpha, \beta + 1)$.*

The next result requires the slightly stronger hypothesis that L is a "strict Sturmian majorant" of l . This means, in addition to the conditions $a_n \geq A_n$ and $c_n \leq C_n$, that either

- $c_n \neq C_n$ for some n in $(\alpha, \beta + 1)$, or
- (if $c_n \equiv C_n$) that $a_n > A_n$ and $c_n \neq 0$ for some n .

Theorem 2.9. *Suppose that L is a strict Sturmian majorant of l . Then if there exists a nontrivial real solution u_n of $l[u_n] = 0$ satisfying $u_\alpha = u_{\beta+1} = 0$, every real solution of $L[v_n] = 0$ has a zero in $(\alpha, \beta + 1)$.*

This follows from Theorem 2.5 since $V[u_n] > 0$ is a consequence of the strict Sturmian hypothesis.

In the special case that the difference equations (E_1) and (E_2) coincide, we obtain the classical Sturm separation theorem as a special case of Theorem 2.8.

Theorem 2.10 (Sturm Separation Theorem). *The zeros of linearly independent solutions of (E_1) separate each other.*

The following example illustrates that Theorem 2.5 (or Theorem 2.6) is stronger than the Sturm-Picone theorem 2.2 (and also Theorems 2.8 and 2.9).

Example 2.11. We consider

$$(2.12) \quad \Delta^2 u_n + u_{n+1} = 0,$$

$$(2.13) \quad \Delta^2 v_n + 2v_{n+1} = 0.$$

In this example, $a_n = A_n = c_n = 1$ and $C_n = 2$. It can be checked that the conditions of Theorem 2.1 is satisfied, and the $u_n = \sin\left(\frac{\pi}{3}n\right)$ of (2.12) satisfies $u_0 = u_3 = 0$. Therefore, every solution of (2.13) has a zero in $(0, 3)$. In fact, (2.13) has an oscillatory solution $v_n = \cos\left(\frac{\pi}{2}n\right)$, which satisfies $v_1 = 0$.

Example 2.12. In the case that $a_n = A_n = c_n = 1$ and $C_n = n + 1 - k$, $0 < k < \frac{3}{2}$, $0 \leq n \leq 3$, the difference equations (E_1) , (E_2) become

$$(2.14) \quad \Delta^2 u_n + u_{n+1} = 0,$$

$$(2.15) \quad \Delta^2 v_n + (n + 1 - k)v_{n+1} = 0,$$

respectively. The solution $u_n = \sin\left(\frac{\pi}{3}n\right)$ of (2.14) satisfies $u_0 = u_3 = 0$. The variation (2.11) reduces to

$$\begin{aligned} V[u_n] &= \sum_{i=0}^3 (i-k) \sin^2\left(\frac{\pi}{3}i\right) \\ &= \frac{3}{2} \left(\frac{3}{2} - k\right) > 0. \end{aligned}$$

According to Theorem 2.5, every solution of (2.15) has a zero in $(0, 3)$. This cannot be concluded from the Sturm-Picone theorem 2.2 (or from Theorems 2.8 and 2.9) since the condition $c_n \leq C_n$ does not hold throughout the interval $0 < n < 3$.

3. COMPARISON THEOREMS FOR HALF-LINEAR DIFFERENCE EQUATIONS

This work was motivated by some papers [2], [13], [14] dealing with the comparison theory of the second order half-linear difference equation. In [2] we can get following lemmas playing an important role in the proof of Theorem 2.16.

Lemma 2.13 (Picone type identity [2]). *Let u_n and v_n be defined on $[\alpha, \beta + 1]$ and assume $v_n \neq 0$ for $n \in [\alpha, \beta + 1]$. Then for $n \in [\alpha, \beta + 1]$, the following equality holds:*

$$\begin{aligned} (2.16) \quad & \Delta \left\{ \frac{u_n}{\Psi_r(v_n)} \left[\Psi_r(v_n) a_n \Psi_r(\Delta u_n) - \Psi_r(u_n) A_n \Psi_r(\Delta v_n) \right] \right\} \\ &= (a_n - A_n) |\Delta u_n|^{r+1} + (C_n - c_n) |u_{n+1}|^{r+1} \\ &+ \left(\frac{u_{n+1}}{\Psi_r(v_{n+1})} \right) \left\{ \Psi_r(v_{n+1}) \cdot l_r[u_n] - \Psi_r(u_{n+1}) \cdot L_r[v_n] \right\} \\ &+ A_n \left(\frac{v_n}{v_{n+1}} \right) H(u_n, v_n) \end{aligned}$$

holds for $n \in [\alpha, \beta]$, where

$$H(u_n, v_n) \equiv \frac{v_n}{v_{n+1}} |\Delta u_n|^{r+1} - \frac{v_{n+1} \Psi_r(\Delta v_n)}{v_n \Psi_r(v_{n+1})} |u_{n+1}|^{r+1} + \frac{v_{n+1} \Psi_r(\Delta v_n)}{v_n \Psi_r(v_n)} |u_n|^{r+1}.$$

Lemma 2.14 (see [2]). *Let u_n, v_n be defined on $[\alpha, \beta + 1]$ and assume $v_n \neq 0$ for $n \in (\alpha, \beta + 1)$. Then $H(u_n, v_n) \geq 0$ for $n \in [\alpha, \beta]$, where equality holds if and only if $\Delta u_n = (u_n \Delta v_n) / v_n$.*

Remark 2.15. If we put $r = 2$ (i.e. linear case), we obtain

$$H(u_n, v_n) = \left(\Delta u_n - \frac{\Delta v_n}{v_n} u_n \right)^2.$$

For our next result based on the identity (2.16) we define the following:

$$J_r[\eta] = \sum_{i=\alpha}^{\beta} (A_i |\Delta \eta_i|^{r+1} - C_i |\eta_{i+1}|^{r+1}).$$

Theorem 2.16 (Wirtinger). *If there exist a solution v_n of $L_r[v_n] = 0$ such that $v_n \neq 0$ for $n \in [\alpha, \beta + 1]$, then for all $\eta_n \in D$*

$$(2.17) \quad J_r[\eta_n] \geq 0,$$

where equality holds if and only if η_n is a constant multiple of v_n .

Proof. From Picone's identity (2.16) applied to the case $a_n \equiv A_n$, $c_n \equiv C_n$ and $u_n = \eta_n$ we obtain

$$\begin{aligned} & \Delta \left\{ \eta_n \cdot A_n \Psi_r(\Delta \eta_n) - \eta_n \Psi_r(\eta_n) \cdot A_n \frac{\Psi_r(\Delta v_n)}{\Psi_r(v_n)} \right\} \\ = & A_n \left(\frac{v_n}{v_{n+1}} \right) H(\eta_n, v_n) + \eta_{n+1} l_r[\eta_n] - \frac{\eta_{n+1} \Psi_r(\eta_{n+1})}{\Psi_r(v_{n+1})} L_r[v_n]. \end{aligned}$$

By using the fact that v_n is a solution of $L_r[v_n] = 0$ and using the fact that

$$\begin{aligned} \Delta \left\{ \eta_n \cdot A_n \Psi_r(\Delta \eta_n) \right\} &= A_n |\Delta \eta_n|^{r+1} + \eta_{n+1} \Delta(A_n \Psi_r(\Delta \eta_n)) \\ &= A_n |\Delta \eta_n|^{r+1} + \eta_{n+1} \left\{ l_r[\eta_n] - C_n \Psi_r(\eta_{n+1}) \right\}, \end{aligned}$$

we have

$$(2.18) \quad \begin{aligned} & A_n |\Delta \eta_n|^{r+1} - C_n |\eta_{n+1}|^{r+1} \\ = & \Delta \left\{ \eta_n \Psi_r(\eta_n) \frac{A_n \Psi_r(\Delta v_n)}{\Psi_r(v_n)} \right\} + A_n \left(\frac{v_n}{v_{n+1}} \right) H(\eta_n, v_n). \end{aligned}$$

If both $v_\alpha \neq 0$ and $v_{\beta+1} \neq 0$, then summing (2.18) from α to β and using Lemma 4, we obtain

$$\sum_{i=\alpha}^{\beta} \{ A_n |\Delta \eta_n|^{r+1} - C_n |\eta_{i+1}|^{r+1} \} \geq 0,$$

which implies inequality (2.17). If $v_\alpha = 0$, then $\Delta v_\alpha \neq 0$. By a similar argument as before we consider solutions $v_{\alpha^{[\varepsilon]}}$ satisfying $v_{\alpha^{[\varepsilon]}} = \varepsilon$ for $\varepsilon > 0$ is sufficiently small. Then it follows that $\alpha^{[\varepsilon]} \rightarrow \alpha$ and $v_{\alpha^{[\varepsilon]}} \rightarrow v_\alpha$ as $\varepsilon \rightarrow 0$. We have

$$\lim_{\varepsilon \rightarrow 0} \Psi_r \left(\frac{\eta_{\alpha^{[\varepsilon]}}}{v_{\alpha^{[\varepsilon]}}} \right) = \lim_{\varepsilon \rightarrow 0} \Psi_r \left(\frac{1}{\frac{e^{\eta_{\alpha^{[\varepsilon]}} \log(1+v_{\alpha^{[\varepsilon]}})} \eta_{\alpha^{[\varepsilon]}}}{-1}} \right) = 1.$$

So it is obvious that

$$\lim_{\varepsilon \rightarrow 0} \eta_{\alpha^{[\varepsilon]}} A_{\alpha^{[\varepsilon]}} \Psi_r(\Delta v_{\alpha^{[\varepsilon]}}) \Psi_r \left(\frac{\eta_{\alpha^{[\varepsilon]}}}{v_{\alpha^{[\varepsilon]}}} \right) = 0.$$

Likewise, if $v_{\beta^{[\varepsilon]}+1} = 0$ then

$$\lim_{\varepsilon \rightarrow 0} \eta_{\beta^{[\varepsilon]}+1} A_{\beta^{[\varepsilon]}+1} \Psi_r(\Delta v_{\beta^{[\varepsilon]}+1}) \Psi_r \left(\frac{\eta_{\beta^{[\varepsilon]}+1}}{v_{\beta^{[\varepsilon]}+1}} \right) = 0.$$

for $\beta^{[\varepsilon]} \rightarrow \beta$ ($\varepsilon \rightarrow 0$). Thus, summing (2.18) for $[\alpha^{[\varepsilon]}, \beta^{[\varepsilon]} + 1]$, letting $\varepsilon \rightarrow 0$ and using Lemma 2.14, we again obtain (2.17). Obviously, equality in (2.17) holds if and only if

$H(\eta_n, v_n) = 0$, which according to Lemma 2.14 is possible only if $\Delta\eta_n = \eta_n \cdot \Delta v_n / v_n$, or equivalently, if η_n is a constant multiple of v_n . \square

Corollary 2.17. *If there exists an $\eta_n \in D$ such that*

$$(2.19) \quad J_r[\eta_n] \leq 0,$$

then every solution v_n of $L_r[v_n] = 0$ has a zero in $(\alpha, \beta + 1)$ except possibly when $v_n = c\eta_n$ for some nonzero constant c .

Now we define

$$V_r[\eta_n] = \sum_{i=\alpha}^{\beta} \left\{ (a_i - A_i) |\Delta\eta_i|^{r+1} + (C_i - c_i) |\eta_{i+1}|^{r+1} \right\},$$

then the following comparison theorem is obtained, which is the main result of this section.

Theorem 2.18 (Leighton). *If there exists a nontrivial real solution u_n of $l_r[u_n] = 0$ on $[\alpha, \beta + 1]$ such that $u_\alpha = u_{\beta+1} = 0$ and*

$$(2.20) \quad V_r[u_n] \geq 0,$$

then every real solution v_n of $L_r[v_n] = 0$ has a zero on $(\alpha, \beta + 1)$ except possibly it is a constant multiple of u_n .

Proof. Assume for the sake of contradiction that (E_3) has a solution which is nonzero on $(\alpha, \beta + 1)$. Then from Picone's identity (2.16) it follows that

$$(2.21) \quad \Delta \left\{ \frac{u_n}{\Psi_r(v_n)} \left[\Psi_r(v_n) \cdot a_n \Psi_r(\Delta u_n) - \Psi_r(u_n) \cdot A_n \Psi_r(\Delta v_n) \right] \right\} \\ = (a_n - A_n) |\Delta u_n|^{r+1} + (C_n - c_n) |u_{n+1}|^{r+1} + A_n \left(\frac{v_n}{v_{n+1}} \right) H(u_n, v_n).$$

where we have used that u_n and v_n are solutions of (E_3) and (E_4) , respectively. As in the proof of Theorem 2.16 we can show that the function. \square

Corollary 2.19 (Sturm-Picone). *Suppose $a_n \geq A_n$ and $c_n \leq C_n$ on $n \in (\alpha, \beta + 1)$. If there exists a nontrivial real solution u_n of $l_r[u_n] = 0$ such that $u_\alpha = u_{\beta+1} = 0$, then every real solution of $L_r[v_n] = 0$ except a constant multiple of u_n has at least one zero on $[\alpha, \beta + 1]$.*

Example 2.20. We consider a pair of half-linear equations

$$(2.22) \quad \Delta(|\Delta u_n|^{r-1} \Delta u_n) + 2^{r+2} |u_{n+1}|^{r-1} u_{n+1} = 0$$

and

$$(2.23) \quad \Delta(|\Delta v_n|^{r-1} \Delta v_n) + 2^{r+2} |v_{n+1}|^{r-1} v_{n+1} = 0,$$

where $r > 0$. The solution $u_n = \sin\left(\frac{\pi}{2}n\right)$ of (2.22) satisfies $u_0 = u_2 = 0$. Since the conditions of Corollary 2.19 holds, every solution of (2.23) has a zero in $n = 1$, or v_n is a constant multiple of u_n . In fact, $v_n = \cos\left(\frac{\pi}{2}n\right)$ is a solution of (2.23) satisfies $v_1 = 0$.

Acknowledgment. The authors thank the referee for making several helpful suggestions that significantly improved this paper.

REFERENCES

- [1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Second Edition, Pure and Applied Mathematics, M. Dekker, New York Basel - Hong Kong, 2000.
- [2] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing Corporation, Cairo, 2005.
- [3] D. Aharonov, M. Bohner and U. Elias, Discrete sturm comparison theorems on finite and infinite intervals, *J. Diff. Equ. Appl.* **18** (2012), 1763–1771.
- [4] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems : Difference Equations, Continuous Fractions, and Riccati Equations*, Volume 16 of Kluwer Texts in the Mathematical Sciences, Kluwer Academic Publishers, Boston, MA, 1996.
- [5] O. Došlý, Oscillation theory of linear difference equations, *Archivum Math.* **36** (2000), 329–342.
- [6] O. Došlý, Nonoscillation criteria for half-linear second-order difference equations, *Computers Math. Appl.* **42** (2012), 453–464.
- [7] O. Došlý, J. R. Graef and J. Jaroš, Forced oscillation of second order linear and half-linear difference equations, *Proc. Amer. Math. Soc.* **131** (2002), 2859–2867.
- [8] L. H. Erbe and P. Yan, Disconjugacy for linear hamiltonian difference systems, *J. Math. Anal. Appl.* **167** (1992), 355–367.
- [9] P. Hasil and M. Veselý, Oscillation and non-oscillation criteria for linear and half-linear difference equations, *J. Math. Anal. Appl.* **452** (2017), 401–428.
- [10] J. Jaroš and T. Kusano, A Picone type identity for half-linear differential equations, *Acta Math. Univ. Comenianae* **68** (1999), 127–151.
- [11] W. G. Kelly and A. C. Peterson, *Difference Equations : An introduction with Applications*, 2nd ed., Academic Press, San Diego, CA, 2001.
- [12] M. Ma, Dominant and recessive solutions for second order self-adjoint linear difference equations, *Applied Math. Lett.* **18** (2005), 179–185.
- [13] P. Řehák, Oscillatory properties of second order half-linear difference equations, *Czechoslovak Math. J.* **126** (2001), 303–321.
- [14] P. Řehák, Comparison theorems and strong oscillation in the half-linear discrete oscillation theory, *Rocky Mountain J. Math.* **33** (2003), 333–352.
- [15] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.