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IMPULSIVE FRACTIONAL BOUNDARY VALUE DIFFERENTIAL INCLUSIONS HAVING A CAPUTO-HADAMARD DERIVATIVE

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ABSTRACT. In this paper, the authors investigate the existence of solutions to a class of boundary value problems for impulsive fractional differential inclusions with Caputo-Hadamard derivative.

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1. INTRODUCTION

This paper is concerned with the existence of solutions to the boundary value problem (BVP) for impulsive differential inclusions of the form

(1.1)
$$ {}^c_H D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, T], \; t \neq t_k, \; k = 1, ..., m, \quad 0 < r \leq 1, $$

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, ..., m, \tag{1.2} $$

$$ ay(1) + by(T) = c, \tag{1.3} $$

where $T > 1$, $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}^{Hc}D^r$ is the Caputo–Hadamard fractional derivative of order $0 < r \leq 1$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, a, b and c are real constants with $a + b \neq 0$, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$, are continuous functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ and $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$ where $y(t_k^+) = \lim_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim_{\varepsilon \to 0^-} y(t_k + \varepsilon)$ are the right and left hand limits of y at $t = t_k$, $k = 1, \ldots, m$.

Differential equations of fractional order provide models for many phenomena in various fields of science and engineering including viscoelasticity, electrochemistry, control processes, porous media, electromagnetism, and others. As a consequence,

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there has been a significant development in the theory of fractional calculus and fractional ordinary and partial differential equations in recent years; see, for example, the monographs of Abbas *et al.* [1], Hilfer [27], Kilbas *et al.* [30], and Podlubny [35], as well as the papers of Agarwal *et al.* [3], Benchohra *et al.* [9], and Momani *et al.* [34]. Applied problems require the definitions of fractional derivatives to allow the utilization of physically interpretable initial data (e.g., $y(0)$, $y'(0)$, etc.), and Caputo's fractional derivative is quite useful in such instances. For details concerning geometric and physical interpretation of fractional derivatives of Riemann-Liouville and Caputo type, see [35]. However, the literature on Hadamard-type fractional differential equations has not undergone as extensive a development; see [4, 38]. Hadamard's fractional derivative [22] differs from other fractional derivatives in that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function of exponential order. Detailed descriptions of the Hadamard fractional derivative and integral can be found, for example, in [13, 14, 15]. Hadamard fractional calculus has recently been gaining attention in the study of fractional analysis. The papers [4, 13, 14, 15, 29, 32, 38] contain some major developments in the fundamental theory of Hadamard fractional calculus. A Caputo-type modification of the Hadamard fractional derivative, which is called the Caputo-Hadamard fractional derivative, was given in [28], and some of its basic properties were proved in [2, 18].

Igor Podlubny's website, http://people.tuke.sk/igor.podlubny/, contains more information on fractional calculus and its applications, and so is very useful for those who are interested in this research area. Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has been a significant development in impulsive theory, especially in the area of impulsive differential equations with fixed moments; see, for instance, the monographs by Bainov and Simeonov [6], Benchohra *et al.* [10], Graef *et al.* [19] Lakshmikantham *et al.* [33], and Samoilenko and Perestyuk [37], and the references therein. In [11], Benchohra and Slimani initiated the study of Caputo fractional differential equations with impulses. In 2018, Belhannache *et al.* [7] studied initial value problems for Caputo-Hadamard fractional differential inclusions (of order $0 < r \leq 1$) with impulses.

We have organized this paper as follows. In Section 2, we introduce some preliminary results needed in the following sections. In Section 3, we present an existence result for the problem (1.1)–(1.3) in the case where the right hand side is convex valued by using the nonlinear alternative of Leray-Schauder type. Section 4 contains two existence results for nonconvex valued right hand sides. The first result is based on a fixed point theorem due to Covitz and Nadler [17], and the second one uses the nonlinear alternative of Leray-Schauder [21] for single-valued maps combined with a selection theorem due to Bressan and Colombo [12] for lower semicontinuous multivalued maps with decomposable values. In Section 5, we present a result on the topological structure of the set of solutions of (1.1)–(1.3). An example is given in the last section.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts that are used in the remainder of this paper. Much of what follows is standard, but is included here for the convenience of the reader.

On the interval $[a,b]$, let $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : a \leq t \leq b\},$$

and let $L^1([a,b],\mathbb{R})$ be Banach space of functions $y : [a,b] \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_a^b |y(t)|dt.$$

Here, $AC([a,b],R)$ is the space of functions $y : [a,b] \to \mathbb{R}$ that are absolutely continuous. For any Banach space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multivalued map $G : X \to \mathcal{P}(X)$ is said to be *convex (closed)* valued if $G(X)$ is convex (closed) for all $x \in X$, and we say that G is *bounded on bounded sets* if $G(B) = \underset{x\in B}{\cup} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

A map G is *upper semi-continuous* (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N in X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The set of all fixed points set of the multivalued operator G will be denoted by $Fix\, G$.

A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. We say that a subset A of $[0,T] \times \mathbb{R}$ is $l \otimes \beta$ measurable if A belongs to the σ-algebra generated by all sets of the form $J \times D$, where J is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is *decomposable* if for all u, $v \in A$ and a measurable set $J \subset [0,T]$, we have $u\chi_J + v\chi_{[a,b]-J} \in A$, where χ stands for the characteristic function.

Definition 2.1. A function $F : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Caratheódory if:

(1) $t \to F(t,u)$ is measurable for each $u \in \mathbb{R}$;
(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in [a,b]$.

For each $y \in C([a,b],\mathbb{R})$, the set of selections of F is given by

$$S_{F,y} = \{v \in L^1([a,b],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ a.e. } t \in [a,b]\}.$$

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [31]).

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called:

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in X;$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemma will be used in the sequel.

Lemma 2.3. (Covitz-Nadler [17]) *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $Fix\, N \neq \emptyset$.*

Definition 2.4. ([30]) The Hadamard fractional integral of order $\alpha > 0$ for a function $h : [a,b] \to \mathbb{R}$, where $a \ge 0$, is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds,$$

provided the integral exists.

Definition 2.5. ([28]) Let $AC_\delta^n[a,b] = \{g : [a,b] \to \mathbb{C} \text{ and } \delta^{n-1}g \in AC[a,b]\}$, where $\delta = t\dfrac{d}{dt}$, $0 < a < b < \infty$ and let $\alpha \in \mathbb{C}$ with $Re(\alpha) \ge 0$. For a function $g \in AC_\delta^n[a,b]$, the Caputo-Hadamard derivative of fractional order α is defined as follows:

(i): If $\alpha \notin \mathbb{N}$, and $n-1 < \alpha < n$ such that $n = [Re(\alpha)] + 1$, then

$$({}^{CH}D_a^\alpha g)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n g(s) \frac{ds}{s};$$

(ii): If $\alpha = n \in \mathbb{N}$, then $({}^{CH}D_a^{\alpha} g)(t) = \delta^n g(t)$.

Here, $[Re(\alpha)]$ denotes the integer part of the real number $Re(\alpha)$ and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.6. *Let $y \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $\alpha \in \mathbb{C}$. Then*

$$I_a^{\alpha}({}^{CH}D_a^{\alpha} y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k. \tag{2.1}$$

3. THE CONVEX CASE

In this section, we assume that F is a compact and convex valued multivalued map. Consider the set of functions

$$\begin{aligned} PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} \mid y \in & C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m, \text{and there exist} \\ & y(t_k^+) \text{ and } y(t_k^-),\ k = 1, \dots, m,\ \text{with } y(t_k^-) = y(t_k)\}. \end{aligned}$$

This forms a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} |y(t)|.$$

Set $J' = J \setminus \{t_1, \dots, t_m\}$.

Definition 3.1. A function $y \in PC(J,\mathbb{R})$ is a solution of (1.1)–(1.3) if there exists a function $\rho \in L^1([a,T],\mathbb{R})$ such that $\rho(t) \in F(t,y)$ a.e. $t \in J$,and ρ satisfies ${}^{CH}D^{\alpha} y(t) = \rho(t)$ on J' and conditions (1.2)–(1.3).

To prove the existence of a solution to (1.1)–(1.3), we need the following lemma relating solutions of a fractional BVP to solutions of a corresponding integral equation.

Lemma 3.2. *Let $0 < r \leq 1$ and let $\rho : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \left(\dfrac{-1}{a+b}\right) \left[b \displaystyle\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(r)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \left(\log \dfrac{t_i}{s}\right)^{r-1} \rho(s) \dfrac{ds}{s} \right. \\ \left. + \dfrac{b}{\Gamma(r)} \displaystyle\int_{t_m}^{T} \left(\log \dfrac{T}{s}\right)^{r-1} \rho(s) \dfrac{ds}{s} - c \right] \\ + \displaystyle\sum_{i=1}^{k} I_i(y(t_i^-)) + \dfrac{1}{\Gamma(r)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\log \dfrac{t_i}{s}\right)^{r-1} \rho(s) \dfrac{ds}{s} \\ + \dfrac{1}{\Gamma(r)} \displaystyle\int_{t_k}^{t} \left(\log \dfrac{t}{s}\right)^{r-1} \rho(s) \dfrac{ds}{s}, \qquad \text{if } t \in [t_k, t_{k+1}], \end{cases} \tag{3.1}$$

where $k = 1, ..., m$, if and only if y is a solution of the fractional BVP

$$ {}^{CH}D_{t_k}^{r} y(t) = \rho(t), \quad \text{for each} \quad t \in J_k, \tag{3.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, ..., m, \tag{3.3}$$

$$ay(1) + by(T) = c. \tag{3.4}$$

Proof. Assume that y satisfies the impulsive boundary value problem (3.2)–(3.4). If $t \in [1, t_1]$, then

$$^{CH}D_t^r y(t) = \rho(t), \quad t \in [1, t_1],$$

and by virtue of Lemma 2.6, we see that

$$y(t) = c_0 + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}.$$

If $t \in (t_1, t_2]$, then again by Lemma 2.6,

$$\begin{aligned}
y(t) &= y(t_1^+) + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&= I_1(y(t_1^-)) + \left[c_0 + \frac{1}{\Gamma(r)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s}\right] \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&= c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(r)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_1}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}.
\end{aligned}$$

If $t \in (t_2, t_3]$,

$$\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(r)} \int_{t_2}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(r)} \int_{t_2}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&= I_2(y(t_2^-)) + \left[c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(r)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s}\right. \\
&\quad \left. + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \rho(s) \frac{ds}{s}\right] + \frac{1}{\Gamma(r)} \int_{t_2}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} \\
&= c_0 + [I_1(y(t_1^-)) + I_2(y(t_2^-))] + \left[\frac{1}{\Gamma(r)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s}\right. \\
&\quad \left. + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \rho(s) \frac{ds}{s}\right] + \frac{1}{\Gamma(r)} \int_{t_2}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}.
\end{aligned}$$

And in general, if $t \in (t_k, t_{k+1}]$, where $k = 1, ..., m$,

$$y(t) = c_0 + \sum_{i=1}^k I_i(y(t_i^-)) + \frac{1}{\Gamma(r)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \rho(s) \frac{ds}{s}$$

$$+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s}.$$

The impulsive conditions and boundary conditions, (3.3) and (3.4), imply that

$$ay(1)+by(T)=c_0(a+b)+b\sum_{i=1}^{k}I_i(y(t_i^-))+\frac{b}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\rho(s)\frac{ds}{s}$$
$$+\frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}\rho(s)\frac{ds}{s}=c,$$

so

$$c_0=\left(\frac{-1}{a+b}\right)\left[b\sum_{i=1}^{m}I_i(y(t_i^-))+\frac{b}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\rho(s)\frac{ds}{s}\right.$$
$$\left.+\frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}\rho(s)\frac{ds}{s}-c\right].$$

Hence, we obtain the solution

$$y(t)=\left(\frac{-1}{a+b}\right)\left[b\sum_{i=1}^{m}I_i(y(t_i^-))+\frac{b}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\rho(s)\frac{ds}{s}\right.$$
$$\left.+\frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}\rho(s)\frac{ds}{s}-c\right]$$
$$+\sum_{i=1}^{k}I_i(y(t_i^-))+\frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\rho(s)\frac{ds}{s}$$
$$+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s}.$$

Conversely, assume that y satisfies (3.1). By a direct computation, it follows that the solution given by (5) satisfies (3.2)–(3.4). This completes the proof of the lemma. □

Theorem 3.3. *Assume the following conditions hold:*

- **(H1):** $F: J\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map.*
- **(H2):** *There exist* $p\in C(J,\mathbb{R}^+)$ *and a continuous and nondecreasing function* $\psi:[0,\infty)\to(0,\infty)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}}=sup\{|v|,v\in F(t,u)\}\leq p(t)\psi(|u|)\ \textit{for}\ t\in J\ \textit{and}\ u\in\mathbb{R}.$$

- **(H3):** *There exists a continuous and nondecreasing function* $\psi^*:[0,\infty)\to(0,\infty)$ *such that*

$$\|I_k(u)\|\leq\psi^*(|u|)\ \textit{for each}\ u\in\mathbb{R}.$$

(H4): *There exists $M > 0$ such that*

$$\frac{M}{\left(\left|\frac{b}{a+b}\right|+1\right)\left[\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(M)+m\psi^*(M)\right]+\left|\frac{c}{a+b}\right|} > 1, \tag{3.5}$$

where $p^* = \sup_{t\in J}\{p(t)\}$.

(H5): *There exists $l \in L^1(J, \mathbb{R}^+)$ such that*

$$H_d(F(t,u), F(t,\bar{u})) \le l(t)|u-\bar{u}| \text{ for all } u, \bar{u} \in \mathbb{R}.$$

Then the BVP (1.1)–(1.3) has at least one solution on J.

Proof. We transform the problem (1.1)–(1.3) into a fixed point problem by defining the multivalued operator $N : PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ by

$$N(y) = \{h \in PC(J,\mathbb{R}) : y(t)$$

where

$$y(t) = \begin{cases} \left(\frac{-1}{a+b}\right)\left[b\sum_{i=1}^{m} I_i(y(t_i^-))\right. \\ +\frac{b}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ \left. +\ \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s} - c\right] \\ +\sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ +\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s}, \quad \text{if } \ t\in[t_k,t_{k+1}],\ v\in S_{F,y}. \end{cases}$$

In view of Lemma 3.2, we see that the fixed points of N are solutions to (1.1)–(1.3). We shall show that N satisfies the assumptions of the nonlinear Leray-Schauder alternative. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in PC(J,E)$.* Let h_1, h_2 belong to $N(y)$; then there exist v_1, $v_2 \in S_{F,y}$ such that, for each $t \in J$ and $i = 1, 2$,

$$\begin{aligned} h_i(t) &= \left(\frac{-1}{a+b}\right)\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1} v_i(s)\frac{ds}{s}\right. \\ &\quad \left. + \ \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1} v_i(s)\frac{ds}{s} - c\right] + \sum_{i=1}^{k} I_i(y(t_i^-)) \\ &\quad + \frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1} v(s)\frac{ds}{s} + \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1} v_i(s)\frac{ds}{s}. \end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then, for $t \in J$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m} I_k(y(t_k^-)) + \frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}(\lambda v_1(s)\right.$$
$$\left. + (1-\lambda)v_2(s))\frac{ds}{s} + \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}(\lambda v_1(s) + (1-\lambda)v_2(s))\frac{ds}{s} - c\right]$$
$$+ \sum_{k=1}^{m} I_k(y(t_k^-)) + \frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}(\lambda v_1(s) + (1-\lambda)v_2(s))\frac{ds}{s}$$
$$+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}(\lambda v_1(s) + (1-\lambda)v_2(s)\frac{ds}{s}.$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\lambda h_1 + (1-\lambda)h_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.* Let $B_{\mu_*} = \{y \in PC(J,\mathbb{R}) : \|y\|_\infty \leq \mu_*\}$ be a bounded set in $PC(J,\mathbb{R})$ and $y \in B_{\mu_*}$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = \left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m} I_k(y(t_k^-)) + \frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v(s)\frac{ds}{s}\right.$$
$$\left. + \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}v(s)\frac{ds}{s} - c\right]$$
$$+ \sum_{k=1}^{m} I_k(y(t_k^-)) + \frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v(s)\frac{ds}{s}$$
$$+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}v(s)\frac{ds}{s}.$$

By (H2)–(H3), we have for each $t \in J$,

$$|h(t)| \leq \left|\frac{b}{a+b}\right|\left[\sum_{i=1}^{m}|I_k(y(t_i^-))| + \frac{1}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|v(s)|\frac{ds}{s}\right.$$
$$\left. + \frac{1}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}|v(s)|\frac{ds}{s}\right] + \left|\frac{c}{a+b}\right|$$
$$+ \frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|v(s)|\frac{ds}{s}$$
$$+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|f(s,y(s))|\frac{ds}{s} + \sum_{i=1}^{k}|I_i(y(t_i^-)|$$
$$\leq \left|\frac{b}{a+b}\right|\left[m\psi^*(\|y\|_\infty) + \frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\|y\|_\infty)\right] + \left|\frac{c}{a+b}\right|$$

$$+\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\|y\|_\infty)+m\psi^*(\|y\|_\infty)$$
$$\leq\left(\left|\frac{b}{a+b}\right|+1\right)\left[\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\mu_*)+m\psi^*(\mu_*)\right]+\left|\frac{c}{a+b}\right|.$$

Thus,

$$\|h\|_\infty\leq\left(\left|\frac{b}{a+b}\right|+1\right)\left[\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\mu_*)+m\psi^*(\mu_*)\right]+\left|\frac{c}{a+b}\right|:=\ell,$$

so $N(B_{\mu_*})$ is bounded.

Step 3: *N maps bounded sets into equicontinuous sets in* $PC(J,\mathbb{R})$. Let λ_1, $\lambda_2\in J$, $\lambda_1<\lambda_2$, and let B_{μ^*} be a bounded set in $PC(J,\mathbb{R})$ as in Step 2. Let $y\in B_{\mu^*}$ and $h\in N(y)$. Then,

$$\begin{aligned}
&|h(\lambda_2)-h(\lambda_1)|\\
&\quad=\frac{1}{\Gamma(r)}\int_1^{\lambda_1}\left[\left(\log\frac{\lambda_2}{s}\right)^{r-1}-\left(\log\frac{\lambda_1}{s}\right)^{r-1}\right]|v(s)|\frac{ds}{s}\\
&\quad\quad+\sum_{1<t_k<\lambda_2-\lambda_1}|I_k(y(t_k^-))|+\frac{1}{\Gamma(r)}\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{r-1}|v(s)|\frac{ds}{s}\\
&\quad\leq\frac{p^*\psi(\mu_*)}{\Gamma(r+1)}[2\log(\lambda_2-\lambda_1)]^r+(\log\lambda_2)^r-(\log\lambda_1)^r]+(\lambda_2-\lambda_1)\psi^*(\mu_*).
\end{aligned}$$

As $\lambda_1\to\lambda_2$, the right hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzelà-Ascoli theorem, we conclude that $N:PC(J,\mathbb{R})\to\mathcal{P}(PC(J,\mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.* Let $y_n\to y_*$, $h_n\in N(y_n)$ and $h_n\to h_*$; we need to show that $h_*\in N(y_*)$. Now $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$\begin{aligned}
h_n(t)&=\left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^m I_k(y_n(t_k^-))+\frac{b}{\Gamma(r)}\sum_{k=1}^m\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right.\\
&\quad\left.+\frac{b}{\Gamma(r)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}-c\right]+\sum_{k=1}^m I_k(y_n(t_k^-))\\
&\quad+\frac{1}{\Gamma(r)}\sum_{k=1}^m\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v_n(s)\frac{ds}{s}+\frac{1}{\Gamma(r)}\int_{t_k}^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}.
\end{aligned}$$

We must show that there exists $v_*\in S_{F,y_*}$ such that, for each $t\in J$,

$$\begin{aligned}
h_*(t)&=\left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^m I_k(y^*(t_k^-))+\frac{b}{\Gamma(r)}\sum_{k=1}^m\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v_*(s)\frac{ds}{s}\right.\\
&\quad\left.+\frac{b}{\Gamma(r)}\int_{t_m}^T\left(\log\frac{T}{s}\right)^{r-1}v_*(s)\frac{ds}{s}-c\right]+\sum_{k=1}^m I_k(y^*(t_k^-))
\end{aligned}$$

$$+\frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v_*(s)\frac{ds}{s}+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}v_*(s)\frac{ds}{s}.$$

Since $F(t,\cdot)$ is upper semi-continuous, for every $\epsilon>0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n\geq n_0$, we have

$$v_n(t)\in F(t,y_n(t))\subset F(t,y_*(t))+\epsilon B(0,1) \text{ a.e. } t\in J.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot)\to v_*(\cdot) \text{ as } m\to\infty,$$

and

$$v_*(t)\in F(t,y_*(t)),\ \text{ a.e. } t\in J.$$

For every $w\in F(t,y_*(t))$, we have

$$|v_{n_m}(t)-v_*(t)|\leq|v_{n_m}(t)-w|+|w-v_*(t)|.$$

Then,

$$|v_{n_m}(t)-v_*(t)|\leq d(v_{n_m}(t),F(t,y_*(t)).$$

We can obtain an analogous relation by interchanging the roles of v_{n_m} and v_*. It then follows that

$$|v_{n_m}(t)-v_*(t)|\leq H_d(F(t,y_n(t)),F(t,y_*(t)))\leq l(t)\|y_n-y_*\|_\infty.$$

Therefore,

$$\begin{aligned}
|h_{n_m}(t)-h_*(t)|\leq{}&\left|\frac{b}{a+b}\right|\Bigg[\sum_{i=1}^{m}|I_k(y_{n_m}(t_k^-))-I_k(y_*(t_k^-))|\\
&+\frac{1}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|v_{n_m}(s)-v_*(s)|\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}|v_{n_m}(s)-v_*(s)|\frac{ds}{s}\Bigg]\\
&+\frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|v_{n_m}(s)-v_*(s)|\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|v_{n_m}(s)-v_*(s)|\frac{ds}{s}\\
&+\sum_{i=1}^{k}|I_k(y_{n_m}(t_k^-))-I_ky_*(t_k^-))|\\
\leq{}&\left|\frac{b}{a+b}\right|\Bigg[\sum_{i=1}^{m}|I_k(y_{n_m}(t_k^-))-I_k(y_*(t_k^-))|\\
&+\frac{1}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|l(s)|\|y_{n_m}-y_*\|\frac{ds}{s}
\end{aligned}$$

$$+\frac{1}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}|l(s)|\|y_{n_m}-y_*\|\frac{ds}{s}\Bigg]$$
$$+\frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|l(s)|\|y_{n_m}-y_*\|\frac{ds}{s}$$
$$+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|l(s)|\|y_{n_m}-y_*\|\frac{ds}{s}]$$
$$+\sum_{i=1}^{k}|I_k(y_{n_m}(t_k^-))-I_k(y_*(t_k^-))|.$$

As $m\to\infty$,

$$\begin{aligned}
&\|h_{n_m}(t)-h_*(t)\|_\infty\\
&\le\left|\frac{b}{a+b}\right|\Bigg[\sum_{i=1}^{m}|I_k(y_{nm}(t_k^-))-I_k(y_*(t_k^-))|\\
&\quad+\frac{1}{\Gamma(r)}\|l\|_{L^1}\|y_{nm}-y_*\|\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(r)}\|l\|_{L^1}\|y_{nm}-y_*\|\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}\frac{ds}{s}\Bigg]\\
&\quad+\frac{1}{\Gamma(r)}\|l\|_{L^1}\|y_{nm}-y_*\|\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(r)}\|l\|_{L^1}\|y_{nm}-y_*\|\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s}\\
&\quad+\sum_{i=1}^{k}|I_k(y_{nm}(t_k^-))-I_k(y_*(t_k^-))|\to 0,
\end{aligned}$$

which is what we needed to show.

Step 5: *A priori bounds on solutions.* Let $y\in PC(J,\mathbb{R})$ be such that $y\in\lambda N(y)$ with $\lambda\in(0,1]$. Then, there exists $v\in S_{F,y}$ such that, for each $t\in J$,

$$\begin{aligned}
|y(t)|&\le\left|\frac{b}{a+b}\right|[m\psi^*(\|y\|_\infty)+\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\|y\|_\infty)]+\left|\frac{c}{a+b}\right|\\
&\quad+\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\|y\|_\infty)+m\psi^*(\|y\|_\infty)\\
&\le\left(\left|\frac{b}{a+b}\right|+1\right)\left[\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(\mu_*)+m\psi^*(\mu_*)\right]+\left|\frac{c}{a+b}\right|,
\end{aligned}$$

Thus,

$$\frac{||y||_\infty}{\left(\left|\frac{b}{a+b}\right|+1\right)[\frac{(m+1)}{\Gamma(r+1)}(\log T)^r p^*\psi(||y||_\infty)+m\psi^*(||y||_\infty)]+\left|\frac{c}{a+b}\right|}\le 1.$$

By condition (H5), we have that $\|y\|_\infty \neq M$. Let $U = \{y \in PC(J, \mathbb{R}) : \|y\|_\infty < M\}$. The operator $N : \overline{U} \to \mathcal{P}(PC(J, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1]$. As a consequence of the nonlinear Leray-Schauder alternative, we conclude that N has a fixed point $y \in \overline{U}$ that is a solution of the problem (1.1)–(1.3). This completes the proof of the theorem. □

4. THE NONCONVEX CASE

In this section we consider problem (1.1)–(1.3) with a nonconvex valued right hand side. Our result is based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [17] (see Lemma 2.3 above).

Theorem 4.1. *Assume that (H5) and the following conditions hold:*

- **(H6):** $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable, convex valued, and integrably bounded for each* $u \in \mathbb{R}$.
- **(H7):** *There exists a constant* $l^* > 0$ *such that*

$$|I_k(u) - I_k(\overline{u})| \leq l^*|u - \overline{u}| \quad \text{for} \quad u, \ \overline{u} \in \mathbb{R} \quad \text{and} \quad k = 1, 2, \ldots, m.$$

If

$$\left(\left|\frac{b}{a+b}\right| + 1\right)\left[ml^* + \frac{(m+1)\|l\|_{L^1}(\log T)^r}{\Gamma(r+1)}\right] + ml^* < 1, \tag{4.1}$$

then the BVP (1.1)–(1.3) has at least one solution on J.

Proof. We shall show that N satisfies the assumptions of Lemma 2.3; we do this in two steps.

Step 1: $N(y) \in \mathcal{P}_{cl}(PC(J, \mathbb{R}))$ *for each* $y \in PC(J, \mathbb{R})$. Let $\{y_n\}_{n \geq 1} \subset N(y)$ be such that $y_n \to \overline{y}$ in $PC(J, \mathbb{R})$. Then, $\overline{y} \in PC(J, \mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y_n(t) = {} & \left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m} I_k(y_n(t_k^-)) + \frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right. \\ & \left. + \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s} - c\right] \\ & + \sum_{k=1}^{m} I_k(y_n(t_k^-)) + \frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v_n(s)\frac{ds}{s} \\ & + \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s}. \end{aligned}$$

From (H5) and the fact that F has compact values, we may pass to a subsequence if necessary to obtain that v_n converges weakly to v in $L^1_w(J, \mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem [19, Lemma 2.66] implies that $\{v_n\}$ converges strongly to v, and hence $v \in S_{F,y}$. Then for each $t \in J$,

$$\begin{aligned} y_n(t) \to \overline{y}(t) = &\left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m} I_k(\overline{y}(t_k^-)) + \frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v(s)\frac{ds}{s}\right. \\ &\left. + \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s} - c\right] \\ &+ \sum_{k=1}^{m} I_k(\overline{y}(t_k^-)) + \frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s}. \end{aligned}$$

Hence, $\overline{y} \in N(y)$.

Step 2: *There exists* $\gamma < 1$ *such that*

$$H_d(N(y), N(\overline{y})) \le \gamma\|y - \bar{y}\|_\infty \quad \textit{for} \;\; y, \overline{y} \in PC(J, \mathbb{R}).$$

Let $y, \overline{y} \in PC(J, \mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1 \in F(t, y(t))$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) = &\left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m} I_k(y(t_k^-)) + \frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v_1(s)\frac{ds}{s}\right. \\ &\left. + \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1} v_1(s)\frac{ds}{s} - c\right] \\ &+ \sum_{k=1}^{m} I_k(y(t_k^-)) + \frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v_1(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1} v_1(s)\frac{ds}{s}. \end{aligned}$$

From (H5) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \le l(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in F(t, \overline{y}(t))$ such that

$$|v_1(t) - w| \le l(t)|y(t) - \overline{y}(t)|, \; t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable, there exists a measurable selection $v_2(t)$ for V (see [12]). Hence, $v_2(t) \in F(t, \overline{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \le l(t)|y(t) - \overline{y}(t)|, \; t \in J.$$

For $t \in J$, let

$$h_2(t) = \left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m} I_k(\overline{y}(t_k^-)) + \frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v_2(s)\frac{ds}{s} + \frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1} v_2(s)\frac{ds}{s} - c\right] + \sum_{k=1}^{m} I_k(\overline{y}(t_k^-)) + \frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1} v_2(s)\frac{ds}{s} + \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1} v_2(s)\frac{ds}{s}.$$

Then for each $t \in J$,

$$\begin{aligned}
|h_1(t) - h_2(t)| \le{}& \left|\frac{b}{a+b}\right|\left[\sum_{i=1}^{m}|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| + \frac{1}{\Gamma(r)}\right.\\
&+ \sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|v_1(s) - v_2(s)|\frac{ds}{s}\\
&\left.+ \frac{1}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}|v_1(s) - v_2(s)|\frac{ds}{s}\right]\\
&+ \frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|v_1(s) - v_2(s)|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|v_1(s) - v_2(s)|\frac{ds}{s} + \sum_{i=1}^{m}|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))|\\
\le{}& \left|\frac{b}{a+b}\right|[\sum_{i=1}^{m} l^*|y(s) - \overline{y}(s)| + \frac{1}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|l(s)||y(s) - \overline{y}(s)|\frac{ds}{s}\\
&\left.+ \frac{1}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}|l(s)||y(s) - \overline{y}(s)|\frac{ds}{s}\right]\\
&+ \frac{1}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}|l(s)||y(s) - \overline{y}(s)|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}|l(s)||y(s) - \overline{y}(s)|\frac{ds}{s} + \sum_{i=1}^{m} l^*|y(s) - \overline{y}(s)|\\
\le{}& \left|\frac{b}{a+b}\right|\left[ml^*\|y - \overline{y}\|_\infty + \frac{\|l\|_{L^1}\|y - \overline{y}\|_\infty}{\Gamma(r)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s}\right.\\
&\left.+ \frac{\|l\|_{L^1}\|y - \overline{y}\|_\infty}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}\frac{ds}{s}\right]\\
&+ \frac{\|l\|_{L^1}\|y - \overline{y}\|_\infty}{\Gamma(r)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}\frac{ds}{s}
\end{aligned}$$

$$+\frac{\|l\|_{L^1}\|y-\overline{y}\|_\infty}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s}+ml^*\|y-\overline{y}\|_\infty$$
$$\leq\left[\left(\left|\frac{b}{a+b}\right|+1\right)\left[ml^*+\frac{(m+1)\|l\|_{L^1}(\log T)^r}{\Gamma(r+1)}\right]+ml^*\right]\|y-\overline{y}\|_\infty.$$

Therefore,

$$\|h_1-h_2\|_\infty\leq\left[\left(\left|\frac{b}{a+b}\right|+1\right)\left[ml^*+\frac{(m+1)l^*\|l\|_{L^1}(\log T)^r}{\Gamma(r+1)}\right]+ml^*\right]\|y-\overline{y}\|_\infty.$$

From an analogous relation obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N(y),N(\bar{y}))\leq\left[\left(\left|\frac{b}{a+b}\right|+1\right)\left[ml^*+\frac{(m+1)\|l\|_{L^1}(\log T)^r}{\Gamma(r+1)}\right]+ml^*\right]\|y-\overline{y}\|_\infty.$$

By (4.1), N is a contraction and so by Lemma 2.3, N has a fixed point y that is solution to (1.1)–(1.3). This completes the proof of the theorem. □

5. TOPOLOGICAL STRUCTURE OF THE SOLUTION SET

In this section we present a result on the topological structure of the set of solutions to problem (1.1)–(1.3).

Theorem 5.1. *In addition to conditions (H1), (H4), and (H5), assume that:*

(H8): *There exists $f_1\in C(J,\mathbb{R})$ such that*

$$\|F(t,u)\|_{\mathcal{P}}\leq f_1(t),\ \text{for all } t\in J \text{ and } u\in\mathbb{R};$$

(H9): *There exists $\zeta\in\mathbb{R}_+^*$ such that*

$$|I_k(u)|\leq\zeta \text{ for } u\in\mathbb{R}.$$

Then the set of solutions of problem (1.1)–(1.3) is nonempty and compact in $PC(J,\mathbb{R})$.

Proof. Let

$$S=\{y\in PC(J,\mathbb{R})\mid y\text{ is a solution of (1.1)–(1.3)}\}.$$

First notice that conditions (H8) and (H9) imply that (H2) and (H3) hold. Then from Theorem 3.3, $S\neq\emptyset$. Now let $\{y_n\}_{n\in\mathbb{N}}\subset S$. Then there exists $v_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$y_n(t)=\left(\frac{-1}{a+b}\right)\left[b\sum_{k=1}^{m}I_k(y_n(t_k^-))+\frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right.$$
$$\left.+\frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}-c\right]$$
$$+\sum_{k=1}^{m}I_k(y(t_k^-))+\frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v_n(s)\frac{ds}{s}$$

$$+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}.$$

From (H1), (H8), and (H9), it follows that there exists $M_1>0$ such that $\|y_n\|\leq M_1$ for all $n\geq 1$. It is also the case that $\{y_n\}_{n\geq 1}$ is equicontinuous in $PC(J,\mathbb{R})$, and so there exists a subsequence, relabeled $\{y_n\}$, such that $\{y_n\}$ converges to y in $PC(J,\mathbb{R})$.

Applying (H1), for each $\varepsilon>0$, there exists $n_0(\varepsilon)\in\mathbb{N}$ such that $v_n(t)\in F(t,y_n(t))\subset F(t,y(t))+\varepsilon B(0,1)$, for each $n\geq n_0$ and for a.e. $t\in J$. But since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $\{v_{n_m}(\cdot)\}$ such that $v_{n_m}\to v(\cdot)$ as $m\to\infty$ and $v(t)\in F(t,y(t))$ for a.e. $t\in J$. We also have $|v_{n_m}(t)|\leq f_1(t)$ a.e. $t\in J$. By the Lebesgue dominated convergence theorem, we obtain $v\in L^1(J,\mathbb{R})$, and so $v\in S_{F,y}$. Therefore,

$$\begin{aligned}y(t)=&\left(\frac{-1}{a+b}\right)\Bigg[b\sum_{k=1}^{m}I_k(y(t_k^-))+\frac{b}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v(s)\frac{ds}{s}\\&+\frac{b}{\Gamma(r)}\int_{t_m}^{T}\left(\log\frac{T}{s}\right)^{r-1}v(s)\frac{ds}{s}-c\Bigg]\\&+\sum_{k=1}^{m}I_k(y(t_k^-))+\frac{1}{\Gamma(r)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}v(s)\frac{ds}{s}\\&+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}v(s)\frac{ds}{s}.\end{aligned}$$

and so S is compact. □

6. AN EXAMPLE

Here we apply Theorem 3.3 to the fractional differential inclusion

(6.1)

$${}_H^cD_{\frac{1}{2}}^r y(t)\in F(t,y(t)),\ \text{for a.e. } t\in J=[1,e],\ t\neq\frac{3}{2},\ t\neq t_k,\ k=1,...,m,\ 0<r\leq 1,$$

with

$$\Delta y|_{t=\frac{3}{2}}=\frac{1}{3+|y(\frac{3}{2}^-)|},\tag{6.2}$$

$$y(1)+y(e)=0,\tag{6.3}$$

where ${}_H^cD^r$ is the Caputo-Hadamard fractional derivative, and $F:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map satisfying

$$F(t,y)=\{v\in\mathbb{R}: f_1(t,y)\leq v\leq f_2(t,y)\},$$

and where $f_1, f_2 : [1,e] \times \mathbb{R} \mapsto \mathbb{R}$ are such that, for each $t \in [1,e]$, $f_1(t,\cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t,y) > \mu_1\}$ is open for each $\mu_1 \in \mathbb{R}$), and such that, for each $t \in [1,e]$, $f_2(t,\cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t,y) < \mu_2\}$ is open for each $\mu_2 \in \mathbb{R}$). Assume that there exist $p \in C([J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \mapsto (0,\infty)$ such that

$$\max\{|f_1(t,y)|, |f_2(t,y)|\} \leq p(t)\psi(|y|) \text{ for } t \in J \text{ and } y \in \mathbb{R},$$

and where $a = b = 1$, $c = 0$, $T = e$, and

$$I_k(y) = \frac{1}{3+|y|},\ k = 1, ..., m.$$

It is clear that F is compact and convex valued, and is upper semi-continuous, so (H1)–(H3) hold.

Take $\psi^* : [0,\infty) \mapsto (0,\infty)$ to be $\psi^*(t) = \frac{1}{3}t$. Then we have $m = 2$, and if we assume there exists a number $M > 0$ such that

$$\frac{M}{\frac{3}{2}\left(\frac{3}{\Gamma(r+1)}p^*\psi(M) + \frac{2}{3}M\right)} > 1,$$

where $p^* = \sup_{t \in J}\{p(t)\}$, then (H4) and (H5) hold, and so all the conditions of Theorem 3.3 are satisfied. Therefore, problem (6.1)–(6.3) has at least one solution y on $[1,e]$.

We also see that (H8) holds, and (H9) holds with $\zeta = 1/3$. Thus, by Theorem 5.1, the set of solutions to (6.1)–(6.3) is nonempty and compact.

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