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GLOBAL SMOOTHNESS AND APPROXIMATION BY ACTIVATED SMOOTH SINGULAR OPERATORS

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ABSTRACT. In this work we continue with the study of smooth activated singular integral operators over the real line regarding their simultaneous global smoothness preservation property with respect to the L_p norm, $1 \leq p \leq \infty$, by involving higher order moduli of smoothness. Also we treat their activated simultaneous approximation to the unit operator with rates involving the modulus of smoothness. The derived Jackson type inequalities are almost sharp containing elegant constants, and they reflect the high order of differentiability of the engaged function. We involve five different activation functions.

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1. Introduction

The global smoothmess preservation property of singular integrals has been studied initally in [6] and later in [14]. The rate of convergence of singular integrals has been studied initially in [19], [17], [18] and later in [7], [12] and [13].

Here we continue with the study of smooth activated singular integral operators over $\mathbb{R}$ acting on highly smooth functions. We study first their activated simultaneos global smoothness preservation property with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$, by using higher order moduli of smoothness. Then we study their simultaneous pointwise and uniform approximation to the unit operator with rates by using also the higher order moduli of smoothness. The established estimates are almost optimal and contain elegant constants. The modulus of smoothness in the estimates is with respect to the higher order derivative of the engaged function. The discussed operators are not in general positive.

Our main motivation is the classic monograph [15].

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Also of great interest and motivating the author are the articles [1]-[5]. In recent intense mathematical activity by the use of neural networks in solving numerically differential equations our current work is expected to play a pivotal role, as in the classic case played the earlier versions of singular integrals.

For the history of the topic we mention about our monograph [15] of 2012, which was the first complete source to deal exclusively with the classic theory of the approximation of singular integrals to the identity-unit operator. The authors there studied quantitatively the basic approximation properties of the general Picard, Gauss-Weierstrass and Poisson-Cauchy singular integral operators over the real line, which are not positive linear operators. In particular they researched the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. This is given via inequalities and with the use of high order modulus of smoothness of the high order derivative of the engaged function. Some of these inequalites are proven to be sharp. Also, they studied the global smoothness preservation property of these operators. Furthermore they proved the asymptotic expansions of Voronovskaya type for the error of approximation. They continued with the study of related properties of the general fractional Gauss-Weierstrass and Poisson-Cauchy singular integral operators. These properties were established with respect to L_p norm, $1 \leq p \leq \infty$. The case of Lipschitz type functions approximation was given separately and in detail. Furthermore they presented the corresponding general approximation theory of general singular integral operators with lots of applications to, the under focused till then, trigonometric singular integral.

2. **Background on General Global Smoothness Preservation and Approximation**

The next in this section are all coming from [15], Chapter 18, pp. 339-348.

Here we talk about the global smoothness preservation properties and differentiability, also approximations, of smooth general singular integral operators $\Theta_{r,\xi}(f;x)$, defined as follows.

Let $\xi > 0$ and μ_ξ be Borel probability measures on $\mathbb{R}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we put

$$\alpha_j := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, ..., r. \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, we define for $x \in \mathbb{R}$,

$$\Theta_{r,\xi}(f;x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) d\mu_\xi(t). \tag{2}$$

We suppose $\Theta_{r,\xi}(f;x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}$.

Let $f \in C(\mathbb{R})$, for $m \in \mathbb{N}$ the mth modulus of smoothness for $1 \le p \le \infty$, is given by

$$\omega_m(f,h)_p := \sup_{0 \le t \le h} \left\| \Delta_t^m f(x) \right\|_{p,x}, \tag{3}$$

where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+jt), \tag{4}$$

see also [16, p, 44].

Denote

$$\omega_m(f,h)_\infty = \omega_m(f,h). \tag{5}$$

We mention the main global smoothness preservation result:

Theorem 2.1. *Let $h > 0$, $f \in C(\mathbb{R})$.*

i) Assume $\Theta_{r,\xi}(f;x) \in \mathbb{R}$, $\xi > 0$, $\forall\, x \in \mathbb{R}$ and $\omega_m(f,h) < \infty$. Then

$$\omega_m(\Theta_{r,\xi} f, h) \le \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h). \tag{6}$$

ii) Assume $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, then

$$\omega_m(\Theta_{r,\xi} f, h)_1 \le \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h)_1. \tag{7}$$

iii) Assume $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p > 1$. Then

$$\omega_m(\Theta_{r,\xi} f, h)_p \le \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h)_p. \tag{8}$$

Next we discuss about the derivatives of $\Theta_{r,\xi}(f;x)$ and their impact to simultaneous global smoothness preservation and convergence of these operators.

Theorem 2.2. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n, r \in \mathbb{N}$. Furthemore suppose that for each $x \in \mathbb{R}$ the function $f^{(i)}(x+jt) \in L_1(\mathbb{R}, \mu_\xi)$ as a function of t, for all $i = 0, 1, ..., n-1$; $j = 1, ..., r$. Suppose that there exist $g_{i,j} \ge 0$, $i = 1, ..., n$; $j = 1, ..., r$, with $g_{i,j} \in L_1(\mathbb{R}, \mu_\xi)$ such that for each $x \in \mathbb{R}$ we have*

$$\left| f^{(i)}(x+jt) \right| \le g_{i,j}(t), \tag{9}$$

for μ_ξ-almost all $t \in \mathbb{R}$, all $i = 1, ..., n$; $j = 1, 2, ..., r$. Then $f^{(i)}(x + jt)$ defines a μ_ξ-integrable function with respect to t for each $x \in \mathbb{R}$, all $i = 1, ..., n$; $j = 1, ..., r$, and

$$(\Theta_{r,\xi}(f; x))^{(i)} = \Theta_{r,\xi}\left(f^{(i)}; x\right), \tag{10}$$

for all $x \in \mathbb{R}$, all $i = 1, ..., n$.

We have

Theorem 2.3. *Let $h > 0$ and the assumptions of Theorem 2.2 valid.*

i) Suppose that $\omega_m\left(f^{(i)}, h\right) < \infty$, all $i = 0, 1, ..., n$, then

$$\omega_m\left((\Theta_{r,\xi} f)^{(i)}, h\right) \le \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right), \tag{11}$$

for all $i = 0, 1, ..., n$.

ii) Assume $f^{(i)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $i = 0, 1, ..., n$, then

$$\omega_m\left((\Theta_{r,\xi} f)^{(i)}, h\right)_1 \le \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right)_1, \tag{12}$$

for all $i = 0, 1, ..., n$.

iii) Assume $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p > 1$, $i = 0, 1, ..., n$, then

$$\omega_m\left((\Theta_{r,\xi} f)^{(i)}, h\right)_p \le \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right)_p, \tag{13}$$

$i = 0, 1, ..., n$.

Next we mention some simultaneous approximation results of operators $\Theta_{r,\xi}$.

Theorem 2.4. *Let $f \in C^{n+\rho}(\mathbb{R})$, $n, \rho \in \mathbb{Z}^+$ and $\omega_r\left(f^{(n+i)}, h\right) < \infty$, $\forall\ h > 0$ for $i = 0, 1, ..., \rho$. Suppose $\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty$. Set $\delta_k := \sum_{j=1}^{r} \alpha_j j^k$, and existing $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, ..., n \in \mathbb{N}$. We consider the assumptions of Theorem 2.2 valid for $n = \rho$ there. Then*

$$\left\| (\Theta_{r,\xi}(f; x))^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x} \le$$

$$\frac{\omega_r\left(f^{(n+i)}, \xi\right)}{n!} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t). \tag{14}$$

When $n = 0$ the sum in the left of (14) collapses.

We mention

Theorem 2.5. *Let* $f \in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume* $\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1} d\mu_\xi(t) < \infty$, *and* $c_{k,\xi} \in \mathbb{R}$, $k = 1, ..., n$. *We consider the assumptions of Theorem 2.2 as valid for* $n = \rho$ *there. Then*

$$\left\| (\Theta_{r,\xi}(f;x))^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(k+i)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{p,x} \leq$$

$$\frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}$$

$$\left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1} d\mu_\xi(t) \right]^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r \left(f^{(n+i)}, \xi\right)_p . \tag{15}$$

We also mention

Proposition 2.6. Let $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t) < \infty$. We consider the assumptions of Theorem 2.2 valid for $n = \rho$ there. Then

$$\left\| (\Theta_{r,\xi}(f))^{(i)} - f^{(i)} \right\|_p \leq$$

$$\omega_r \left(f^{(i)}, \xi\right)_p \left(\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t) \right)^{\frac{1}{p}}, \tag{16}$$

for all $i = 0, 1, ..., \rho$.

We need

Theorem 2.7. *Let* $f \in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$. *Assume that* $\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) < \infty$, *and* $c_{k,\xi} \in \mathbb{R}$, $k = 1, ..., n$. *We consider the assumptions of Theorem 2.2 valid for* $n = \rho$ *there. Then*

$$\left\| (\Theta_{r,\xi}(f;x))^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(k+i)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{1,x} \leq \frac{1}{(n-1)!(r+1)}$$

$$\left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) \right] \xi \omega_r \left(f^{(n+i)}, \xi\right)_1 , \tag{17}$$

for all $i = 0, 1, ..., \rho$.

We also need

Proposition 2.8. Let $f^{(i)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$. Assume $\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty$. We consider the assumptions of Theorem 2.2 valid for $n = \rho$ there. Then

$$\left\| (\Theta_{r,\xi}(f))^{(i)} - f^{(i)} \right\|_1 \leq$$

$$\omega_r\left(f^{(i)}, \xi\right)_1 \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t)\right], \tag{18}$$

for all $i = 0, 1, ..., \rho$.

3. Background on Activation functions

Here all come from [10].

3.1. About Richards's curve.

Here we follow [9], Chapter 1.

A Richards's curve is

$$\varphi(x) = \frac{1}{1 + e^{-\mu x}}; \quad x \in \mathbb{R}, \ \mu > 0, \tag{19}$$

which is strictly increasing on $\mathbb{R}$, and it is a sigmoid function, in particular this is a generalized logistic function. And it is an activation function in neural networks, see [9], chapter 1.

It is

$$\lim_{x \to +\infty} \varphi(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} \varphi(x) = 0. \tag{20}$$

We consider the function

$$G(x) = \frac{1}{2}(\varphi(x+1) - \varphi(x-1)), \ x \in \mathbb{R}, \tag{21}$$

which is $G(x) > 0$, all $x \in \mathbb{R}$.

It is

$$\varphi(0) = \frac{1}{2}, \quad \varphi(x) = 1 - \varphi(-x), \tag{22}$$

and

$$G(x) = G(-x), \forall \ x \in \mathbb{R}. \tag{23}$$

We also have

$$G(0) = \frac{e^\mu - 1}{2(e^\mu + 1)}. \tag{24}$$

We also get

$$\lim_{x \to +\infty} G(x) = \lim_{x \to -\infty} G(x) = 0, \tag{25}$$

and G is a bell symmetric function with maximum

$$G(0) = \frac{e^\mu - 1}{2(e^\mu + 1)}. \tag{26}$$

Theorem 3.1. *It holds*

$$\sum_{i=-\infty}^{\infty} G(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{27}$$

Theorem 3.2. *It holds*

$$\int_{-\infty}^{\infty} G(x)\, dx = 1. \tag{28}$$

So G is a density function.

We make

Remark 3.3. So we have

$$G(x) = \frac{1}{2}\left(\varphi(x+1) - \varphi(x-1)\right), \quad \forall\, x \in \mathbb{R}. \tag{29}$$

i) Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we get:

$$G(x) = \frac{1}{2} 2\varphi'(\eta) = \varphi'(\eta) = \frac{\mu e^{-\mu\eta}}{\left(1+e^{-\mu\eta}\right)^2}, \quad \mu > 0, \tag{30}$$

where $0 \leq x-1 < \eta < x+1$.

Notice that

$$G(x) < \mu e^{-\mu\eta} < \mu e^{-\mu(x-1)}, \ \forall\, x \geq 1. \tag{31}$$

ii) Let now $x \leq -1$. That is $x-1 < x+1 \leq 0$. Applying again the mean value theorem we get:

$$G(x) = \frac{1}{2} 2\varphi'(\eta) = \varphi'(\eta) = \frac{\mu e^{-\mu\eta}}{\left(1+e^{-\mu\eta}\right)^2}, \tag{32}$$

where $x-1 < \eta < x+1 \leq 0$.

Hence, we derive that

$$G(x) < \mu e^{-\mu\eta} < \mu e^{-\mu(x-1)}, \ \forall\, x \leq -1. \tag{33}$$

Consequently, we proved that

$$G(x) < \mu e^{-\mu(x-1)}, \ \forall\, x \in (-\infty, -1] \cup [1, +\infty) = \mathbb{R} - (-1, 1). \tag{34}$$

Let $0 < \xi \leq 1$, it holds

$$G\left(\frac{x}{\xi}\right) < \mu e^{-\mu\left(\frac{x}{\xi}-1\right)}, \ \forall\, x \geq \xi, \text{ or } \forall\, x \leq -\xi. \tag{35}$$

Clearly, by Theorem 3.2 we have that

$$\frac{1}{\xi}\int_{-\infty}^{\infty} G\left(\frac{x}{\xi}\right) dx = 1. \tag{36}$$

So that $\frac{1}{\xi} G\left(\frac{x}{\xi}\right)$ is a density function, and let $d\mu_\xi(x) := \frac{1}{\xi} G\left(\frac{x}{\xi}\right) dx$, that is μ_ξ is a Borel probability measure.

We give the following important result.

Theorem 3.4. *Let* $0 < \xi \leq 1$, *and*

$$c_{k,\xi}^{*} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^{k} G\left(\frac{x}{\xi}\right) dx, \quad k = 1, ..., n \in \mathbb{N}. \tag{37}$$

Then $c_{k,\xi}^{*}$ *are finite and* $c_{k,\xi}^{*} \to 0$, *as* $\xi \to 0$.

Infact it holds

$$\left|c_{k,\xi}^{*}\right| \leq \left[1 + 2\mu^{-k} e^{\mu} k!\right] \xi^{k} < \infty, \tag{38}$$

for $k = 1, ..., n$.

Next we give.

Theorem 3.5. *It holds*

$$\int_{-\infty}^{\infty} |t|^{n} \left(1 + \frac{|t|}{\xi}\right)^{r} d\mu_{\xi}(t) < \infty; \; r, n \in \mathbb{N}, \tag{39}$$

for

$$d\mu_{\xi}(x) = \frac{1}{\xi} G\left(\frac{x}{\xi}\right) dx, \quad 0 < \xi \leq 1. \tag{40}$$

Also this integral converges to zero, as $\xi \to 0$.

Infact it holds

$$\frac{1}{\xi}\int_{-\infty}^{\infty} |x|^{n} \left(1 + \frac{|x|}{\xi}\right)^{r} G\left(\frac{x}{\xi}\right) dx \leq$$

$$2^{r-1}\left[\left(1 + 2\mu^{-n} e^{\mu} n!\right) + \left(1 + 2\mu^{-(n+r)} e^{\mu} (n+r)!\right)\right] \xi^{n} < \infty. \tag{41}$$

3.2. **About the q-Deformed and λ-Parametrized Hyperbolic tangent function $g_{q,\lambda}$.** We consider the activation function $g_{q,\lambda}$ and study its related properties, all the basics come from [9], ch. 17.

Let the activation function

$$g_{q,\lambda}(x) = \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \, x \in \mathbb{R}. \tag{42}$$

It is

$$g_{q,\lambda}(0) = \frac{1-q}{1+q},$$

and

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \quad \forall \, x \in \mathbb{R}, \tag{43}$$

with

$$g_{q,\lambda}(+\infty) = 1, \;\; g_{q,\lambda}(-\infty) = -1.$$

We consider the function

$$M_{q,\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{44}$$

$\forall\ x \in \mathbb{R},\ q, \lambda > 0$. We have $M_{q,\lambda}(\pm\infty) = 0$, so that the x-axis is a horizontal asymptote.

It holds

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall\ x \in \mathbb{R},\ q, \lambda > 0, \tag{45}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \quad \forall\ x \in \mathbb{R}.$$

The $M_{q,\lambda}$ maximum is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \tag{46}$$

Theorem 3.6. *We have that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \ \forall\ x \in \mathbb{R},\ \forall\ \lambda, q > 0. \tag{47}$$

Theorem 3.7. *It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0. \tag{48}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

Remark 3.8. i) Let $x \geq 1$. That is $0 \leq x-1 < x+1$. By mean value theorem we obtain

$$M_{q,\lambda}(x) = \frac{1}{4}\left[g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right] = \frac{1}{4} \cdot 2 \cdot \frac{4q\lambda e^{2\lambda\xi}}{\left(e^{2\lambda\xi}+q\right)^2} = \frac{2q\lambda e^{2\lambda\xi}}{\left(e^{2\lambda\xi}+q\right)^2}, \tag{49}$$

for some $0 \leq x-1 < \xi < x+1$; $\lambda, q > 0$.

But $e^{2\lambda\xi} < e^{2\lambda\xi} + q$, and

$$M_{q,\lambda}(x) < \frac{2q\lambda\left(e^{2\lambda\xi}+q\right)}{\left(e^{2\lambda\xi}+q\right)^2} = \frac{2q\lambda}{\left(e^{2\lambda\xi}+q\right)} < \frac{2q\lambda}{\left(e^{2\lambda(x-1)}+q\right)} < \frac{2q\lambda}{e^{2\lambda(x-1)}}, \tag{50}$$

$x \geq 1$.

That is

$$M_{q,\lambda}(x) < 2q\lambda e^{-2\lambda(x-1)}, \quad \forall\ x \geq 1. \tag{51}$$

Set $\mu := 2\lambda$, then

$$M_{q,\lambda}(x) < q\mu e^{-\mu(x-1)}, \quad \forall\ x \geq 1. \tag{52}$$

ii) Let now $x \leq -1$. That is $x-1 < x+1 \leq 0$. Again we have

$$M_{q,\lambda}(x) < \frac{2q\lambda}{\left(e^{2\lambda\xi}+q\right)}, \tag{53}$$

$x-1 < \xi < x+1 \leq 0$; $\lambda, q > 0$.

We have

$$e^{2\lambda(x-1)} < e^{2\lambda\xi} < e^{2\lambda(x+1)},$$

and

$$e^{2\lambda(x-1)} + q < e^{2\lambda\xi} + q < e^{2\lambda(x+1)} + q. \tag{54}$$

Hence

$$\frac{1}{e^{2\lambda\xi} + q} < \frac{1}{e^{2\lambda(x-1)} + q}. \tag{55}$$

Therefore it holds

$$M_{q,\lambda}(x) < \frac{2q\lambda}{e^{2\lambda(x-1)} + q} < \frac{2q\lambda}{e^{2\lambda(x-1)}}, \quad x \leq -1. \tag{56}$$

That is

$$M_{q,\lambda}(x) < 2q\lambda e^{-2\lambda(x-1)}, \quad \forall\ x \leq -1. \tag{57}$$

Set $\mu := 2\lambda$, then

$$M_{q,\lambda}(x) < q\mu e^{-\mu(x-1)}, \quad \forall\ x \leq -1. \tag{58}$$

We have proved that

$$M_{q,\lambda}(x) < q\mu e^{-\mu(x-1)}, \tag{59}$$

$\forall\ x \in (-\infty, -1] \cup [1, +\infty) = \mathbb{R} - (-1, 1)$.

Let $0 < \xi \leq 1$, it holds

$$M_{q,\lambda}\left(\frac{x}{\xi}\right) < q\mu e^{-\mu\left(\frac{x}{\xi}-1\right)}, \quad \forall\ x \geq \xi, \text{ or } \forall\ x \leq -\xi. \tag{60}$$

By Theorem 3.7 we have

$$\frac{1}{\xi}\int_{-\infty}^{\infty} M_{q,\lambda}\left(\frac{x}{\xi}\right) dx = 1. \tag{61}$$

So that $\frac{1}{\xi}M_{q,\lambda}\left(\frac{x}{\xi}\right)$ is a density function and let

$$d\mu_\xi(x) := \frac{1}{\xi}M_{q,\lambda}\left(\frac{x}{\xi}\right) dx, \tag{62}$$

that is μ_ξ is a Borel probability measure.

We give

Theorem 3.9. *Let*

$$\overline{c}_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k M_{q,\lambda}\left(\frac{x}{\xi}\right) dx, \quad k = 1, ..., n \in \mathbb{N}. \tag{63}$$

Then $\overline{c}_{k,\xi}$ *are finite and* $\overline{c}_{k,\xi} \to 0$, *as* $\xi \to 0$.

In fact it holds

$$|\overline{c}_{k,\xi}| \leq \left[1 + \left(q + \frac{1}{q}\right)\mu^{-k}e^{\mu}k!\right]\xi^k < \infty, \quad k = 1, ..., n. \tag{64}$$

It also follows

Theorem 3.10. *It holds* ($\lambda, q > 0$; $r, n \in \mathbb{N}$; $0 < \xi \leq 1$)

$$\frac{1}{\xi}\int_{-\infty}^{\infty}|t|^{n}\left(1+\frac{|t|}{\xi}\right)^{r}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt \leq$$

$$2^{r-1}\left[\left[1+\left(q+\frac{1}{q}\right)\mu^{-n}e^{\mu}n!\right]+\left[1+\left(q+\frac{1}{q}\right)\mu^{-(n+r)}e^{\mu}\left(n+r\right)!\right]\right]\xi^{n} < \infty, \tag{65}$$

and it converges to zero, as $\xi \to 0$.

3.3. **About the Gudermannian generated activation function.** Here we follow [8], Ch. 2.

Let the related normalized generator sigmoid function:

$$f\left(x\right) := \frac{8}{\pi}\int_{0}^{x}\frac{1}{e^{t}+e^{-t}}dt, \quad x \in \mathbb{R}, \tag{66}$$

and the neural network activation function:

$$\psi\left(x\right) := \frac{1}{4}\left(f\left(x+1\right)-f\left(x-1\right)\right) > 0, \ \ x \in \mathbb{R}. \tag{67}$$

We mention

Theorem 3.11. *It holds*

$$\int_{-\infty}^{\infty}\psi\left(x\right)dx = 1. \tag{68}$$

So that $\psi\left(x\right)$ *is a density function.*

By [8], p. 49, we found that

$$\psi\left(x\right) < \frac{2}{\pi\cosh\left(x-1\right)}, \quad \forall\ x \geq 1. \tag{69}$$

But

$$\frac{1}{\cosh\left(x-1\right)} = \frac{2}{e^{x-1}+e^{-(x-1)}} < \frac{2}{e^{x-1}} = 2e^{-(x-1)}, \tag{70}$$

$\forall\ x \in \mathbb{R}$.

Therefore it is

$$\psi\left(x\right) < \frac{4}{\pi}e^{-(x-1)} = \frac{4}{\pi}ee^{-x}, \quad \forall\ x \geq 1. \tag{71}$$

So here it is

$$d\mu_{\xi}\left(x\right) = \frac{1}{\xi}\psi\left(\frac{x}{\xi}\right)dx, \quad 0 < \xi \leq 1,$$

the related Borel probability measure.

We give the following results.

Theorem 3.12. *Let* $0 < \xi \leq 1$, *and*

$$\gamma_{k,\xi} := \frac{1}{\xi} \int_{-\infty}^{\infty} x^k \psi \left(\frac{x}{\xi} \right) dx, \quad k = 1, ..., n \in \mathbb{N}. \tag{72}$$

Then $\gamma_{k,\xi}$ *are finite and* $\gamma_{k,\xi} \to 0$, *as* $\xi \to 0$.

Theorem 3.13. *It holds*

$$\frac{1}{\xi} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi} \right)^r \psi \left(\frac{t}{\xi} \right) dt < \infty; \tag{73}$$

$r, n \in \mathbb{N}$; $0 < \xi \leq 1$.

Also this integral converges to zero, as $\xi \to 0$.

3.4. **About the q-deformed and λ-parametrized logistic type activation function.** Here all come from [9], Ch. 15.

The activation function now is

$$\varphi_{q,\lambda}(x) := \frac{1}{1 + qe^{-\lambda x}}, \quad x \in \mathbb{R}, \tag{74}$$

where $q, \lambda > 0$.

The density function here will be

$$G_{q,\lambda}(x) := \frac{1}{2} \left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1) \right) > 0, \ \ x \in \mathbb{R}. \tag{75}$$

We mention

Theorem 3.14. *It holds*

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x) \, dx = 1. \tag{76}$$

By [9], p. 373, we have

$$G_{q,\lambda}(x) < q\lambda e^{-\lambda(x-1)}, \quad \forall \ x \geq 1.$$

So here it is

$$d\mu_\xi(x) = \frac{1}{\xi} G_{q,\lambda} \left(\frac{x}{\xi} \right) dx, \quad 0 < \xi \leq 1, \tag{77}$$

the related Borel probability measure.

We give the following results.

Theorem 3.15. *Let*

$$\overline{\delta}_{k,\xi} := \frac{1}{\xi} \int_{-\infty}^{\infty} x^k G_{q,\lambda} \left(\frac{x}{\xi} \right) dx, \quad k = 1, ..., n \in \mathbb{N}. \tag{78}$$

Then $\overline{\delta}_{k,\xi}$ *are finite and* $\overline{\delta}_{k,\xi} \to 0$, *as* $\xi \to 0$.

Theorem 3.16. *It holds*

$$I_{G_{q,\lambda},\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} |t|^n \left(1+\frac{|t|}{\xi}\right)^r G_{q,\lambda}\left(\frac{t}{\xi}\right) dt < \infty; \tag{79}$$

where $\lambda, q > 0$; $r, n \in \mathbb{N}$; $0 < \xi \leq 1$.

Also $I_{G_{q,\lambda},\xi} \to 0$, *as* $\xi \to 0$.

3.5. **About the q-Deformed and β-Parametrized Half Hyperbolic Tangent function $\varphi_{q,\beta}$.** Here all come from [9], Ch. 19.

The activation function now is

$$\varphi_{q,\beta}(x) := \frac{1-qe^{-\beta t}}{1+qe^{-\beta t}}, \quad \forall\, t \in \mathbb{R}, \tag{80}$$

where $q, \beta > 0$.

The corresponding density function will be

$$\Phi_{q,\beta}(x) := \frac{1}{4}\left(\varphi_{q,\beta}(x+1) - \varphi_{q,\beta}(x-1)\right) > 0, \ \forall\, x \in \mathbb{R}. \tag{81}$$

It holds

Theorem 3.17.

$$\int_{-\infty}^{\infty} \Phi_{q,\beta}(x)\, dx = 1. \tag{82}$$

By [9], p. 481, we have that

$$\Phi_{q,\beta}(x) < \beta q e^{-\beta(x-1)}, \quad \forall\, x \geq 1. \tag{83}$$

Thus here it is

$$d\mu_\xi(x) = \frac{1}{\xi}\Phi_{q,\beta}\left(\frac{x}{\xi}\right) dx, \quad 0 < \xi \leq 1, \tag{84}$$

the related Borel probability measure.

We state the following results.

Theorem 3.18. *Let*

$$\varepsilon_{k,\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} x^k \Phi_{q,\beta}\left(\frac{x}{\xi}\right) dx, \quad k = 1, ..., n \in \mathbb{N}. \tag{85}$$

Then $\varepsilon_{k,\xi}$ *are finite and* $\varepsilon_{k,\xi} \to 0$, *as* $\xi \to 0$.

Theorem 3.19. *It holds*

$$I_{\Phi_{q,\beta},\xi} := \frac{1}{\xi}\int_{-\infty}^{\infty} |t|^n \left(1+\frac{|t|}{\xi}\right)^r \Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt < \infty; \tag{86}$$

where $q, \beta > 0$; $r, n \in \mathbb{N}$; $0 < \xi \leq 1$.

Also $I_{\Phi_{q,\beta},\xi} \to 0$, *as* $\xi \to 0$.

4. **More on Activation Probability measures**

Here all come from [11].

We mention the following results.

Theorem 4.1. *Let* $p>1$, $r\in\mathbb{N}$, $0<\xi\leq 1$, $n\in\mathbb{N}$, $l:=\max(r,n)$, $\lceil\cdot\rceil$ *ceiling of the number, and* $h:=2(l\lceil p\rceil+1)$. *It holds*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1}-1\right)|t|^{np-1}G\left(\frac{t}{\xi}\right)dt\leq$$

$$2^{h}\left\{1+\left[1+2\mu^{-h}e^{\mu}h!\right]\right\}<+\infty. \tag{87}$$

Proposition 4.2. Let $r\in\mathbb{N}$. It holds

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r}G\left(\frac{t}{\xi}\right)dt\leq$$

$$2^{r-1}\left[1+\left[1+2\mu^{-r}e^{\mu}r!\right]\right]<+\infty. \tag{88}$$

Theorem 4.3. *Let* $r,n\in\mathbb{N}$, $0<\xi\leq 1$. *It holds*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}G\left(\frac{t}{\xi}\right)dt\leq$$

$$2^{r+n}\left[1+\left[1+2\mu^{-(r+n)}e^{\mu}(r+n)!\right]\right]<+\infty. \tag{89}$$

Proposition 4.4. Let $r\in\mathbb{N}$, $p>1$, $\lambda:=r\lceil p\rceil\in\mathbb{N}$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp}G\left(\frac{t}{\xi}\right)dt\leq \tag{90}$$

$$2^{\lambda-1}\left[1+\left[1+2\mu^{-\lambda}e^{\mu}\lambda!\right]\right]<+\infty.$$

Similar results are needed and follow.

Theorem 4.5. *All as in Theorem 4.1. Then*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1}-1\right)|t|^{np-1}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq \tag{91}$$

$$2^{h}\left\{1+\left[1+\left(q+\frac{1}{q}\right)\mu^{-h}e^{\mu}h!\right]\right\}<+\infty,$$

where $q,\lambda>0$.

Theorem 4.6. *Let* $r,n\in\mathbb{N}$, $0<\xi\leq 1$. *Then*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt\leq \tag{92}$$

$$2^{r+n}\left[1+\left[1+\left(q+\frac{1}{q}\right)\mu^{-(r+n)}e^{\mu}(r+n)!\right]\right]<+\infty.$$

Proposition 4.7. Let $r \in \mathbb{N}$. It holds

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r} M_{q,\lambda}\left(\frac{t}{\xi}\right)dt \leq \tag{93}$$

$$2^{r-1}\left[1+\left[1+\left(q+\frac{1}{q}\right)\mu^{-r}e^{\mu}r!\right]\right] < +\infty.$$

Proposition 4.8. Let $r \in \mathbb{N}$, $p > 1$, $\lambda := r\lceil p\rceil$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp} M_{q,\lambda}\left(\frac{t}{\xi}\right)dt \leq \tag{94}$$

$$2^{\lambda-1}\left[1+\left[1+\left(q+\frac{1}{q}\right)\mu^{-\lambda}e^{\mu}\lambda!\right]\right] < +\infty.$$

We continue with more related results.

Theorem 4.9. *Let $p > 1$, $r \in \mathbb{N}$, $0 < \xi \leq 1$, $n \in \mathbb{N}$. Then, there exists $\lambda_1 > 0$ such that:*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1}-1\right)|t|^{np-1}\,\psi\left(\frac{t}{\xi}\right)dt \leq \lambda_1 \in \mathbb{R}. \tag{95}$$

More needed results are listed.

Theorem 4.10. *Let $r, n \in \mathbb{N}$, $0 < \xi \leq 1$. Then, there exists $\lambda_2 > 0$ such that:*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,\psi\left(\frac{t}{\xi}\right)dt \leq \lambda_2 \in \mathbb{R}. \tag{96}$$

Proposition 4.11. Let $r \in \mathbb{N}$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r}\psi\left(\frac{t}{\xi}\right)dt \leq \lambda_3 \in \mathbb{R}. \tag{97}$$

Proposition 4.12. Let $r \in \mathbb{N}$, $p > 1$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp}\psi\left(\frac{t}{\xi}\right)dt \leq \lambda_4 \in \mathbb{R}. \tag{98}$$

More needed results:

Theorem 4.13. *Let $p > 1$, $r \in \mathbb{N}$, $0 < \xi \leq 1$, $n \in \mathbb{N}$, $q, \lambda > 0$. Then, there exists $\rho_1 > 0$:*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1}-1\right)|t|^{np-1}\,G_{q,\lambda}\left(\frac{t}{\xi}\right)dt \leq \rho_1 \in \mathbb{R}. \tag{99}$$

Theorem 4.14. *Let $r, n \in \mathbb{N}$, $0 < \xi \leq 1$. Then, there exists $\rho_2 > 0$:*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,G_{q,\lambda}\left(\frac{t}{\xi}\right)dt \leq \rho_2 \in \mathbb{R}. \tag{100}$$

Proposition 4.15. Let $r \in \mathbb{N}$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r} G_{q,\lambda}\left(\frac{t}{\xi}\right) dt \leq \rho_3 \in \mathbb{R}. \tag{101}$$

Proposition 4.16. Let $r \in \mathbb{N}$, $p > 1$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp} G_{q,\lambda}\left(\frac{t}{\xi}\right) dt \leq \rho_4 \in \mathbb{R}. \tag{102}$$

Furthermore we have the following:

Theorem 4.17. *Let $p > 1$, $r \in \mathbb{N}$, $0 < \xi \leq 1$, $n \in \mathbb{N}$; $q, \beta > 0$. Then, there exists $\psi_1 > 0$:*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{rp+1}-1\right)|t|^{np-1}\,\Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt \leq \psi_1. \tag{103}$$

Theorem 4.18. *Let $r, n \in \mathbb{N}$, $0 < \xi \leq 1$. Then, there exists $\psi_2 > 0$:*

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}\,\Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt \leq \psi_2. \tag{104}$$

Proposition 4.19. Let $r \in \mathbb{N}$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{r} \Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt \leq \psi_3 \in \mathbb{R}. \tag{105}$$

Proposition 4.20. Let $r \in \mathbb{N}$, $p > 1$. Then

$$\frac{1}{\xi}\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp} \Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt \leq \psi_4 \in \mathbb{R}. \tag{106}$$

5. Main Results

Here we describe the activated approximation and simultaneous approximation properties of the following activated singular integral operators which are special cases of $\Theta_{r,\xi}(f,x)$, see (2). Their definitions are based on Sections 3, 4. Basically we apply our listed results in Section 2.

Definition 5.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function and α_j as in (1), $x \in \mathbb{R}$, $0 < \xi \leq 1$.

We call

1)

$$\Theta_{1,r,\xi}(f,x) = \frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right) G\left(\frac{t}{\xi}\right) dt, \tag{107}$$

2)

$$\Theta_{2,r,\xi}(f,x) = \frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right) M_{q,\lambda}\left(\frac{t}{\xi}\right) dt, \quad q, \lambda > 0, \tag{108}$$

3)

$$\Theta_{3,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)\psi\left(\frac{t}{\xi}\right)dt, \tag{109}$$

4)

$$\Theta_{4,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)G_{q,\lambda}\left(\frac{t}{\xi}\right)dt, \quad q,\lambda>0, \tag{110}$$

and

5)

$$\Theta_{5,r,\xi}(f,x)=\frac{1}{\xi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j f(x+jt)\right)\Phi_{q,\beta}\left(\frac{t}{\xi}\right)dt, \quad q,\beta>0. \tag{111}$$

We start with activated global smoothness presentation results.

Theorem 5.2. *Let $h>0$, $f\in C(\mathbb{R})$; $j^*=1,2,3,4,5$.*

i) Assume $\Theta_{j^*,r,\xi}(f;x)\in\mathbb{R}$, $\xi>0$, $\forall\ x\in\mathbb{R}$ and $\omega_m(f,h)<\infty$. Then

$$\omega_m(\Theta_{j^*,r,\xi}f,h)\le\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h). \tag{112}$$

ii) Assume $f\in(C(\mathbb{R})\cap L_1(\mathbb{R}))$, then

$$\omega_m(\Theta_{j^*,r,\xi}f,h)_1\le\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h)_1. \tag{113}$$

iii) Assume $f\in(C(\mathbb{R})\cap L_p(\mathbb{R}))$, $p>1$. Then

$$\omega_m(\Theta_{j^*,r,\xi}f,h)_p\le\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h)_p. \tag{114}$$

Proof. By Theorem 2.1 □

Next comes about differentiation of $\Theta_{j^*,r,\xi}(f;x)$, $j^*=1,2,3,4,5$.

Theorem 5.3. *Here all are as in Theorem 2.2, with $d\mu_\xi$ to be $\frac{1}{\xi}G\left(\frac{t}{\xi}\right)dt$, $\frac{1}{\xi}M_{q,\lambda}\left(\frac{t}{\xi}\right)dt$, $\frac{1}{\xi}\psi\left(\frac{t}{\xi}\right)dt$, $\frac{1}{\xi}G_{q,\lambda}\left(\frac{t}{\xi}\right)dt$ and $\frac{1}{\xi}\Phi_{q,\beta}\left(\frac{t}{\xi}\right)dt$, respectively for $j^*=1,2,3,4,5$.*

Then $f^{(i)}(x+jt)$ defines a μ_ξ-integrable function with respect to t for each $x\in\mathbb{R}$, all $i=1,...,n$; $j=1,...,r$, and

$$\left(\Theta_{j^*,r,\xi}(f;x)\right)^{(i)}=\Theta_{j^*,r,\xi}\left(f^{(i)};x\right), \tag{115}$$

for all $x\in\mathbb{R}$, all $i=1,...,n$; $j^*=1,2,3,4,5$.

Proof. By Theorem 2.2 □

It follows activated simultaneous global smoothness preservation of $\Theta_{j^*,r,\xi}$ operators, $j^* = 1, 2, 3, 4, 5$.

Theorem 5.4. *Let $h > 0$ and the assumptions of Theorem 5.3 are valid; $j^* = 1, 2, 3, 4, 5$.*

i) Suppose that $\omega_m\left(f^{(i)}, h\right) < \infty$, all $i = 0, 1, ..., n$, then

$$\omega_m\left(\left(\Theta_{j^*,r,\xi}f\right)^{(i)}, h\right) \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right), \tag{116}$$

for all $i = 0, 1, ..., n$.

ii) Assume $f^{(i)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $i = 0, 1, ..., n$, then

$$\omega_m\left(\left(\Theta_{j^*,r,\xi}f\right)^{(i)}, h\right)_1 \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right)_1, \tag{117}$$

for all $i = 0, 1, ..., n$.

iii) Assume $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p > 1$, $i = 0, 1, ..., n$, then

$$\omega_m\left(\left(\Theta_{j^*,r,\xi}f\right)^{(i)}, h\right)_p \leq \left(\sum_{j=0}^{r} |\alpha_j|\right) \omega_m\left(f^{(i)}, h\right)_p, \tag{118}$$

$i = 0, 1, ..., n$.

Proof. By Theorem 2.3 □

We need the following.

Definition 5.5. We call

$$\begin{gathered} d\mu_{1\xi} = \frac{1}{\xi} G\left(\frac{t}{\xi}\right) dt, \\ d\mu_{2\xi} = \frac{1}{\xi} M_{q,\lambda}\left(\frac{t}{\xi}\right) dt, \\ d\mu_{3\xi} = \frac{1}{\xi} \psi\left(\frac{t}{\xi}\right) dt, \\ d\mu_{4\xi} = \frac{1}{\xi} G_{q,\lambda}\left(\frac{t}{\xi}\right) dt, \text{ and} \\ d\mu_{5\xi} = \frac{1}{\xi} \Phi_{q,\beta}\left(\frac{t}{\xi}\right) dt. \end{gathered} \tag{119}$$

Also we need.

Definition 5.6. We set

$$\begin{gathered} c_{1,k,\xi} = c^*_{k,\xi}, \\ c_{2,k,\xi} = \overline{c}_{k,\xi}, \\ c_{3,k,\xi} = \gamma_{k,\xi}, \\ c_{4,k,\xi} = \overline{\delta}_{k,\xi}, \text{ and} \\ c_{5,k,\xi} = \varepsilon_{k,\xi}. \end{gathered} \tag{120}$$

Next come simultaneous activated approximation results.

Theorem 5.7. *Let $f \in C^{n+\rho}(\mathbb{R})$, $n, \rho \in \mathbb{Z}^+$ and $\omega_r\left(f^{(n+i)}, h\right) < \infty$, $\forall\ h > 0$ for $i = 0, 1, ..., \rho$; $\delta_k = \sum_{j=1}^{r} \alpha_j j^k$. We consider the assumptions of Theorems 2.2, 5.3 valid for $n = \rho$ there. Then*

$$\left\| \left(\Theta_{j^*,r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k c_{j^*,k,\xi} \right\|_{\infty,x} \leq$$

$$\frac{\omega_r\left(f^{(n+i)}, \xi\right)}{n!} \int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_{j^*\xi}(t). \tag{121}$$

When $n = 0$ the sum in the left of (121) collapses; $j^* = 1, 2, 3, 4, 5$.

Proof. By Theorem 2.4, Definitions 5.5, 5.6, and Theorems 3.4, 3.5; Theorems 3.9, 3.10; Theorems 3.12, 3.13; Theorems 3.15, 3.16; and Theorems 3.18, 3.19 □

We continue with the next L_p result.

Theorem 5.8. *Let $f \in C^{n+\rho}(\mathbb{R})$, with $f^{(n+i)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We consider the assumptions of Theorems 2.2, 5.3 valid for $n = \rho$ there. Then*

$$\left\| \left(\Theta_{j^*,r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(k+i)}(x)}{k!} \delta_k c_{j^*,k,\xi} \right\|_{p,x} \leq$$

$$\frac{1}{((n-1)!)\left(q(n-1)+1\right)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}$$

$$\left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1\right) |t|^{np-1} d\mu_{j^*\xi}(t)\right]^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r\left(f^{(n+i)}, \xi\right)_p, \tag{122}$$

for $j^* = 1, 2, 3, 4, 5$.

Proof. By Theorem 2.5, Definitions 5.5, 5.6, and Theorems 3.4, 3.9, 3.12, 3.15, 3.18; and Theorems 4.1, 4.5, 4.9, 4.13, 4.17. □

A related results follows:

Proposition 5.9. Let $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We consider the assumptions of Theorems 2.2, 5.3 valid for $n = \rho$ there. Then

$$\left\| \left(\Theta_{j^*,r,\xi}(f)\right)^{(i)} - f^{(i)} \right\|_p \leq$$

$$\omega_r\left(f^{(i)}, \xi\right)_p \left(\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_{j^*\xi}(t)\right)^{\frac{1}{p}}, \tag{123}$$

for all $i = 0, 1, ..., \rho$; $j^* = 1, 2, 3, 4, 5$.

Proof. By Proposition 2.6, Definitions 5.5, 5.6, and Propositions 4.4, 4.8, 4.12. 4.16, 4.20. □

Next we cover the case $p = 1$.

Theorem 5.10. *Let $f \in C^{n+\rho}(\mathbb{R})$, with $f^{(n+i)} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$. We consider the assumptions of Theorems 2.2, 5.3 valid for $n = \rho$ there. Then*

$$\left\| \left(\Theta_{j^*,r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(k+i)}(x)}{k!} \delta_k c_{j^*,k,\xi} \right\|_{1,x} \leq \frac{1}{(n-1)!\,(r+1)}$$

$$\left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{r+1} - 1 \right) |t|^{n-1} d\mu_{j^*\xi}(t) \right] \xi \omega_r \left(f^{(n+i)}, \xi\right)_1, \tag{124}$$

for all $i = 0, 1, ..., \rho$; $j^* = 1, 2, 3, 4, 5$.

Proof. By Theorem 2.7, Definitions 5.5, 5.6, and Theorems 3.4, 4.3; Theorems 3.9, 4.6; Theorems 3.12, 4.10; Theorems 3.15, 4.14; and Theorems 3.18, 4.18. □

We finish with a basic simultaneous activated approximation result.

Proposition 5.11. Let $f^{(i)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $i = 0, 1, ..., \rho \in \mathbb{Z}^+$. We consider the assumptions of Theorems 2.2, 5.3 valid for $n = \rho$ there. Then

$$\left\| \left(\Theta_{j^*,r,\xi}(f)\right)^{(i)} - f^{(i)} \right\|_1 \leq$$

$$\omega_r \left(f^{(i)}, \xi\right)_1 \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{r} d\mu_{j^*\xi}(t) \right], \tag{125}$$

for all $i = 0, 1, ..., \rho$; $j^* = 1, 2, 3, 4, 5$.

Proof. By Proposition 2.8, Definitions 5.5, 5.6, and Propositions 4.2, 4.7, 4.11, 4.15, 4.19. □

REFERENCES

[1] Ali Aral, *On a generalized Gauss Weierstrass singular integral*, Fasc. Math. No. 35 (2005), 23-33.

[2] Ali Aral, *Pointwise approximation by the generalization of Picard and Gauss-Weierstrass singular integrals*, J. Concr. Appl. Math. 6 (2008), no. 4, 327-339.

[3] Ali Aral, *On generalized Picard integral operators*, Advances in summability and approximation theory, 157-168, Springer, Singapore, 2018.

[4] Ali Aral, E. Deniz, H. Erbay, *The Picard and Gauss-Weierstrass singular integrals in (p,q)-calculus*, Bull, Malays. Math. Sci. Soc. 43 (2020), no. 2, 1569-1583.

[5] Ali Aral, Sorin G. Gal, *q-generalizations of the Picard and Gauss-Weiertrass singular integrals*, Taiwanese J. Math. 12 (2008), no. 9, 2501-2515.

[6] G.A. Anastassiou, *Global smoothness preservation by singular integrals*, Proyecciones, 14, No. 2 (1995), 83-88.

[7] G.A. Anastassiou, *Quantitative Approximations*, Chapman & Hall / CRC, Boca Raton, New York, 2001.

[8] G.A. Anastassiou, *Banach space valued Neural Network*, Springer, Heidelberg, New York, 2023.

[9] G.A. Anastassiou, *Parametrized, Deformed and General Neural Networks*, Springer, Heidelberg, New York, 2023.

[10] G.A. Anastassiou, *Quantitative uniform approximation by activated singular operators*, Mathematics, 2023, 12, 2152, https://doi.org/10.3390/math12142152.

[11] G.A. Anastassiou, *Quantitative L_p Approximation by Activared Singular integral Operators*, submitted, 2024.

[12] G.A. Anastassiou and S. Gal, *Convergence of generalized singular integrals to the unit, univariate case*, Math. Inequalities & Applications, 3, No. 4 (2000), 511-518.

[13] G.A. Anastassiou and S. Gal, *Convergence of generalized singular integrals to the unit, multivariate case*, Applied. Math. Rev., Vol. 1, World Sci. Publ. Co., Singapore, 2000, pp. 1-8.

[14] G.A. Anastassiou and S. Gal, *Approximation Theory*, Birkhäuser, Boston, Berlin, 2000.

[15] G. Anastassiou, R. Mezei, *Approximation by Singular Integrals*, Cambridge Scientific Publishers, Cambridge, UK, 2012.

[16] R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Vol. 303, Berlin, New York, 1993.

[17] S.G. Gal, *Remark on the degree of approximation of continuous functions by singular integrals*, Math. Nachr., 164 (1993), 197-199.

[18] S.G. Gal, *Degree of approximation of continuous functions by some singular integrals*, Rev. Anal. Numér, Théor. Approx., (Cluj), Tome XXVII, No. 2 (1998), 251-261.

[19] R.N. Mohapatra and R.S. Rodriguez, *On the rate of convergence of singular integrals for Hölder continuous functions*, Math. Nachr., 149 (1990), 117-124.