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ASYMPTOTIC STABILITY OF GENERALIZED HOMOGENEOUS NONAUTONOMOUS DYNAMICAL SYSTEMS

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ABSTRACT. The goal of the paper is to study the relationship between asymptotic stability and exponential stability of the solutions of generalized infinite-dimensional homogeneous nonautonomous dynamical systems. This problem is studied and solved within the framework of general non-autonomous (cocycle) dynamical system. The application of our general results for functional differential and semi-linear parabolic equations is given.

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1. INTRODUCTION

This paper is dedicated to the study of the problem of asymptotic stability of a class of infinite-dimensional nonautonomous dynamical systems with some property of symmetry. Namely, we study this problem for so-called generalized homogeneous nonautonomous dynamical systems, that is, a class of nonautonomous dynamical systems invariant with respect to a group of translations called dilations. We establish our main results in the framework of general nonautonomous (cocycle) dynamical systems.

The motive for writing of this article was the works of Bacciotti A. and Rosier L. [2], Polyakov A. [27], Zubov V. I. [33] (see also the bibliography therein) and the works of Cheban D. N. [11]-[13], [14, Ch.II] and [18]. We prove the equivalence of the uniform asymptotic stability and exponential stability for this class of nonautonomous dynamical systems. If the phase space Y of the driving system $(Y,\mathbb{T},\sigma)$ for the cocycle dynamical systems $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is compact, then we prove that the asymptotic stability and uniform asymptotic stability are equivalent. If additionally the driving system $(Y,\mathbb{S},\sigma)$ with compact phase space Y is minimal, then for the asymptotic stability, the uniform stability and the existence of a positive number a and an element $y_0 \in Y$ such that $\lim\limits_{t\to+\infty}|\varphi(t,u,y_0)| = 0$ for any $u \in B[0,a]$ are sufficient. We apply these results for functional-differential and semi-linear parabolic equations.

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The paper is organized as follows. In the second Section, we collect some known notions and facts from dynamical systems that we use in this paper. Namely, we present the construction of shift dynamical systems, definitions of Poisson stable motions and some facts about compact global attractors of dynamical systems. In the third Section we establish the relation between uniformly asymptotic stability and exponential stability for general nonautonomous (cocycle) dynamical systems. The fourth Section is dedicated to the study the homogeneous dissipative dynamical systems. In the fifth Section we establish the relation between uniform stability, attraction and uniform exponential stability for the infinite-dimensional nonautonomous dynamical systems with driving system $(Y,\mathbb{S},\sigma)$ of the phase space Y is compact. The sixth Section is dedicated to the study the relation between asymptotic stability and exponential stability for the nonautonomous dynamical systems with the compact and minimal phase space of their driving system. Finally, in the seventh Section we apply our general results, obtained in Sections 3-6 to functional-differential and semi-linear parabolic equations.

2. PRELIMINARIES

Throughout the paper, we assume that X (respectively, Y) is a metric space with the metric ρ_X (respectively, ρ_Y). For simplicity we will use the same notation ρ to denote the metrics on them, which we think would not lead to confusion. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ := \{s \in \mathbb{S} |\ s \geq 0\}$ and $\mathbb{T} \subseteq \mathbb{S}$ be a sub-semigroup of $\mathbb{S}$ such that $\mathbb{S}_+ \subseteq \mathbb{T}$.

Let $(X,\mathbb{T},\pi)$ be a dynamical system on X, $M \subseteq X$, $\mathfrak{M}$ be some family of subsets from X.

Denote by

$$\omega(M) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \pi(t, M)}.$$

Lemma 2.1. [16, Ch.I] *Let $B \subseteq X$ then the following conditions are equivalent:*

1. *for every $\{x_k\} \subseteq B$ and $t_k \to +\infty$ sequence $\{\pi(t_k, x_k)\}$ is relatively compact;*
2. (a) *$\omega(B)$ is not empty and is compact;*
   (b) *$\omega(B)$ is invariant and the equality below takes place*
   $$\lim_{t\to +\infty} \sup_{x\in B} \rho(\pi(t,x), \omega(B)) = 0,$$
   where $\rho(x,M) := \inf_{m\in M} \rho(x,m)$ is a compact subset of X;
3. *there exists a non empty compact subset $K \subseteq X$ such that*
   $$\lim_{t\to +\infty} \sup_{x\in B} \rho(\pi(t,x), K) = 0.$$

Theorem 2.2. *Let $M \subseteq X$ be a nonempty compact positively invariant and asymptotically stable subset of a dynamical system $(X,\mathbb{T},\pi)$, then the following affirmations hold:*

1. *$\omega(M) \subseteq M$;*
2. *the set $\omega(M)$ is invariant;*
3. *$\omega(M) = \bigcap_{t\geq 0} \pi(t,M)$;*
4. *$\omega(M)$ is a maximal compact invariant set in $W^s(M)$.*

Definition 2.3. A dynamical system $(X,\mathbb{T},\pi)$ is said to be $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi(t,M) \subseteq B(K,\varepsilon)$ for any $t \geq L(\varepsilon, M)$, where K is a subset from X depending only on $\mathfrak{M}$. In this case K we will call an attractor for $\mathfrak{M}$.

The most important for the applications are the cases when K is a bounded or compact set and $\mathfrak{M} = \{\{x\} \mid x \in X\}$ or $\mathfrak{M} = C(X)$, or $\mathfrak{M} = \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, where

1. $C(X)$ is the family of all compact subsets of X;
2. $B(x_0, \delta) := \{x \in X |\ \rho(x, x_0) < \delta\}$.

Definition 2.4. A dynamical system $(X, \mathbb{T}, \pi)$ is called:

– pointwise dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{t\to+\infty} \rho(\pi(t, x), K) = 0; \tag{2.1}$$

– compactly dissipative if the equality (2.1) takes place uniformly with respect to (w.r.t.) x on the compact subsets from X;
– locally dissipative if for any point $p \in X$ there exists $\delta_p > 0$ such that the equality (2.1) takes place uniformly w.r.t. $x \in B(p, \delta_p)$.

Let $(X, \mathbb{T}, \pi)$ be a compactly dissipative dynamical system and K be a nonempty compact set, that is, an attractor for compact subsets of X. Then for every compact subset $M \subseteq X$ the equality

$$\lim_{t\to+\infty} \sup_{x\in M} \rho(\pi(t, x), K) = 0$$

holds. It is possible to show [16, Ch.I] that the set J defined by the equality

$$J := \omega(K) \tag{2.2}$$

does not depend on the choice of the set K attracting any compact subset from X.

Definition 2.5. The set $W^s(\Lambda)$ (respectively, $W^u(\Lambda)$), defined by the equality

$$W^s(\Lambda) := \{x \in X |\ \lim_{t\to+\infty} \rho(\pi(t, x), \Lambda) = 0\}$$

is called a stable manifold of set $\Lambda \subseteq X$.

Definition 2.6. A set M is called:

- orbital stable, if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, M) < \delta$ implies $\rho(xt, M) < \varepsilon$ for any $t \geq 0$;
- attracting, if there exists $\gamma > 0$ such that $B(M, \gamma) \subset W^s(M)$, where $B(M, \gamma) := \{x \in X |\ \rho(x, M) < \gamma\}$;
- asymptotic stable, if it is orbital stable and attracting;
- globally asymptotically stable, if it is asymptotically stable and $W^s(M) = X$;
- uniformly attracting, if there exists $\gamma > 0$ such that

$$\lim_{t\to+\infty} \sup_{x\in B[M,\gamma]} \rho(\pi(t, x), M) = 0,$$

where $B[M, \gamma] := \{x \in X |\ \rho(x, M) \leq \gamma\}$.

Definition 2.7. The set J defined by the equality (2.2) is called [16, Ch.I] the center of Levinson of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Theorem 2.8. [16, Ch.I] *Let $(X, \mathbb{T}, \pi)$ be a compact dissipative system and let J be its Levinson center. Then:*

1. *J is a compact and invariant set;*
2. *J is orbitally stable;*
3. *J is the attractor of the family of all compact subsets of X;*

4. *J is the maximal compact invariant set of $(X, \mathbb{T}, \pi)$.*

Definition 2.9. The dynamical system $(X, \mathbb{T}, \pi)$ is said to be:

- locally completely continuous (or locally compact) if for every point $p \in X$ there exist $\delta = \delta(p) > 0$ and $l = l(p) > 0$ such that $\pi(l, B(p, \delta))$ is precompact;
- weakly dissipative if there exists a nonempty compact $K \subseteq X$ such that for every $\varepsilon > 0$ and $x \in X$ there is $\tau = \tau(\varepsilon, x) > 0$ for which $\pi(\tau, x) \in B(K, \varepsilon)$. In this case we will call K a weak attractor.

Theorem 2.10. [16, Ch.I] *For the locally compact dynamical systems the weak, point, compact and local dissipativity are equivalent.*

Definition 2.11. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be asymptotically compact if for any bounded positively invariant set $M \subset X$ there exists a compact set $K = K(M)$ such that

$$\lim_{t \to +\infty} \beta(\pi(t, M), K) = 0,$$

where $\beta(A, B) := \sup_{a \in A} \rho(a, B)$.

Theorem 2.12. [16, Ch.I] *If the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and asymptotically compact, then it is locally dissipative.*

Theorem 2.13. [16, Ch.I] *For the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$ to be locally dissipative, it is necessary and sufficient that its Levinson center J would be uniformly attracting, i.e., there exists a positive number γ such that*

$$\lim_{t \to +\infty} \beta(\pi(t, B[J, \gamma]), J) = 0.$$

Definition 2.14. Recall [25] that a bundle is a triplet $\xi := (X, h, Y)$, where $h : X \mapsto Y$ is a map. The space Y is called the base space, the space X is called the total space, and the map h is called the projection of bundle. For each $y \in Y$, the space $X_y := h^{-1}(y)$ is called the fibre of bundle over $y \in Y$.

Example 2.15. Let $X := W \times Y$, the triplet $\xi = (X, h, Y)$, where $h := pr_2$ is the projection on the second factor, is a bundle which is called the product (or trivial) bundle over Y with fibre W.

Definition 2.16. A space W is the fibre of a bundle $\xi = (X, h, Y)$ provided every fibre $h^{-1}(y)$ for $y \in Y$ is homeomorphic to W. A bundle $\xi = (X, h, Y)$ is trivial with fibre W provided $\xi = (X, h, Y)$ is isomorphic to the product bundle $(W \times Y, h', Y)$, where $h' = pr_2$.

Definition 2.17. A bundle ξ over Y is locally trivial with fibre W provided ξ is locally isomorphic [25] with the product bundle $(W \times Y, pr_2, Y)$.

Remark 2.18. *In this paper we will consider only the locally trivial bundles.*

Definition 2.19. Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ be two dynamical systems. A mapping $h : X \to Y$ is called a *a homomorphism* (respectively, *isomorphism*) of dynamical system $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (respectively, homeomorphic) and $h(\pi(x, t)) = \sigma(h(x), t)$ (for any $t \in \mathbb{T}_1$ and $x \in X$). In this case a dynamical system $(X, \mathbb{T}_1, \pi)$ is called an *extension of dynamical system* $(Y, \mathbb{T}_2, \sigma)$ by a homomorphism h, but a dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called a *factor of dynamical system* $(X, \mathbb{T}_1, \pi)$ by a homomorphism h. The dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called also a *base (driving system) of extension* $(X, \mathbb{T}_1, \pi)$.

Definition 2.20. Let (X, h, Y) be a bundle. The triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$ is called a *nonautonomous dynamical system* (shortly NDS).

Remark 2.21. *In the works of I. U. Bronshtein and his collaborators (see, for example,* [3]-[9] *and* [22] *) an* extension *is called a triplet* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, h), h\rangle$*, i.e., the object which we name here a* nonautonomous dynamical system*.*

Definition 2.22. A nonautonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be convergent if the following conditions are valid:

1. the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative;
2. the set $J_X \bigcap X_y$ contains no more than one point for any $y \in J_Y$, where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and J_X (respectively, J_Y) is the Levinson's center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$).

Theorem 2.23. [16, Ch.II] *Let* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ *be a non-autonomous dynamical system,* M *be a nonempty compact and positively invariant set. Suppose that the following conditions are fulfilled:*

1. $h(M) = Y$*;*
2. $M \bigcap X_y$ *contains a single point for all* $y \in Y$*;*
3. M *is globally asymptotically stable, i.e., for any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $\rho(x, p) < \delta$ $(x \in X_y, p \in M_y := M \bigcap X_y)$ *implies* $\rho(xt, pt) < \varepsilon$ *for any* $t \geq 0$ *and* $\lim_{t\to+\infty} \rho(xt, M_{h(x)t}) = 0$ *for all* $x \in X$*.*

Then the non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ *is convergent.*

Lemma 2.24. *Let* $F : X \to Y$ *be a continuous mapping from the compact metric space* X *onto metric space* Y*. If* $F^{-1}(y) := \{x \in X |\ F(x) = y\}$ *consists of a single point for any* $y \in Y$*, then the mapping* F^{-1} *from* Y *on* X *is continuous.*

Proof. For any $y \in Y$ we denote by $x_y := F^{-1}(y)$. Let $y_0 \in Y$and $y_k \to y_0$ as $k \to \infty$. We will show that

$$\lim_{k\to\infty} x_{y_k} = x_{y_0}. \tag{2.3}$$

If we assume that (2.3) is not true, then there exist a positive number ε_0 and a subsequence $\{y_{k_m}\}$ of $\{y_k\}$ such that

$$\rho(x_{y_{k_m}}, x_{y_0}) \geq \varepsilon_0 \tag{2.4}$$

for any $m \in \mathbb{N}$. Since the metric space X is compact, then without loss of generality we can suppose that the sequence $\{x_{y_{k_m}}\}$ converges. Denote its limit by $\bar{x}$. Passing to the limit in (2.4) as $m \to \infty$ we obtain

$$\rho(\bar{x}, x_{y_0}) \geq \varepsilon_0. \tag{2.5}$$

On the other hand we have $x_{y_{k_m}} = F^{-1}(y_{k_m})$ and, consequently,

$$y_{k_m} = F(x_{k_m}) \tag{2.6}$$

for any $m \in \mathbb{N}$. Passing to the limit in (2.6) as $m \to \infty$ we obtain $y_0 = F(\bar{x})$ and, consequently,

$$\bar{x} = x_{y_0}. \tag{2.7}$$

The relations (2.5) and (2.7) are contradictory. The obtained contradiction proves our statement. To finish the proof of Lemma it suffices take into account that the point y_0 is an arbitrary point of Y. Lemma is proved. □

Lemma 2.25. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system, M be a nonempty compact and positively invariant set. Suppose that the following conditions are fulfilled:*

1. *$h(M)=Y$;*
2. *$M\bigcap X_y$ contains a single point for any $y\in Y$;*
3. *M is uniformly stable, i.e., for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\rho(x,p)<\delta$ ($x\in X_y$, $p\in M_y:=M\bigcap X_y$) implies $\rho(xt,pt)<\varepsilon$ for any $t\ge 0$.*

Then the subset M is positively orbitally stable and $D^+(M)=M$.

Proof. This fact can be proved using absolutely the same arguments as in the proof of Theorem 2.23. □

Lemma 2.26. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system, M be a nonempty compact and positively invariant set. Suppose that the following conditions are fulfilled:*

1. *$h(M)=Y$;*
2. *$M\bigcap X_y$ contains a single point for any $y\in Y$.*

If the set Y is invariant, then the set M is so.

Proof. Let x be an arbitrary point from M and $y:=h(x)$. Since the set Y is invariant, then for any $t\in\mathbb{T}_2$ there exists a point $y_t\in Y$ such that $\sigma(t,y_t)=y$. For any $t\in\mathbb{T}_1$ denote by $x_t:=M_{y_t}=h^{-1}(y_t)$. Then we have $\pi(t,x_t)=x$. Indeed,

$$h(\pi(t,x_t))=\sigma(t,h(x_t))=\sigma(t,y_t)=y,$$

i.e., $\pi(t,x_t)\in M_y=\{x\}$. Lema is proved. □

Denote by $\Omega_X:=\overline{\cup\{\omega_x|x\in X\}}$ and

$$D^+(M):=\bigcap_{\varepsilon>0}\overline{\bigcup\{\pi(t,B(M,\varepsilon))|t\ge 0\}}.$$

Theorem 2.27. [16, Ch.I] *Let $(X,\mathbb{T},\pi)$ be a compactly dissipative dynamical system and J be its Levinson center. Then the following statements are equivalent:*

1. *a compact positively invariant subset M of X is orbitally stable if and only if $D^+(M)=M$;*
2. *$J=D^+(\Omega_X)$.*

Theorem 2.28. *Under the conditions of Theorem 2.23 the following statements hold:*

1. *the dynamical system $(X,\mathbb{T},\pi)$ is compactly dissipative and its Levinson center $J\subseteq M$;*
2. *if the set M is invariant, then $J=M$.*

Proof. By Theorem 2.23 the dynamical system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative. Let x be an arbitrary point from X, then under the conditions of Theorem 2.23 we have $\lim\limits_{t\to+\infty}\rho(\pi(t,x),x_{\sigma(t,h(x))})=0$. From this relation we have $\omega_x\subseteq M$ because the set M is positively invariant and the mapping $y\to x_y$ is continuous (see Lema 2.24). Thus we obtain $\Omega_X\subseteq M$ and, consequently, $D^+(\Omega_X)\subseteq D^+(M)$. By Lemma 2.25 the set M is positively orbitally stable and by Theorem 2.27 (item (i)) $D^+(M)=M$. From the above we obtain $D^+(\Omega_X)\subseteq M$. Since $J=D^+(\Omega_X)$ (see Theorem 2.27 item (ii)), then $J\subseteq M$.

To prove the second statement it is sufficient to show that $M \subseteq J$. We note that by Lemma 2.26 the set M is invariant, if the set Y is so. Thus M is a compact and invariant subset of $(X, \mathbb{T}, \pi)$. On the other hand the Levinson center J of dynamical system $(X, \mathbb{T}, \pi)$ is the maximal compact invariant of this system and, consequently, $M \subseteq J$. Theorem is completely proved. □

Definition 2.29. (*Cocycle on the state space E with the base* $(Y, \mathbb{S}, \sigma)$) A triplet $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ (briefly ϕ) is said to be a *cocycle* over $(Y, \mathbb{S}, \sigma)$ with the fibre E if the mapping $\phi : \mathbb{T}_+ \times Y \times E \to E$ satisfies the following conditions:

1. $\phi(0, u, y) = u$ for any $u \in E$ and $y \in Y$;
2. $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for any $t, \tau \in \mathbb{T}_+, u \in E$ and $y \in Y$;
3. the mapping ϕ is continuous.

Remark 2.30. If $\varphi(t_0, u_1, y_0) = \varphi(t_0, u_2, y_0)$ ($t_0 > 0$, $u_1, u_2 \in E$ and $y_0 \in Y$), then $\varphi(t, u_1, y_0) = \varphi(t, u_2, y_0)$ for any $t \geq t_0$.

Condition (**C**). (*Strong uniqueness condition*) If $\varphi(t_0, u_1, y_0) = \varphi(t_0, u_2, y_0)$ ($t_0 > 0$, $u_1, u_2 \in E$ and $y_0 \in Y$), then $\varphi(t, u_1, y_0) = \varphi(t, u_2, y_0)$ for any $t \in \mathbb{T}_+$.

Remark 2.31. Everywhere below in this paper we consider only the cocycles φ satisfying *Condition* (**C**).

Definition 2.32. (Skew-product dynamical system) Let $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle on E, $X := E \times Y$ and π be a mapping from $\mathbb{T}_+ \times X$ to X defined by $\pi := (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for any $t \in \mathbb{T}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{T}_+, \pi)$ is an autonomous dynamical system and it is called a *skew-product dynamical system.*

Let $x \in X$. Denote by $\Sigma_x^+ := \{\pi(t, x) : \ t \geq 0\}$ (respectively, $\Sigma_x := \{\pi(t, x_0) : \ t \in \mathbb{T}\}$) the positive semi-trajectory (respectively, the trajectory) of the point x and $H^+(x) := \overline{\Sigma}_x^+$ (respectively, $H(x) := \overline{\Sigma}_x$) the semi-hull of x (respectively, the hull of x), where by bar the closure of Σ_x^+ (respectively, Σ_x) in X is denoted.

3. UNIFORM ASYMPTOTIC STABILITY of INFINITE-DIMENSIONAL NONAUTONOMOUS GENERALIZED HOMOGENEOUS DYNAMICAL SYSTEMS: GENERAL CASE

Let $E := \mathbb{R}^n$ with the euclidian norm $|x| := \sqrt{x_1^2 + \ldots + x_n^2}$. Denote by

$$|x|_{r,p} := \left(\Sigma_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}},$$

where $r := (r_1, \ldots, r_n)$, $r_i > 0$ for any $i = 1, \ldots, n$ and $p \geq \max\{r_i |\ 1 \leq i \leq n\}$. Denote by $\rho(x) := |x|_{r,p}$,

$$\mathcal{K} := \{\alpha \in C(\mathbb{R}_+, \mathbb{R}_+)|\ \alpha(0) = 0 \ \text{ and } \alpha \ \text{ is strictly increasing}\}$$

and

$$\mathcal{K}_\infty := \{\alpha \in \mathcal{K}|\ \alpha(t) \to +\infty \ \text{ as } \ t \to +\infty\}.$$

There exist $a, b \in \mathcal{K}_\infty$ such that

$$a(|x|_{r,p}) \leq |x| \leq b(|x|_{r,p}) \tag{3.1}$$

for any $x \in \mathbb{R}^n$ (see for example [21]).

A generalized weight is a vector $r = (r_1, \dots, r_n)$ with $r_i > 0$ for any $i = 1, \dots, n$. The dilation associated to the generalized weight r is the action of the multiplicative group $\mathbb{R}_+ \setminus \{0\}$ on $\mathbb{R}^n$ given by:

$$\Lambda^r : (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}^n \quad \big((\mu, x) \to \Lambda^r_\mu x\big),$$

where $\Lambda^r_\mu := diag(\mu^{r_i})_{i=1}^n$.

Remark 3.1. The following statements hold:

1. $\Lambda^r_1 = I$, where $I := diag(1, \dots, 1)$;
2. $\Lambda^r_{\mu_1}\Lambda^r_{\mu_2} = \Lambda^r_{\mu_1\mu_2}$ for any $\mu_1, \mu_2 \in \mathbb{R}_+ \setminus \{0\}$;
3. the matrix Λ^r_μ $(\mu > 0)$ is invertible and $\Lambda^r_{\mu^{-1}}$ is its inverse, i.e., $\Lambda^r_{\mu^{-1}} = \big(\Lambda^r_\mu\big)^{-1}$, because $L^r_\mu \Lambda^r_{\mu^{-1}} = \Lambda^r_1 = I$ for any $\mu > 0$;
4. $||\Lambda^r_\mu|| \to 0$ as $\mu \to 0$, because $\|\Lambda^r_\mu\| \le \mu^\kappa$ (where $\kappa := \max\{r_i |\ 1 \le i \le n\}$);
5. $$|\Lambda^r_\mu x| \ge \mu^\nu |x|$$ for any $x \in \mathbb{R}^n$ and $\mu > 0$, where $\nu := \min\{r_i |\ i = 1, \dots, n\} > 0$;
6. $$\rho(\Lambda^r_\mu x) = \mu\rho(x)$$ for any $(\mu, x) \in (0, +\infty) \times \mathbb{R}^n$, where $\rho(x) := |x|_{r,p}$;
7. $\Lambda^{(1,\dots,1)}_\mu = diag(\mu, \dots, \mu) = \mu I$ for any $\mu > 0$.

Let P be the field of complex $\mathbb{C}$ or real $\mathbb{R}$ numbers and E be a Banach space over the field P.

Definition 3.2. A dynamical system $(E, \mathbb{T}, \lambda)$ is said to be linear if $\lambda(\tau, \alpha u + \beta v) = \alpha\lambda(\tau, u) + \beta\lambda(\tau, v)$ for any $(t, \tau) \in \mathbb{T} \times \mathbb{R}$, $u, v \in E$ and $\alpha, \beta \in P$.

Remark 3.3. If $(E, \mathbb{T}, \lambda)$ is a linear dynamical system, then $\pi(t, 0) = 0$ for any $t \in \mathbb{T}$.

Let $(E, \mathbb{R}, \lambda)$ be a linear dynamical system on E. Everywhere in this paper we suppose that the trivial motion of the two-sided dynamical system $(E, \mathbb{S}, \lambda)$ is uniformly attracting in the negative direction (or negatively uniformly attracting), i.e.,

$$\lim_{\tau \to -\infty} \|\lambda^\tau\| = 0.$$

Definition 3.4. A function $f : E \to \mathbb{R}$ (respectively, $f : E \to E$) is said to be λ-homogeneous of the degree m if $f(\lambda(\tau, x)) = e^{m\tau} f(x)$ (respectively, $f(\lambda(\tau, x)) = e^{m\tau}\lambda(\tau, f(x))$) for any $\tau \in \mathbb{R}$ and $x \in E$.

Everywhere below in this Section we suppose that $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$.

Definition 3.5. A dynamical system $(E, \mathbb{T}, \pi)$ is said to be λ-homogeneous of the degree m if for any $x \in E$, $t \in \mathbb{T}$ and $\tau \in \mathbb{R}$ we have $\pi(t, \lambda(\tau, x)) = \lambda(\tau, \pi(te^{m\tau}, x))$.

Remark 3.6. 1. If $(E, \mathbb{T}, \lambda)$ is a linear dynamical system and $d(t) := \lambda(t, \cdot)$ $(t \in \mathbb{T})$, then $\{d(t)\}_{t\in\mathbb{T}}$ is a strongly continuous semigroup of linear bounded operators acting on the Banach space E and $d(0) = I$, where I is a unit linear operator acting on E.

2. If $\mathbb{T} = \mathbb{R}$, then $\{d(t)\}_{t\in\mathbb{T}}$ is a strongly continuous group of linear bounded operators acting on E.

Lemma 3.7. *Let $\{d(t)\}_{t\in\mathbb{T}}$ be a strongly continuous semigroup (respectively, a group if $\mathbb{T}=\mathbb{R}$) of linear bounded operators acting on the Banach space E and $d(0)=I$.*

Then the following statements hold:

1. *if K is a compact subset of $\mathbb{T}$, then there exists a positive constant $M=M(K)$ such that*

$$\|d(t)\| \leq M \tag{3.2}$$

   for any $t\in K$;
2. *the triplet $(E,\mathbb{T},\pi)$ is a linear dynamical system (respectively, two-sided dynamical system if $\mathbb{T}=\mathbb{R}$) on E, where $\lambda(\tau,x):=d(\tau)x$ for any $\tau\in\mathbb{T}$ and $x\in E$.*

Proof. Let K be an arbitrary compact subset of $\mathbb{T}$. Consider the family of linear bonded operators $\mathcal{A}:=\{d(t)|\ t\in\mathbb{K}\}$. Since the semigroup $\{d(t)\}_{t\in\mathbb{T}}$ is strongly continuous and K is a compact subset of $\mathbb{T}$, then there exists a constant $C_x>0$ such that

$$|d(t)x| \leq C_x$$

for any $t\in\mathbb{K}$. By Banach-Steinhaus theorem there exists a positive constant $M=M(K)$ such that the inequality (3.2) holds.

It easy to check that $\lambda(0,x)=x$ and $\lambda(t+\tau,x)=\lambda(t,\lambda(\tau,x))$ for any $t,\tau\in\mathbb{T}$ and $x\in E$. To finish the proof of the second statement of Lemma it is sufficient to show that the mapping $\lambda:\mathbb{T}\times E\to E$ is continuous. Let (t_0,x_0) be an arbitrary point from $\mathbb{T}\times E$ and $(t_k,x_k)\to(t_0,x_0)$ as $k\to\infty$. Since $K_0:=\{t_k|\ k\in\mathbb{N}\}\bigcup\{t_0\}$ is a compact subset of $\mathbb{T}$, then by the first statement of Lemma there exists a positive constant $M_0=M(K_0)$ such that

$$\|d(t_k)\| \leq M_0 \tag{3.3}$$

for any $k\in\mathbb{N}$. On the other hand, taking into consideration (3.3), we have

$$\begin{aligned} |\lambda(t_k,x_k)-\lambda(t_0,x_0)| = |d(t_k)x_k-d(t_0)x_0| \leq |d(t_k)x_k-d(t_k)x_0| + \\ |d(t_k)x_0-d(t_0)x_0| \leq M_0|x_k-x_0| + |d(t_k)x_0-d(t_0)x_0| \end{aligned} \tag{3.4}$$

for any $k\in\mathbb{N}$. Since the semigroup of linear operators $\{d(t\}_{t\in\mathbb{T}}$ is strongly continuous, then passing to the limit in (3.4) as $k\to\infty$ we obtain

$$\lim_{k\to\infty}\lambda(t_k,x_k)=\lambda(t_0,x_0).$$

Lemma is proved. □

Lemma 3.8. [16, Ch.II] *Let $\mathfrak{D}$ be a family of functions $\eta:\mathbb{R}_+\to\mathbb{R}_+$ satisfying the conditions:*

a. *there exists $M>0$ such that $0<\eta(t)\leq M$ for any $t\geq 0$ and $\eta\in\mathfrak{D}$;*
b. *$\eta(t)\to 0$ as $t\to+\infty$ uniformly in $\eta\in\mathfrak{D}$, i.e., for any $\varepsilon>0$ there exists $L(\varepsilon)>0$ such that $\eta(t)<\varepsilon$ for any $t\geq L(\varepsilon)$ and $\eta\in\mathfrak{D}$.*

Then we have the following statements:

1. *if $\eta(t+\tau)\leq\eta(t)\eta(\tau)$ for any $t,\tau\geq 0$ and $\eta\in\mathfrak{D}$, then there exit positive numbers $\mathcal{N}$ and ν such that*

$$\eta(t)\leq\mathcal{N}e^{-\nu t}$$

   for any $t\geq 0$ and $\eta\in\mathfrak{D}$;

2. *if $\eta(t+\tau) \leq \eta(t)\eta(\tau\eta^m(t))$ $(m>0)$ for any $t,\tau \geq 0$ and $\eta \in \mathfrak{D}$, then there exist positive numbers a and b such that*
$$\eta(t) \leq M(a+bt)^{-\frac{1}{m}}$$
for any $t \geq 0$ and $\eta \in \mathfrak{D}$.

Definition 3.9. Let $(E,\mathbb{S},\lambda)$ be a linear dynamical system. Following [26]-[28], [33] a cocycle $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ over dynamical system $(Y,\mathbb{T},\sigma)$ with the fibre E is said to be λ-homogeneous of the degree $m \in \mathbb{R}$ if
$$\varphi(t,\lambda^\tau u,y) = \mu^m \lambda^\tau \varphi(t,u,y)$$
for any $\tau \in \mathbb{S}$ and $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$, where $\lambda^\tau = \lambda(\tau,\cdot)$.

Let $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a cocycle over dynamical system $(Y,\mathbb{T},\sigma)$ with the fibre E. Assume that φ admits a trivial motion $u=0$, that is, $\varphi(t,0,y)=0$ for any $(t,y)\mathbb{T}_+ \times Y$ (0 is the null element of E).

In this Section we suppose that the phase space Y of the driving system $(Y,\mathbb{R},\sigma)$, generally speaking, is not compact.

Lemma 3.10. *If the trivial motion $u=0$ of the cocycle φ is uniformly attracting, then it is uniformly stable.*

Proof. Let γ be a positive number such that
$$\lim_{t\to+\infty} \sup_{|u|\leq\gamma, y\in Y} |\varphi(t,u,y)| = 0.$$
Suppose that the trivial motion $u=0$ of the cocycle φ is not uniformly stable, then there exist positive numbers ε_0, $\delta_k \to 0$ $(\delta_k > 0)$ as $k\to\infty$ and sequences $\{u_k\}$ (with $|u_k| \leq \delta_k$ for any $k \in \mathbb{N}$), $\{y_k\} \subset Y$ and $t_k \geq k$ such that
$$|\varphi(t_k,u_k,y_k)| \geq \varepsilon_0 \tag{3.5}$$
for any $k \in \mathbb{N}$. Since the trivial motion $u=0$ of the cocycle φ is uniformly attracting then for $\varepsilon > 0$ there exists a positive number $L(\varepsilon_0)$ such that
$$|\varphi(t,u,y)| < \frac{\varepsilon_0}{2}$$
for any $t \geq L(\varepsilon_0)$, $|u| \leq \gamma$ and $y \in Y$. We choose $k_0 \in \mathbb{N}$ such that $\delta_k \leq \gamma$ and $t_k \geq L_0$ for any $k \geq k_0$ and, consequently,
$$|\varphi(t_k,u_k,y_k)| < \frac{\varepsilon_0}{2} \tag{3.6}$$
for any $k \geq k_0$. The inequalities (3.5)and (3.6) are contradictory. The obtained contradiction proves our statement. □

Corollary 3.11. *The trivial motion $u=0$ of the cocycle φ is uniformly asymptotically stable if and only if it is uniformly attracting.*

Proof. This statement follows directly from Lemma 3.10 . □

Definition 3.12. The trivial motion $u=0$ of the cocycle φ is said to be:

1. uniformly stable if for arbitrary positive number ε there exists a positive number $\delta=\delta(\varepsilon)$ such that $|u| < \delta$ implies
$$|\varphi(t,u,y)| < \varepsilon$$
for any $(t,y) \in \mathbb{T}_+ \times Y$;

2. uniformly attracting if there exists a positive number γ such that

$$\lim_{t\to+\infty} \sup_{|u|\le\gamma, y\in Y} |\varphi(t,u,y)| = 0; \tag{3.7}$$

3. globally uniformly attracting if the equality (3.7) holds for any $\gamma > 0$;
4. uniformly asymptotically stable if it is uniformly stable and uniformly attracting;
5. globally uniformly asymptotically stable if it is uniformly stable and globally uniformly attracting.

Definition 3.13. Following [20], [27, Ch.VII] a function $\rho : E \to \mathbb{R}_+$ we will call a λ-homogeneous norm on the Banach space E if the following conditions are fulfilled:

1. the mapping $\rho : E \to \mathbb{R}_+$ is continuous;
2.
$$\rho(\lambda^\tau u) = e^\tau \rho(u) \tag{3.8}$$
for any $\tau \in \mathbb{T}$ and $u \in E \setminus \{0\}$;
3. there are functions $a, b \in \mathcal{K}_\infty$ such that
$$a(\rho(u)) \le |u| \le b(\rho(u)) \tag{3.9}$$
for any $u \in E$.

Remark 3.14. Everywhere below we assume that $\rho : E \to \mathbb{R}_+$ is a λ-homogeneous of the degree zero norm on the Banach space E.

Remark 3.15. Let $\rho : E \to \mathbb{R}_+$ and there exist $a, b \in \mathcal{K}_\infty$ such that (3.9) holds. Then $\rho(u) = 0$ if and only if $u = 0$, i.e., the function $\rho : E \to \mathbb{R}_+$ is positive defined.

Lemma 3.16. *Let $(E, \mathbb{S}, \lambda)$ be a two-sided dynamical system on the Banach space E, $\rho : E \to \mathbb{R}_+$ be a λ-homogeneous norm on E and $\lambda(t, 0) = 0$ for any $t \in \mathbb{S}$. Then the trivial motion $u = 0$ of the dynamical system $(E, \mathbb{S}, \lambda)$ is globally uniformly asymptotically stable in the negative direction.*

Proof. Let ε be an arbitrary positive number, $\delta(\varepsilon) := a(b^{-1}(\varepsilon))$ and $|u| < \delta$, then

$$|\lambda^\tau u| \le b(\rho(\lambda^\tau u)) = b(e^\tau \rho(u)) \le b(\rho(u)) \le$$
$$b(a^{-1}(|u|)) < b(a^{-1}(\delta(\varepsilon))) = \varepsilon$$

for any $\tau \le 0$ and $|u| < \delta$, i.e., the trivial motion of the dynamical system $(E, \mathbb{S}, \sigma)$ is negatively uniformly stable.

Let γ be an arbitrary positive number and $|u| \le \gamma$. Note that

$$|\lambda^\tau u| \le b(\rho(\lambda^\tau u)) = b(e^\tau \rho(u)) \le b(e^\tau a^{-1}(\gamma))$$

for any $\tau \in \mathbb{S}$ and, consequently,

$$\sup_{|u|\le\gamma} |\lambda^\tau u| \le b(e^\tau a^{-1}(\gamma)) \tag{3.10}$$

for any $\tau \in \mathbb{S}$. Passing to the limit in (3.10) as $\tau \to -\infty$ we obtain

$$\lim_{\tau\to-\infty} \sup_{|u|\le\gamma} |\lambda^\tau u| = 0.$$

Lemma is completely proved. □

Corollary 3.17. *Under the conditions of Lemma 3.16 if the dynamical system $(E, \mathbb{S}, \lambda)$ is linear, then*

$$\lim_{\tau\to-\infty} \|\lambda^\tau\| = 0.$$

Lemma 3.18. *The trivial motion $u = 0$ of the λ-homogeneous cocycle φ of the degree zero is uniformly stable if and only if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(u) < \delta$ implies $\rho(\varphi(t, u, y)) < \varepsilon$ for any $(t, y) \in \mathbb{T}_+ \times Y$.*

Proof. Let $u = 0$ be uniformly stable motion of φ, $\mu > 0$ and $\Delta(\mu) > 0$ be a positive number figuring in the definition of the uniform stability of $u = 0$. For any $\varepsilon > 0$ we put $\delta(\varepsilon) := b^{-1}(\Delta(a(\varepsilon))) > 0$, where a and b are some functions from $\mathcal{K}_\infty$ figuring in (3.9). If $\rho(u) < \delta$, then we have $|u| \le b(\rho(u)) < \Delta(a(\varepsilon))$ and, consequently, $|\varphi(t, u, y)| < a(\varepsilon)$ for any $t \in \mathbb{T}_+$. Note that $\rho(\varphi(t, u, y)) \le a^{-}(|\varphi(t, u, y)|) < a^{-1}(a(\varepsilon)) = \varepsilon$ for any $t \ge 0$.

The converse statement can be proved using the same arguments as above. Lemma is proved. □

Lemma 3.19. *If the trivial motion $u = 0$ of the cocycle φ is uniformly stable, then there exists a positive number $\widetilde{M}$ such that*

$$|\varphi(t, u, y)| \le \widetilde{M}$$

for any $|u| \le 1$ and $(t, y) \in \mathbb{T}_+ \times Y$.

Proof. Since the trivial motion $u = 0$ of the cocycle φ is uniformly stable, then there exists a positive number $\delta_0 = \delta(1)$ such that $|u| \le \delta_0$ implies

$$|\varphi(t, u, y)| \le 1$$

for any $|u| \le \delta_0$ and $(t, y) \in \mathbb{T}_+ \times Y$. Since $\|\lambda^\tau\| \to 0$ as $\tau \to -\infty$ then there exists a positive number τ_0 such that

$$\|\lambda^\tau\| \le \delta_0 \tag{3.11}$$

for any $\tau \le -\tau_0$. Note that

$$|\varphi(t, u, y)| = |\varphi(t, \lambda^{-\tau}\lambda^\tau u, y)| = |\lambda^{-\tau}\varphi(t, \lambda^\tau u, y)| \le \|\lambda^{-\tau}\||\varphi(t, \lambda^\tau u, y)| \tag{3.12}$$

for any $\tau \le -\tau_0$ and $(t, u, y) \in \mathbb{T}_+ \times E \times Y$. By (3.11) we have

$$|\lambda^{-\tau} u| \le \delta_0$$

for any $|u| \le 1$ and, consequently,

$$\sup_{|u| \le 1} |\varphi(t, \lambda^{-\tau} u, y)| \le 1 \tag{3.13}$$

for any $(t, y) \in \mathbb{T}_+ \times Y$. Finally, we note that from (3.12) and (3.13) we obtain

$$|\varphi(t, u, y)| \le \|\lambda^{-\tau_0}\| := \widetilde{M}$$

for any $|u| \le 1$ and $(t, y) \in \mathbb{T}_+ \times Y$. Lemma is proved. □

Corollary 3.20. *Under the conditions of Lemma 3.19 for any $R > 0$ there exists a positive constant $M(R)$ such that*

$$|\varphi(t, u, y)| \le M(R)$$

for any $u \in E$ with $|u| \le R$ and $(t, y) \in \mathbb{T}_+ \times Y$.

Proof. Let R be an arbitrary positive number. Since $\|\lambda^\tau\| \to 0$ as $\tau \to -\infty$, then there exists a positive number $\tau_0 = \tau_0(R)$ such that

$$\|\lambda^\tau\| \le R^{-1} \tag{3.14}$$

for any $\tau \le -\tau_0$ and, consequently,

$$|\lambda^{\tau} u| \le \|\lambda^{\tau}\||u| \le R^{-1}R = 1 \tag{3.15}$$

for any $|u| \le R$. Note that

$$|\varphi(t,u,y)| = |\varphi(t,\lambda^{-\tau}\lambda^{\tau}u,y)| = |\lambda^{-\tau}\varphi(t,\lambda^{\tau}u,y)| \le \|\lambda^{-\tau}\||\varphi(t,\lambda^{\tau}u,y)| \tag{3.16}$$

for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. According to (3.14)-(3.16) we obtain

$$|\varphi(t,u,y)| \le \|\lambda^{-\tau_0}\||\varphi(t,\lambda^{\tau_0}u,y)| \le \|\lambda^{-\tau_0}\|\widetilde{M} := M(R)$$

for any $|u| \le R$ and $(t,y) \in \mathbb{T}_+ \times Y$. □

Corollary 3.21. *Under the conditions of Lemma 3.19 there exists a positive constant M such that*

$$\rho(\varphi(t,u,y)) \le M$$

for any $u \in E$ with $\rho(u) \le 1$ and $(t,y) \in \mathbb{T}_+ \times Y$.

Proof. Let $u \in E$ with $\rho(u) \le 1$ and $a, b \in \mathcal{K}_\infty$ be the function from (3.9), then we have

$$|u| \le b(\rho(u)) \le b(1).$$

By Corollary 3.20

$$|\varphi(t,u,y)| \le M(b(1))$$

for any $|u| \le b(1)$ and, consequently,

$$a(\rho(\varphi(t,u,y))) \le |\varphi(t,u,y)| \le M(b(1)) \tag{3.17}$$

for any $(t,y) \in \mathbb{T}_+ \times Y$. From (3.17) we obtain

$$\rho(\varphi(t,u,y)) \le a^{-1}(M(b(1)) := M$$

for any $u \in E$ with $\rho(u) \le 1$ and $(t,y) \in \mathbb{T}_+ \times Y$. □

Remark 3.22. If φ is an λ-homogeneous cocycle of the degree zero, then $\varphi(t,u,y) \neq 0$ for any $t \in \mathbb{T}_+$ and $u \neq 0$.

This statement follows directly from the Condition (**C**).

Lemma 3.23. *Let $\langle E, \varphi, (Y,\mathbb{T},\sigma)\rangle$ be a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre E. Assume that φ is an λ-homogeneous cocycle of the degree zero. Then*

1. $$\rho(\varphi(t+s,u,y)) = \rho(\varphi(s,u,y))\rho(\varphi(t,\lambda^{-\tau}\varphi(\tau,u,y),\sigma(\tau,y))$$

   for any $t, s \in \mathbb{T}_+$, where $\tau := \ln \rho(\varphi(s,u,y))$;
2. $$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\lambda^{-\tau}u,y))$$

   for any $u \in \setminus\{0\}$, $t \in \mathbb{T}_+$ and $y \in Y$, where $\tau = \ln \rho(u)$.

Proof. Note that

$$\rho(\varphi(t+s,u,y)) = \rho(\varphi(t,\varphi(s,u,y),\sigma(s,y))) =$$
$$\rho(\varphi(t,\lambda^{\tau}\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))) = \rho(\lambda^{\tau}\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))) =$$
$$e^{\tau}\rho(\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y)))$$

for any $\tau \in \mathbb{S}$, $t,s \in \mathbb{T}_+$ and $(u,y) \in E \times Y$. In particular for $\tau = \ln \rho(\varphi(s,u,y))$ we obtain from (3.18) the following equality

$$\rho(\varphi(t+s,u,y)) = \rho(\varphi(s,u,y))\rho(\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))).$$

The second statement of Lemma follows from the first one if we take $s = 0$. □

Theorem 3.24. *Let $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a λ-homogeneous cocycle of the degree zero. The following statement are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*
2. *there exists a positive number M such that*

$$\rho(\varphi(t,u,y)) \le M\rho(u) \tag{3.18}$$

for any $(t,u,y) \in \mathbb{T}_+ \times E \times Y$.

Proof. To prove this Theorem it is sufficient to show (i) implies (ii) because the inverse implication, taking into account Lemma 3.18, is evident.

Let M be a positive number from Corollary 3.21 and (t,u,y) be an arbitrary element from $\mathbb{T}_+ \times E \times Y$ with $u \neq 0$, then by Lemma 3.23 (item (ii)) we have

$$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\lambda^{-\tau}u,y)), \tag{3.19}$$

where $\tau := \ln \rho(u)$.

Since $\rho(\lambda^{-\tau}u) = \rho(u)^{-1}\rho(u) = 1$, then by Corollary 3.21 we have

$$\rho(\varphi(t,\lambda^{-\tau}u,y)) \le M. \tag{3.20}$$

From (3.19) and (3.20) we obtain (3.18). Theorem is proved. □

Lemma 3.25. *If the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then*

$$\lim_{t\to+\infty} \sup_{|u|\le 1, y\in Y} |\varphi(t,u,y)| = 0. \tag{3.21}$$

Proof. Since the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then there exists a positive number γ such that

$$\lim_{t\to+\infty} \sup_{|u|\le \gamma, y\in Y} |\varphi(t,u,y)| = 0. \tag{3.22}$$

Since $\|\lambda^{-\tau}\| \to 0$ as $\tau \to +\infty$, then there exists a positive number τ_0 such that

$$\||\lambda^{-\tau}\| \le \gamma \tag{3.23}$$

for any $\tau \ge \tau_0$ and, consequently,

$$|\lambda^{-\tau_0}u| \le \|\lambda^{-\tau_0}\||u| \le \gamma \tag{3.24}$$

for any $|u| \le 1$. From (3.12) we have

$$|\varphi(t,u,y)| \le \|\lambda^{\tau_0}\||\varphi(t,\lambda^{-\tau_0}u,y)| \tag{3.25}$$

and taking into account (3.22) -(3.25) we obtain (3.21). Lemma is proved. □

Corollary 3.26. *Assume that the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then*

$$\lim_{t\to+\infty} \sup_{|u|\le R, y\in Y} |\varphi(t,u,y)| = 0 \tag{3.26}$$

for any $R > 0$.

Proof. Let R be an arbitrary (fixed) positive number. Since $\|\lambda^{-\tau}\| \to 0$ as $\tau \to +\infty$ then there exists a positive number τ_0 such that

$$\||\lambda^{-\tau}\| \le R^{-1} \tag{3.27}$$

for any $\tau \ge \tau_0$ and, consequently,

$$|\lambda^{-\tau_0}u| \le \|\lambda^{-\tau_0}\||u| \le R^{-1}R = 1 \tag{3.28}$$

for any $|u| \le R$. Taking into consideration (3.26)-(3.28) we obtain

$$|\varphi(t,u,y)| = |\varphi(t,\lambda^{-\tau_0}\lambda^{\tau_0}u,y)| = |\lambda^{\tau_0}\varphi(t,\lambda^{-\tau_0}u,y)| \le$$
$$\|\lambda^{\tau_0}\||\varphi(t,\lambda^{-\tau_0}u,y)| \le R^{-1} \sup_{|v|\le 1, y\in Y} |\varphi(t,v,y)| \to 0$$

as $t \to +\infty$ uniformly with respect to $|u| \le R$ and $y \in Y$. □

Corollary 3.27. *Under the conditions of Lemma 3.25 we have*

$$\lim_{t\to+\infty} \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,u,y)) = 0. \tag{3.29}$$

Proof. Let $u \in E$ with $\rho(u) \le 1$, then $|u| \le b(1)$. Note that

$$a(\rho(\varphi(t,u,y)) \le |\varphi(t,u,y)| \le \sup_{|u|\le b(1), y\in Y} |\varphi(t,u,y)| := \eta(t)$$

and by Corollary 3.26

$$\lim_{t\to+\infty} \eta(t) = 0. \tag{3.30}$$

From (3.30) we obtain

$$\sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,u,y)) \le a^{-1}(\eta(t)) \tag{3.31}$$

for any $t \in \mathbb{T}_+$. Passing to the limit in (3.31) and taking into account (3.30) we obtain (3.29). □

Theorem 3.28. *Let φ be a λ-homogeneous cocycle over dynamical system $(Y,\mathbb{T},\sigma)$ with the fibre. The following statements are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly asymptotically stable;*
2. *there are positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,y)) \le \mathcal{N}e^{-\nu t}\rho(u) \tag{3.32}$$

for any $(t,u,y) \in \mathbb{T}_+ \times E \times Y$.

Proof. It is evident that 2. implies 1.

Now we will establish that 1. implies 2. Indeed, denote by

$$m(t) := \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,u,y)) \tag{3.33}$$

for every $t \in \mathbb{T}_+$. By (3.33) the mapping $m : \mathbb{T}_+ \to \mathbb{R}_+$ is well defined possessing the following properties:

a. $0 \le m(t) \le M$ for any $t \in \mathbb{T}_+$, where $M := a^{-1}(M(b(1)))$ is the constant from Corollary 3.21;

b. $m(t) \to 0$ as $t \to +\infty$;
c. $m(t+\tau) \le m(t)m(\tau)$ for any $t, \tau \in \mathbb{T}_+$.

The statement a. (respectively, statement b.) follows from Corollary 3.21 (respectively, Corollary 3.27). To prove the statement c. we note that

$$m(t+s) = \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t+s,u,y) =$$
$$\sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,\varphi(s,u,y),\sigma(s,y))) =$$
$$\sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,\lambda^{\tau}\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))) = \tag{3.34}$$
$$\sup_{\rho(u)\le 1, y\in Y} \rho(\lambda^{\tau}\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))),$$

where

$$\tau := \ln \rho(\varphi(s,u,y)). \tag{3.35}$$

By the equality (3.8) we have

$$\sup_{\rho(u)\le 1, y\in Y} \rho(\lambda^{\tau}\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))) = \tag{3.36}$$
$$e^{\tau}\rho(\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))).$$

Note that

$$\rho(\lambda^{-\tau}\varphi(s,u,y)) = e^{-\tau}\rho(\varphi(s,u,y) = 1, \tag{3.37}$$

where $\tau = \ln \rho(\varphi(s,u,y))$ and, consequently,

$$\rho(\varphi(t,\lambda^{-\tau}\varphi(s,u,y),\sigma(s,y))) \le \sup_{\rho(v)\le 1, q\in Y} \rho(\varphi(t,v,q)) = m(t). \tag{3.38}$$

From (3.34)-(3.38) we obtain

$$m(t+s) \le m(s)m(t)$$

for any $t, s \in \mathbb{T}_+$.

According to Lemma 3.23 (item (ii)) we have

$$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\lambda^{-\tau}u,y)) \le m(t)\rho(u)$$

for any $u \in E \setminus \{0\}$ and $(t,y) \in \mathbb{T}_+ \times Y$ because $\rho(\lambda^{-\tau}u) = 1$ ($\tau = \ln \rho(u)$) and, consequently,

$$\rho(\varphi(t,\lambda^{-\tau}u,y)) \le \sup_{\rho(v)\le 1, y\in Y} \rho(\varphi(t,v,y)) = m(t). \tag{3.39}$$

By Lemma 3.8 (item (i)) there are positive numbers $\mathcal{N}$ and ν such that $m(t) \le \mathcal{N}e^{-\nu t}$ for any $t \in \mathbb{T}_+$, and taking into account (3.39) we obtain (3.32). Theorem is proved. □

4. HOMOGENEOUS DISSIPATIVE DYNAMICAL SYSTEMS

Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be an λ-homogeneous cocycle of order m, then φ admits the trivial motion.

Denote by

$$W^s_y(0) := \{u \in E | \lim_{t\to+\infty} |\varphi(t,u,y)| = 0\}.$$

Lemma 4.1. *Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a λ-homogeneous of the degree zero cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre E. Assume that $W^s_y(0)$ is a neighborhood of 0 in E, then $W^s_y(0) = E$.*

Proof. Let $u \in E$ be an arbitrary point. Under the condition of Lemma there exists a positive number δ_y such that $B(0,\delta_y) \subseteq W^s_y(0)$, where $B(0,\delta) := \{u \in E|\ |u| < \delta\}$. Since $\|\lambda^{-\tau}\| \to 0$ as $\tau \to +\infty$ then there exists a positive number τ_0 such that

$$\lambda^{-\tau} u \in B(0,\delta_y) \tag{4.1}$$

for any $\tau > \tau_0$. Note that

$$\varphi(t,u,y) = \varphi(t,\lambda^{\tau}\lambda^{-\tau}u,y)) = \lambda^{\tau}\varphi(t,\lambda^{-\tau}u,y). \tag{4.2}$$

From (4.1)-(4.2) we obtain $u \in W^s_y(0)$, that is, $E = W^s_y(0)$. Lemma is proved. □

Let Y be a metric space and $(Y,\mathbb{T},\sigma)$ be a dynamical system on Y. Everywhere in this Section we suppose that $\sigma(t,Y) = Y$ for any $t \in Y$, i.e., that the set Y is invariant.

Theorem 4.2. *Let $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre E. Assume that the cocycle φ is λ-homogeneous of the degree zero and Y is compact. Then the following conditions are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is attracting;*
2. *the skew-product dynamical system $(X,\mathbb{T}_+,\sigma)$ generated by φ is pointwise dissipative.*

Proof. To prove this statement it is sufficient to show that (i) implies (ii). Let $x = (u,y) \in X = E\times Y$ be an arbitrary point. By Lemma 4.1 we have $W^s_y(0) = E$ and, consequently $u \in W^s_y(0)$, i.e.,

$$\lim_{t\to+\infty} |\varphi(t,u,y)| = 0.$$

Since the space Y is compact, then the motion $\pi(t,x)$ ($x = (u,y)$ and $\pi(t,x) = (\varphi(t,u,y),\sigma(t,y))$) is positively Lagrange stable and $\emptyset \neq \omega_x \subseteq \Theta := \{0\}\times Y$. Thus $\Omega_X \subseteq \Theta$ and, consequently, the dynamical system $(X,\mathbb{T},\sigma)$ is pointwise dissipative. Theorem is proved. □

Theorem 4.3. *Let $\langle E,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre E. Assume that the cocycle φ is λ-homogeneous and Y is compact. Then the following conditions are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is k-attracting, i.e.,*
$$\lim_{t\to+\infty} \sup_{u\in K,\ y\in Y} |\varphi(t,u,y)| = 0 \tag{4.3}$$
for any compact subset K from E;
2. *the skew-product dynamical system $(X,\mathbb{T},\sigma)$ generated by the cocycle φ is compactly dissipative and its Levinson center $J \subseteq \Theta = \{0\}\times Y$.*

Proof. Assume that φ is a λ-homogeneous cocycle and $u = 0$ is its k-attracting trivial motion. We will show that in this case the skew-product dynamical system $(X,\mathbb{T}_+,\sigma)$ generated by φ is compactly dissipative. Let M be an arbitrary compact subset of $X = E\times Y$, then there exists a compact subset K from E such that $M \subseteq K\times Y$. We note that the set $\Sigma^+_M := \{\pi(t,x)|\ x \in M,\ t \in \mathbb{T}_+\}$ is a precompact subset of $X = E\times Y$. Indeed, let $\{\bar{x}_k\}$ be an arbitrary sequence from Σ^+_M then there are $\{t_k\} \subset \mathbb{T}_+$, $\{u_k\} \subset K$ and $\{y_k\} \subset Y$ such that $\bar{x}_k = (\varphi(t_k,u_k,y_k),\sigma(t_k,y_k))$. If the sequence $\{t_k\}$ is bounded, then it easy to see that the sequence $\{\bar{x}_k\}$ is precompact. Assume that the sequence $\{t_k\}$ is unbounded, then without loss of generality we can suppose that $t_k \to +\infty$ as $k \to \infty$. Note that

$$|\varphi(t_k,u_k,y_k)| \leq \sup_{u\in K,\ y\in Y} |\varphi(t_k,u,y)| \tag{4.4}$$

for any $k \in \mathbb{N}$. Passing to the limit in (4.4) as $k \to \infty$ and taking into account (4.3) we obtain $\lim\limits_{k\to\infty} \varphi(t_k,u_k,y_k) = 0$. Since the space Y is compact, then the sequence $\{\pi(t_k,x_k)\}$ is precompact.

Since the set Σ_M^+ is precompact, then $\omega(M)$ is a nonempty, compact and invariant subset of X and taking into account (4.3) we conclude that $\omega(M) \subseteq \Theta = \{0\} \times Y$. Since M is an arbitrary compact subset of X and $\omega(M) \subseteq \Theta$ then $(X, \mathbb{T}_+, \sigma)$ is compactly dissipative and its Levinson center $J \subseteq \Theta$.

Let now K be an arbitrary nonempty compact subset of K. Assume that the equality (4.3) does not take place. Then there are a positive number ε_0 and sequences $\{u_k\} \subset K$, $\{y_k\} \subset Y$ and $t_k \to +\infty$ as $k \to \infty$ such that

$$|\varphi(t_k, u_k, y_k)| \geq \varepsilon_0 \tag{4.5}$$

for any $k \in \mathbb{N}$. Denote by $M := K \times Y$. It is clear that M is a compact subset of X. Since the skew-product dynamical system $(X, \mathbb{T}_+, \sigma)$ is compact dissipative and its Levinson center $J \subseteq \Theta = \{0\} \times Y$, then the sequence $\{(\varphi(t_k, u_k, y_k), \sigma(t_k, y_k))\} \subset \Sigma_M^+$ is precompact. Thus, without loss of generality, we may assume that the sequence $\{\varphi(t_k, u_k, y_k)\}$ converges and its limit $\bar{u}$ equals to zero (null element of E). On the other hand passing to the limit in (4.5) as $k \to \infty$ we obtain $0 = |\bar{u}| \geq \varepsilon_0$. The last relation contradicts to the choice of the number ε_0. The obtained contradiction completes the proof of Theorem. □

Theorem 4.4. *Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre E. Assume that the cocycle φ is λ-homogeneous of the degree zero and Y is compact and invariant. Then the following conditions are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly attracting (u-attracting), i.e., there exists a positive number γ such that*
$$\lim_{t \to +\infty} \sup_{|u| \leq \gamma,\ y \in Y} |\varphi(t, u, y)| = 0;$$
2. *the skew-product dynamical system $(X, \mathbb{T}_+, \sigma)$ generated by the cocycle φ is locally dissipative and its Levinson center $J = \Theta$.*

Proof. Assume that the λ-homogeneous of the degree zero cocycle φ is u-attracting. By Corollary 3.26 for any $R > 0$ we have

$$\lim_{t \to +\infty} \sup_{|u| \leq R,\ y \in Y} |\varphi(t, u, y)| = 0. \tag{4.6}$$

From the equality (4.6) follows that the compact set $\Theta := \{0\} \times Y$ from $X = E \times Y$ attracts every bounded subset from X. This means, in particular, that the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ is locally dissipative and its Levinson center $J \subseteq \Theta$. Since the set Y is compact and invariant then by Theorem 2.28 (item (ii)) $J = \Theta$.

Now we will establish the converse statement. Let $(X, \mathbb{T}_+, \pi)$ be the skew-product dynamical system generated by λ-homogeneous of the degree zero cocycle φ. Assume that $(X, \mathbb{T}_+, \pi)$ is locally dissipative and its Levinson center $J = \Theta$, then by Theorem 2.13 there exists a positive number γ such that

$$\lim_{t \to +\infty} \beta(\pi(t, B[J, \gamma]), J) = 0. \tag{4.7}$$

From these facts it follows that the trivial motion $u = 0$ of the cocycle φ is uniformly stable. Assume that it is not true then there exist $\varepsilon_0 > 0$, $\delta_n \to 0$ ($\delta_n > 0$), $u_n \in E$ with $|u_n| \leq \delta_n$, $y_n \in Y$ and $t_n \to +\infty$ such that

$$|\varphi(t_n, u_n, y_n)| \geq \varepsilon_0 \tag{4.8}$$

for any $n \in \mathbb{N}$. Since the space Y is compact then without loss of generality we can suppose that the sequences $\{x_n\} := \{(u_n, y_n)\} \subset X = E \times Y$ and $\{\sigma(t_n, y_n)\}$ converge. Denote by $x_0 = \lim_{n\to\infty} x_n = (0, y_0)$ and $\bar{y} = \lim_{n\to\infty} \sigma(t_n, y_n)$. Since $x_n \to (0, y_0) \in J = \{0\} \times Y$ and taking into account (4.7) for $\varepsilon_0/2$ we can choose a number $n_0 \in \mathbb{N}$ such that

$$\rho(x_n, J) \le \gamma$$

and

$$\pi(t_n, x_n) \in \pi(t_n, B[J, \gamma]) \subseteq B[J, \varepsilon_0/2]$$

for any $n \ge n_0$.

According to (4.7) and taking into account that $J = \{0\} \times Y$ without loss of generality we may suppose that the sequence $\pi^{t_n} x_n = (\varphi(t_n, u_n, y_n), \sigma(t_n, y_n))$ converges. Denote its limit by

$$\bar{x} = (\lim_{n\to\infty} \varphi(t_n, u_n, y_n), \lim_{n\to\infty} \sigma(t_n, y_n)) = (\bar{u}, \bar{y}).$$

By (4.7) we have $\bar{x} = (\bar{u}, \bar{y}) \in J = \{0\} \times Y$ and, consequently, $\bar{u} = 0$. From the last relation we have a number $\bar{n} \in \mathbb{N}$ (with $\bar{n} \ge n_0$) such that

$$|\varphi(t_n, u_n, y_n)| \le \varepsilon_0/2 \tag{4.9}$$

for any $n \ge \bar{n}$. Note that the relations (4.8) and (4.9) are contradictory. The obtained contradiction proves our statement.

Thus the trivial motion $u = 0$ of the cocycle φ is uniformly stable and, consequently, for $\varepsilon_0 = 1$ there exists a positive number $\delta_0 = \delta(1)$ such that

$$|\varphi(t, u, y)| \le 1 \tag{4.10}$$

for any $(t, y) \in \mathbb{T}_+ \times Y$ and $|u| \le \delta_0$. Denote by $M := \Sigma_K^+$, where $K := B[0, \delta_0] \times Y$. It is clear tat the set M is positively invariant. By condition (4.10) we have $\pi(t, M) \subseteq \widetilde{K}$ for any $t \ge 0$, where $\widetilde{K} := [0, 1] \times Y$. Thus $M = \Sigma_K^+ \subseteq \widetilde{K}$, i.e., the set M is bounded. Since the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ ia asymptotically compact then there exists a nonempty, compact and invariant subset $\mathcal{M}$ ($\mathcal{M} = \omega(M)$) such that

$$\lim_{t\to+\infty} \beta(\pi(t, M), \mathcal{M}) = 0.$$

Taking into account that the Levinson center is a maximal compact invariant set of $(X, \mathbb{T}_+, \pi)$ we conclude that $\mathcal{M} \subseteq J = \{0\} \times Y$. Using the same arguments as above we can prove that

$$\lim_{t\to+\infty} \sup_{|u|\le\delta_0, y\in Y} |\varphi(t, u, y)| = 0,$$

i.e., the trivial motion $u = 0$ of the cocycle φ is uniformly attracting. Theorem is completely proved. □

5. ASYMPTOTIC STABILITY OF INFINITE-DIMENSIONAL NONAUTONOMOUS GENERALIZED HOMOGENEOUS DYNAMICAL SYSTEMS: THE CASE OF THE COMPACT SPACE OF DRIVING SYSTEM

Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre E. Assume that the cocycle φ admits the trivial motion 0, i.e., $\varphi(t, 0, y) = 0$ for any $(t, y) \in \mathbb{T}_+ \times Y$.

Definition 5.1. A trivial motion 0 of the cocycle φ is said to be:

1. attracting if there exists $\gamma > 0$ such that $\lim_{t\to+\infty} |\varphi(t, u, y)| = 0$ for any $|u| < \gamma$ and $y \in Y$;
2. asymptotically stable if it is uniformly stable and attracting;

3. globally asymptotically stable if it is asymptotically stable and $W^s_y(0) = E$ for any $y \in Y$.

Definition 5.2. Let $(X, \mathbb{T}, \pi)$ be a semi-groupe dynamical system. A continuous mapping $\phi : \mathbb{S} \to X$ is said to be a full (entire) trajectory of $(X, \mathbb{T}, \pi)$ if $\pi(t, \phi(s)) = \phi(t+s)$ for any $(t, s) \in \mathbb{T} \times S$.

Theorem 5.3. *Assume that φ is a λ-homogeneous cocycle over $(Y, \mathbb{T}, \sigma)$ of the degree zero, Y is a compact metric space and the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by cocycle φ is locally compact.*

Then the following statements are equivalent:

1. *the trivial motion of φ is attracting;*
2. *the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by cocycle φ is compactly dissipative and its Levinson center $J \subseteq \Theta := \{0\} \times Y$.*

Proof. To prove this statement it is sufficient to show that (i) implies (ii). Indeed, by Lemma 4.1 we have $W^s_y(0) = E \times Y$ or any $y \in Y$. Since the space Y is compact, then the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ is pointwise dissipative. Since the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ is locally compact, then by Theorem 2.10 the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative. Denote by J its Levinson center. Now we will show that $J \subseteq \Theta$. If we suppose that it is not true, then there exists a point $x_0 = (u_0, y_0) \in J \setminus \Theta$. This means that $u_0 \neq 0$ and through the point x_0 passes a full trajectory $\pi(t, x_0) = \{(\varphi(t, u_0, y_0), \sigma(t, y_0)|\ t \in \mathbb{S}\}$ which belongs to J. Since the cocycle φ is λ-homogeneous of the degree zero, then

$$\varphi(t, \lambda^\tau u_0, y_0) = \lambda^\tau \varphi(t, u_0, y_0) \tag{5.1}$$

for any $t \in \mathbb{S}$. From (5.1) it follows that the full trajectory $\{(\varphi(t, \lambda^\tau u_0, y_0), \sigma(t, y_0)|\ t \in \mathbb{S}\}$ is precompact and, consequently,

$$(\lambda^\tau u_0, \sigma(\tau, y_0)) \in J$$

for any $\tau \in \mathbb{R}$. Note that

$$\rho(\lambda^\tau u_0) = e^{-\tau} \rho(u_0) \tag{5.2}$$

for any $\tau \in \mathbb{R}$. Passing to the limit in (5.2) as $\tau \to -\infty$ and taking into account that $u_0 \neq 0$ $(\rho(u_0) > 0)$ we conclude that the set J is not compact. This contradicts to the fact that the Levinson center is the maximal compact invariant set of $(X, \mathbb{T}_+, \pi)$. The obtained contradiction proves our statement. Theorem is completely proved. □

Theorem 5.4. *Assume that the following conditions are fulfilled:*

1. *φ is a λ-homogeneous cocycle of the degree zero over $(Y, \mathbb{T}, \sigma)$;*
2. *Y is a compact and invariant set;*
3. *the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ is asymptotically compact.*

Then the following statements are equivalent:

1. *the trivial motion of φ is asymptotically stable;*
2. *the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ is compactly dissipative and its Levinson center $J = \Theta = \{0\} \times Y$.*

Proof. At first we will show that (i) implies (ii). Indeed, by Lemma 4.1 we have $W^s_y(0) = E \times Y$ or any $y \in Y$. Since the space Y is compact, then the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ is pointwise dissipative and $\Omega_X \subseteq \Theta := \{0\} \times Y$. Since the trivial motion $u = 0$ of the cocycle φ

is uniformly stable, then by Theorems 2.23 and 2.28 the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative and $J = \Theta$.

Now we will establish the converse statement. Suppose that the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ is compactly dissipative and its Levinson center $J = \Theta$. Since the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ is asymptotically compact then by Theorem 2.12 it is locally dissipative. To finish the proof it is sufficient to apply Theorem 4.4. Theorem is completely proved. □

Lemma 5.5. *Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre E, then the following statements hold:*

1. *the trivial motion $u = 0$ of the cocycle φ is positively uniformly stable if and only if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(u) < \delta$ implies $\rho(\varphi(t, u, y)) < \varepsilon$ for any $(t, u) \in \mathbb{T}_+ \times Y$;*
2. $\lim\limits_{t\to+\infty} |\varphi(t, u, y)| = 0$ *if and only if* $\lim\limits_{t\to+\infty} \rho(\varphi(t, u, y)) = 0$.

Proof. Assume that the trivial motion of the cocycle φ is uniformly stable, then for arbitrary $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(u) < \delta$ implies $\rho(\varphi(t, u, y)) < \varepsilon$ for any $(t, u) \in \mathbb{T}_+ \times Y$. If we suppose that it is not true, then there exist $\varepsilon_0 > 0$, $\delta_k \to 0$ $(\delta_k > 0)$, $\rho(u_k) < \delta_k$ $(u_k \in E)$, $(t_k, y_k) \in \mathbb{T}_+ \times Y$ such that

$$\rho(\varphi(t_k, u_k, y_k)) \geq \varepsilon_0 \tag{5.3}$$

for any $k \in \mathbb{N}$. Let $a, b \in \mathcal{K}_\infty$ be the functions figuring in (3.9), then from (5.3) we obtain

$$0 < a(\varepsilon_0) \leq a(\rho(\varphi(t_k, u_k, y_k))) \leq |\varphi(t_k, u_k, y_k)|. \tag{5.4}$$

On the other hand by uniform stability of the trivial motion $u = 0$ for φ we can choose a positive number $\delta(\varepsilon_0)$ such that

$$|\varphi(t, u, y)| < a(\varepsilon_0)$$

for any $|u| < \delta(\varepsilon_0)$ and $(t, y) \in \mathbb{T}_+ \times Y$. Note that $|u_k| \leq b(\rho(u_k)) < b(\delta_k) \to 0$ as $k \to \infty$ and, consequently, there exists a number $k_0 \in \mathbb{N}$ such $|u_k| < \delta(\varepsilon_0)$ for any $k \geq k_0$. Thus we have

$$|\varphi(t, u_k, y)| < a(\varepsilon_0) \tag{5.5}$$

for any $k \geq k_0$ and $(t, y) \in \mathbb{T}_+ \times Y$. In particular, from (5.5) we receive

$$|\varphi(t_k, u_k, y_k)| < a(\varepsilon_0) \tag{5.6}$$

for any $k \geq k_0$. The inequalities (5.4) and (5.6) are contradictory. The obtained contradiction proves our statement. The converse statement can be proved using absolutely the same arguments as above.

Let $(u, y) \in E \times Y$ be so that

$$\lim_{t\to+\infty} |\varphi(t, u, y)| = 0. \tag{5.7}$$

Since $a(\rho(\varphi(t, u, y))) \leq |\varphi(t, u, y)|$ then

$$\rho(\varphi(t, u, y)) \leq a^{-1}(|\varphi(t, u, y)|) \tag{5.8}$$

for any $(t, u, y) \in \mathbb{T}_+ \times E \times Y$. Passing to the limit in (5.8) as $t \to +\infty$ and taking into account (5.7) we obtain $\lim\limits_{t\to+\infty} \rho(\varphi(t, u, y)) = 0$. Let now $\lim\limits_{t\to+\infty} \rho(\varphi(t, u, y)) = 0$, then $|\varphi(t, u, y)| \leq b(\rho(\varphi(t, u, y)))$ and, consequently, $\lim\limits_{t\to+\infty} |\varphi(t, u, y)| = 0$. Lemma is completely proved. □

Theorem 5.6. *Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is λ-homogeneous of the degree zero;*
2. *the space Y is compact and invariant;*
3. *the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$, generated by the cocycle φ, is locally compact.*

Then the following statements are equivalent:

a. *the trivial motion of the cocycle φ is attracting;*
b. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in E$, $y \in Y$ and $t \geq 0$.

Proof. To prove the theorem it is sufficient to establish the implication $a. \Rightarrow b.$, since the converse statement is obvious.

Since the cocycle φ is λ-homogeneous of the degree zero and the trivial motion $u = 0$ is attracting, then from Lemmas 4.1 and 5.5 we have $W^s_y(0) = E$ for any $y \in Y$. Consider the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ. Note that Y is a compact metric space and $W^s_y(0) = E$ for any $y \in Y$. According to Theorem 5.3 we conclude that the dynamical system $(X, \mathbb{T}_+, \pi)$ is locally dissipative and its Levinson center $J \subseteq \Theta$. Since the dynamical system $(X, \mathbb{T}_+, \pi)$ is locally compact, then by Theorem 2.10 $(X, \mathbb{T}_+, \pi)$ is locally compact. By Theorem 4.4 the trivial motion $u = 0$ of the cocycle φ is uniformly attracting and by Lemma 3.10 it is uniformly asymptotically stable. Now to finish the proof of the theorem it suffices apply Theorem 3.28. Theorem is proved. □

Theorem 5.7. *Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is λ-homogeneous of the degree zero;*
2. *the space Y is compact and invariant;*
3. *the skew-product dynamical system $(X, \mathbb{T}, \pi)$, generated by the cocycle φ, is asymptotically compact.*

Then the following statements are equivalent:

a. *the trivial motion of the cocycle φ is asymptotically stable;*
b. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in E$, $y \in Y$ and $t \geq 0$.

Proof. To prove the theorem it is sufficient to establish the implication $a. \Rightarrow b.$

Since the cocycle φ is λ-homogeneous of the degree zero and the trivial motion $u = 0$ is asymptotically stable, then from Lemmas 4.1 and 5.5 we have $W^s_y(0) = E$ for any $y \in Y$. Consider the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ. Taking into account that Y is a compact space and $W^s_y(0) = E$ (for any $y \in Y$) according to Theorem 2.23 the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative and its Levinson center $J \subseteq \Theta := \{0\} \times Y$. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact, then by Theorem 2.12 $(X, \mathbb{T}, \pi)$ is locally dissipative. According to Theorem 4.4 the trivial motion of the cocycle φ is uniformly attracting and by Lemma 3.10 it is uniformly asymptotically stable. Now to finish the proof of the theorem it is sufficient to apply Theorem 3.28. Theorem is proved. □

6. ASYMPTOTIC STABILITY OF INFINITE-DIMENSIONAL NONAUTONOMOUS GENERALIZED HOMOGENEOUS DYNAMICAL SYSTEMS: THE CASE OF THE COMPACT AND MINIMAL PHASE SPACE OF DRIVING SYSTEM

In this Section we suppose that the complete metric space Y is compact and the two-sided dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, i.e., every trajectory $\Sigma_y := \{\sigma(t, y) : \ t \in \mathbb{S}\}$ is dense in Y (this means that $H(y) = Y$ for any $y \in Y$, where $H(y) := \overline{\Sigma}_y$).

Theorem 6.1. [15, Ch.II, pp.94-95] *Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over two-sided dynamical system $(Y, \mathbb{S}, \sigma)$ with the fibre E. Assume that the following conditions are fulfilled:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*
2. *there exist a positive number δ_0 and a point $y_0 \in Y$ such that $B(0, \delta_0) \subset W^s_{y_0}(0)$, where $B(0, r) := \{u \in E|\ |u| < r\}$;*
3. *the skew-product dynamical system $(X, \mathbb{S}_+, \sigma)$, generated by the cocycle φ, is asymptotically compact.*

Then the trivial motion $u = 0$ of the cocycle φ is asymptotically stable, i.e., there exists a positive number β such that $B(0, \beta) \subset W^s_y(0)$ for any $y \in Y$.

Theorem 6.2. *Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over two-sided dynamical system $(Y, \mathbb{S}, \sigma)$ with the fibre E. Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is λ-homogeneous of the degree zero;*
2. *the skew-product dynamical system $(X, \mathbb{S}_+, \sigma)$, generated by the cocycle φ, is asymptotically compact;*
3. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*
4. *there exit a point $y_0 \in Y$ and a positive number δ_{y_0} such that $B(0, \delta_{y_0}) \subset W^s_{y_0}(0)$.*

Then the trivial motion $u = 0$ of the cocycle φ is globally asymptotically stable, i.e., $W^s_y(0) = E$ for any $y \in Y$.

Proof. By Theorem 6.1 there exists a positive number δ_0 such that $B(0, \delta_0) \subset W^s_y(0)$ for any $y \in Y$. According to Lemma 4.1 we have $W^s_y(0) = E$ for any $y \in Y$. Theorem is proved. □

Theorem 6.3. *Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a λ-homogeneous cocycle of the degree zero over two-sided dynamical system $(Y, \mathbb{S}, \sigma)$. Assume that the space Y is compact and the skew-product dynamical system $(X, \mathbb{S}_+, \pi)$, generated by the cocycle φ, is asymptotically compact.*

Then the following statements are equivalent:

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable and there exists a point $y_0 \in Y$ and a positive number δ_{y_0} such that $B(0, \delta_{y_0}) \subset W^s_{y_0}(0)$;*
2. *there exist positive numbers $\mathcal{N}$ and ν such that $\rho(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho(u)$ for any $u \in E, y \in Y$ and $t \geq 0$.*

Proof. According to Theorem 6.2 under the conditions of Theorem 6.3 the trivial motion $u = 0$ of the cocycle φ is (globally) asymptotically stable. To finish the proof of Theorem it is sufficient to apply Theorem 5.7. □

7. APPLICATIONS

7.1. Functional Differential Equations.

7.1.1. *Functional-differential equations with finite delay.* Let us first recall some notions and notations from [23]. Let $r > 0$, $C([a,b],\mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a,b] = [-r,0]$, then we set $\mathcal{C} := C([-r,0],\mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \ge 0$ and $u \in C([\sigma - r, \sigma + A],\mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for any $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t+\theta)$, $-r \le \theta \le 0$. Consider a functional differential equation

$$u'(t) = f(t, u_t), \tag{7.1}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ the space of all continuous mappings $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ equipped with the compact-open topology. On the space $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ is defined (see, e.g. [16, ChI]) a shift dynamical system $(C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n),\mathbb{R},\sigma)$, where $\sigma(\tau, f) := f^\tau$ for any $f \in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ and $\tau \in \mathbb{R}$ and f^τ is τ-translation of f, i.e. $f^\tau(t,u) := f(t+\tau,u)$ for any $(t,u) \in \mathbb{R}\times\mathcal{C}$. Let us set $H(f) := \overline{\{f^\tau : \tau \in \mathbb{R}\}}$.

Along with the equation (7.1) let us consider the family of equations

$$v'(t) = g(t, v_t), \tag{7.2}$$

where $g \in H(f)$.

Definition 7.1. The function f is said to be *regular* if for every equation (7.2) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

Remark 7.2. *Denote by $\tilde{\varphi}(t,u,f)$ the solution of equation (7.1) defined on $[-r,+\infty)$ (respectively, on $\mathbb{R}$) with the initial condition $u \in \mathcal{C}$. By $\varphi(t,u,f)$ we will denote below the trajectory of the equation (7.1), corresponding to the solution $\tilde{\varphi}(t,u,f)$, i.e., a mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t,u,f)(s) := \tilde{\varphi}(t+s,u,f)$ for any $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r,0]$. Below we will use the notions of "solution" and "trajectory" for equation (7.1) as synonymous concepts.*

It is well-known [3, 16] that the mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times H(f) \to \mathcal{C}$ possesses the following properties:

1. $\varphi(0,v,g) = v$ for any $v \in \mathcal{C}$ and $g \in H(f)$;
2. $\varphi(t+\tau,v,g) = \varphi(t,\varphi(\tau,v,g),\sigma(\tau,g))$ for any $t,\tau \in \mathbb{R}_+$, $v \in \mathcal{C}$ and $g \in H(f)$;
3. the mapping φ is continuous.

Thus the equation (7.1) generates a cocycle $\langle \mathcal{C},\varphi,(Y,\mathbb{R},\sigma)\rangle$ and an NDS $\langle (X,\mathbb{R}_+,\pi),\ (Y,\mathbb{R},\sigma),h\rangle$, where $Y := H(f)$, $X := \mathcal{C}\times Y$, $\pi := (\varphi,\sigma)$ and $h := pr_2 : X \to Y$.

Denote by $C(\mathbb{T}\times\mathcal{C},\mathbb{R}^n)$ the family of all continuous functions $f : \mathbb{T}\times\mathcal{C} \to \mathbb{R}^n$ equipped with the compact-open topology. Denote by $(C(\mathbb{T}\times\mathcal{C},\mathbb{R}^n),\mathbb{T},\sigma)$ the shift dynamical system (or called Bebutov dynamical system), i.e., $\sigma(\tau,f) := f^\tau$, where $f^\tau(t,x) := f(t+\tau,x)$ for any $(t,x) \in \mathbb{T}\times\mathbb{R}^n$.

We will say that the function $f \in C(\mathbb{T}\times\mathcal{C},\mathbb{R}^n)$ possesses the property (A), if the motion $\sigma(t.f)$ possesses this property in the shift dynamical system $(C(\mathbb{T}\times\mathcal{C},\mathbb{R}^n),\mathbb{T},\sigma)$. As the property (A) we will consider the Lagrange stability, periodicity in time (respectively, almost periodicity, recurrence and so on).

Note that the function $f \in C(\mathbb{T}\times\mathcal{C},\mathbb{R}^n)$ is Lagrange stable if and only if the function $f_K := f_{|\mathbb{T}\times K}$ is bounded and uniformly continuous on $\mathbb{T}\times K$ for any compact subset K from $\mathcal{C}$ (see, e.g., [31], [32, ChIV]).

Definition 7.3. Following [21] for some $r = (r_1,\dots,r_n) \in (0,+\infty)^n$ we define:

1.
$$\mathbf{\Lambda}^r_\mu \phi := (\mu^{r_1}\phi_1, \dots, \mu^{r_n}\phi_n)$$
for any $\mu > 0$ and $\phi \in \mathcal{C}$, where $\phi = (\phi_1, \dots, \phi_n)$ and $\phi_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$;

2. the mapping $\| \cdot \|_r : \mathcal{C} \to \mathbb{R}_+$ is defined by the equality
$$\|\phi\|_p := \big(\sum_{i=1}^n \|\phi_i\|^{p/r_i}\big)^{1/p},$$
where $p \geq \max\{r_i |\ 1 \leq i \leq n\}$.

Remark 7.4. For any $\mu > 0$ the mapping $\mathbf{\Lambda}^r_\mu : \mathcal{C} \to \mathcal{C}$ is a linear bounded operator and

1.
$$\|\mathbf{\Lambda}^r_\mu\| \leq \mu^k$$
for any $\mu > 0$, where $k := \max\{r_i |\ 1 \leq i \leq n\}$;

2.
$$\|\mathbf{\Lambda}^r_\mu\| \geq \mu^\nu$$
for any $\mu > 0$, where $\nu := \min\{r_i |\ 1 \leq i \leq n\}$.

This statement follows directly from the definition above.

It is easy to check that:

1.
$$\|\mathbf{\Lambda}^r_\mu \phi\|_p = \mu \|\phi\|_p \tag{7.3}$$
for any $r \in (0, +\infty)^n$, $\mu > 0$ and $\phi \in \mathcal{C}$;

2. the mapping $\phi \to \|\phi\|_p$ is continuous.

There are [21] two functions $a, b \in \mathcal{K}_\infty$ such that
$$a(\|\varphi\|_p) \leq \|\phi\| \leq b(\|\phi\|_p)$$
for any $\phi \in \mathcal{C}$.

Definition 7.5. A function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is said to be r homogeneous ($r \in (0, +\infty)^n$) of a degree $m \in \mathbb{R}$ [21] if $f(t, \mathbf{\Lambda}^r_\mu \phi) = \lambda^m \Lambda^r_\mu f(t, \phi)$ for any $(t, \mu, \phi) \in \mathbb{R} \times (0, +\infty) \times \mathcal{C}$.

Remark 7.6. If the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is r homogeneous of the degree $m \geq 0$, then $f(t, 0) = 0$ for any $t \in \mathbb{R}$.

Definition 7.7. A function $F \in C(Y \times \mathcal{C}, \mathbb{R}^n)$ is said to be r-homogeneous of the degree zero if
$$F(y, \mathbf{\Lambda}^r_\mu u) = \Lambda^r_\mu F(y, u)$$
for any $(y, \mu, u) \in Y \times (0, +\infty) \times \mathcal{C}$.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on Y induced from $(C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma)$, i.e., $\sigma(\tau, g) = g^\tau$ for $\tau \in \mathbb{R}$ and $g \in Y$. Then the equation (7.1) generates a cocycle $\langle \mathcal{C}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ and a NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X := \mathcal{C} \times Y$, $\pi := (\varphi, \sigma)$ and $h = pr_2 : X \to Y$.

Remark 7.8. Let F be a mapping from $H(f) \times \mathcal{C} \to \mathbb{R}^n$ defined by the equality
$$F(g, x) = g(0, x) \tag{7.4}$$
for any $(g, x) \in H(f) \times \mathcal{C}$, then F possesses the following properties:

1. F is continuous;
2. $F(g^\tau, x) = g(\tau, x)$ for any $(g, x, \tau) \in H(f) \times \mathcal{C} \times \mathbb{R}$;
3. the equation (7.1) (and its H-class (7.2)) can be rewritten as follows

$$u'(t) = F(\sigma(t, g), u_t) \quad (g \in H(f)).$$

Lemma 7.9. *If the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is r homogeneous of the degree m, then the mapping $F : Y \times \mathcal{C} \to \mathbb{R}^n$ ($Y = H(f)$) defined by the equality (7.4) is also r homogeneous of the degree m with respect to $u \in \mathcal{C}$ uniformly in $y \in Y$.*

Proof. Let $g \in H(f)$, then there exists a sequence $\{t_k\} \subset \mathbb{R}$ such that

$$g(t, u) = \lim_{k\to\infty} f(t + t_k, u)$$

uniformly with respect to (t, u) on every compact subset from $\mathbb{R} \times \mathcal{C}$. Notice that

$$F(g, \mathbf{\Lambda}^r_\mu u) = \lim_{k\to\infty} f(t + t_k, \mathbf{\Lambda}^r_\mu u) = \lambda^m \Lambda^r_\mu \lim_{k\to\infty} f(t + t_k, u) = \lambda^m \Lambda^r_\mu F(g, u)$$

for any $(\mu, g, u) \in (0, +\infty) \times H(f) \times \mathcal{C}$. Lemma is proved. □

Example 7.10. Let $(Y, \mathbb{T}, \sigma)$ be a dynamical system on the metric space Y and $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$. Consider a functional-differential equation

$$u' = F(\sigma(t, y), u_t), \ (y \in Y) \tag{7.5}$$

where $F \in C(Y \times \mathcal{C}, \mathbb{R}^n)$ is a regular function, i.e., for any $(u, y) \in \mathcal{C} \times Y$ there exists a unique solution $\varphi(t, u, y)$ of the equation (7.5) defined on $\mathbb{R}_+$ with initial data $\varphi(0, u, y) = u$. Then (see, for example, [3] and [16]) it is well defined the continuous mapping $\varphi : \mathbb{R}_+ \times \mathcal{C} \times Y \to E$ satisfying the condition $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for any $t, \tau \in \mathbb{R}_+$ and $(u, y) \in \mathcal{C} \times Y$. Then the triplet $\langle \mathcal{C}, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathcal{C}$ (shortly φ) generated by (7.5).

Lemma 7.11. *Assume that the function $F \in C(Y \times \mathcal{C}, \mathbb{R}^n)$ is r-homogeneous of the degree zero, then the cocycle φ generated by (7.5) is also r-homogeneous of the degree zero.*

Proof. To prove this statement we consider the functions $\psi(t) := \mathbf{\Lambda}^r_\mu \varphi(t, u, y)$ and $\tilde{\psi}(t) = \Lambda^r_\mu \tilde{\varphi}(t, u, y)$ for any $(\mu, t, u, y) \in (0, +\infty) \times \mathbb{R}_+ \times \mathcal{C} \times Y$. It is easy to check that

$$\tilde{\psi}'(t) = \Lambda^r_\mu \tilde{\varphi}'(t, u, y) = \Lambda^r_\mu F(\sigma(t, y), \varphi(t, u, y)) =$$
$$F(\sigma(t, y), \mathbf{\Lambda}^r_\mu \varphi(t, u, y)) = F(\sigma(t, y), \psi(t))$$

for any $t \in \mathbb{R}_+$. Since $\psi(0) = \mathbf{\Lambda}^r_\mu u$, then we obtain $\psi(t) = \varphi(t, \mathbf{\Lambda}^r_\mu u, y)$, i.e., $\mathbf{\Lambda}^r_\mu \varphi(t, u, y) = \varphi(t, \mathbf{\Lambda}^r_\mu u, y)$ for any $(\mu, t, u, y) \in (0, +\infty) \times \mathbb{R}_+ \times \mathcal{C} \times Y$. Lemma is proved. □

Remark 7.12. Lemma 7.11 for autonomous equations (7.5) (i.e., if the space Y consists of a single point) was proved in the work [21].

Corollary 7.13. *Assume that the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is r homogeneous of the degree zero, then the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (7.1) is also r homogeneous of the degree zero.*

Proof. This statement follows from Lemmas 7.9 and 7.11. □

Let $f(t, 0) \equiv 0$ and the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be regular.

Definition 7.14. The trivial solution of the equation (7.1) is said to be:

1. *uniformly stable*, if for any positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $\|x\| < \delta$ implies $\|\varphi(t, x, f^{\tau})\| < \varepsilon$ for any $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$;
2. *attracting (respectively, uniformly attracting)*, if there exists a positive number a

$$\lim_{t\to+\infty} \|\varphi(t, x, f^{\tau})\| = 0$$

   for any (respectively, uniformly with respect to) $\|x\| \le a$ and $\tau \in \mathbb{R}$;
3. *asymptotically stable (respectively, uniformly asymptotically stable*, if it is uniformly stable and attracting (respectively, uniformly attracting).

Remark 7.15. If the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is regular and $f(t, 0) = 0$ for any $t \in \mathbb{R}$, then it is easy to show that the trivial solution of equation (7.1) is uniformly attracting if and only if there exists a positive number a such that

$$\lim_{t\to+\infty} \sup_{\|x\|\le a,\ g\in H(f)} \|\varphi(t, u, g)\| = 0.$$

Remark 7.16. 1. Using the same arguments as in the works [1],[29] it is possible to establish the equivalence of standard definition (see, for example, [23, Ch.V]) of uniform stability (respectively, global uniform asymptotically stability) and of the one given above for the equation (7.1) with regular right hand side.

2. Reasoning as in the works of G. Sell [29], [30] we can prove that for the functional-differential equations (7.1) with the regular and Lagrange stable right hand site f the following statements are equivalent:

1. the trivial solution of the equation (7.1) is uniformly asymptotically stable;
2. the trivial motion of the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by (7.1) is uniformly asymptotically stable.

Lemma 7.17. *Let $r \in (0, +\infty)^n$ and $\lambda : \mathbb{R} \times \mathcal{C} \to \mathcal{C}$ be the mapping defined by the equality $\lambda(\tau, \phi) := \mathbf{\Lambda}^r_{e^\tau}\phi$ for any $(\tau, \phi) \in \mathbb{R} \times \mathcal{C}$. Then the following statements hold:*

1. *$\lambda(0, \phi) = \phi$ for any $\phi \in \mathcal{C}$;*
2. *$\lambda(\tau_1 + \tau_2, \phi) = \lambda(\tau_2, \lambda(\tau_1, \phi))$ for any $\tau_1, \tau_2 \in \mathbb{R}$ and $\phi \in \mathcal{C}$;*
3. *the mapping $\lambda : \mathbb{R} \times \mathcal{C} \to \mathcal{C}$ is continuous;*
4. *$\lambda(\tau, \alpha\phi_1 + \beta\phi_2) = \alpha\lambda(\tau, \phi_1) + \beta\lambda(\tau, \phi_2)$ for any $\tau \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and $\phi_1, \phi_2 \in \mathcal{C}$;*
5. $$\|\lambda^{\tau}\| = \|\mathbf{\Lambda}^r_{e^\tau}\|$$
   for any $\tau \in \mathbb{R}$;
6. $$\|\lambda^{\tau}\| \to 0$$
   as $\tau \to -\infty$;
7. $$\|\lambda^{\tau}\| \to +\infty$$
   as $\tau \to +\infty$.

Proof. To prove this Lemma it is sufficient to establish the third statement because the statements (i), (ii) and (iv) are evident. The statements (v)–(vii) follow from Remark 7.4.

Let $\tau_k \to \tau$ and $\phi^k \to \phi$ as $k \to \infty$. Note that

$$\|\lambda(\tau_k, \phi^k) - \lambda(\tau, \phi)\| = \max_{-r \le s \le 0} \sqrt{\sum_{i=1}^{n} |e^{r_i \tau_k} \phi_i^k(s) - e^{r_i \tau} \phi_i(s)|^2} \tag{7.6}$$

and

$$\begin{aligned} |e^{r_i\tau_k}\phi^{k_i}(s) - e^{r_i\tau}\phi_i(s)| &\le |e^{r_i\tau_k}\phi^{k_i}(s) - e^{r_i\tau_k}\phi_i(s)| + |e^{r_i\tau_k}\phi_i(s) - e^{r_i\tau}\phi_i(s)| \le \\ & m\|\phi_i^k - \phi_i\| + |e^{r_i\tau_k} - e^{r_i\tau}|\|\phi_i\| := \omega_i(k), \end{aligned} \tag{7.7}$$

where

$$m := \max_{1 \le i \le n} \sup\{e^{r_i\tau_k} |\ k \in \mathbb{N}\} < +\infty.$$

It follows from (7.7) that

$$\lim_{k \to \infty} \omega_i(k) = 0 \tag{7.8}$$

for any $1 \le i \le n$.

From (7.6) and (7.7) we obtain

$$\|\lambda(\tau_k, \phi^k) - \lambda(\tau, \phi)\| \le \big(\sum_{i=1}^{n} \omega_i^2(k)\big)^{1/2} \tag{7.9}$$

for any $k \in \mathbb{N}$. Passing to the limit in (7.9) as $k \to \infty$ and taking into account (7.8) we obtain

$$\lim_{k \to \infty} \|\lambda(\tau_k, \phi^k) - \lambda(\tau, \phi)\| = 0.$$

Lemma is completely proved. □

Lemma 7.18. *Let $\lambda^\tau : \mathcal{C} \to \mathcal{C}$ be the mapping defined by the equality $\lambda^\tau \phi := \lambda(\tau, \phi)$ for any $\phi \in \mathcal{C}$. Then we have:*

1.
$$\|\lambda^\tau\| \le e^{k\tau} \tag{7.10}$$
for any $\tau < 0$, where $k := \max_{1 \le i \le n} r_i$;
2.
$$\|\lambda^\tau\| \ge e^{\nu\tau} \tag{7.11}$$
for any $\tau > 0$, where $\nu := \min_{1 \le i \le n} r_i$;
3.
$$\rho(\lambda(\tau, \phi)) = e^\tau \rho(\phi)$$
for any $(\tau, \phi) \in \mathbb{R} \times \mathcal{C}$, where $\rho(\phi) := \|\phi\|_p$ ($\phi \in \mathcal{C}$ and $p \ge \max_{1 \le i \le n} r_i$).

Proof. Note that

$$\|\lambda^\tau\| = \sup_{\|\phi\| \le 1} \|\mathbf{\Lambda}^r_{e^\tau} \phi\| = \|\mathbf{\Lambda}^r_{e^\tau}\| \tag{7.12}$$

for any $\tau \in \mathbb{R}$. From (7.12) and (7.3) (respectively (7.3)) we obtain (7.10) (respectively (7.11)).

The second statement follows from (7.3) because $e^\tau = \mu$. □

Lemma 7.19. *If $H(f)$ is compact, then for any point $x \in X := \mathcal{C} \times H(f)$ there exist a neighborhood U_x of the point x and a positive number $l_x > 0$ such that $\pi(l_x, U_x)$ is precompact, i.e., the dynamical system $(X, \mathbb{R}_+, \pi)$ is locally compact.*

Proof. This statement follows from Lemma 3.6.1 and Corollary 3.6.2 in [23, Ch. III] and from the compactness of $H(f)$. □

Theorem 7.20. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and $f(t, 0) = 0$ for any $t \in \mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.1) is uniformly asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in \mathcal{C}$, $g \in H(f)$ and $t \geq 0$, where $\rho(u) = \|u\|_p$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Denote by $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$ the cocycle generated by the functional-differential equation (7.1). Since the function f is r homogeneous of the degree zero, then by Corollary 7.13 the cocycle φ generated by the equation (7.1) is r homogeneous of the degree zero. To finish the proof of Theorem 7.20 it suffices to take Remarks 7.15 and 7.16 into account and apply Theorem 3.28. □

Theorem 7.21. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *$f(t, 0) = 0$ for any $t \in \mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero and Lagrange stable.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.1) is asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in \mathcal{C}$, $g \in H(f)$ and $t \geq 0$, where $\rho(u) = \|u\|_p$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is compact because the function f is Lagrange stable. Since the function f is r homogeneous of the degree zero, then by Corollary 7.13 the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$ generated by the equation (7.1) is also r homogeneous of the degree zero. To finish the proof of Theorem 7.21 it suffices to take Remark 7.16 into account and apply Theorem 5.6. □

Remark 7.22. If the function f is homogeneous of the degree zero (in the classical sense, i.e., $f(t, \mu x) = \mu f(t, x)$ for any $\mu > 0$ and $(t, x) \in \mathbb{R} \times \mathcal{C}$), then the equivalence of the uniform asymptotically stability and exponential stability was established in the work [11].

Recall [31] that the function $f \in C(\mathbb{T} \times \mathcal{C}, \mathbb{R}^n)$ is said to be recurrent in time if the motion $\sigma(t, f)$ generated by f in the shift dynamical system $(C(\mathbb{T} \times \mathcal{C}, \mathbb{R}^n), \mathbb{T}, \sigma)$ is recurrent.

Remark 7.23. Note that the function f is recurrent in time if and only if its hull $H(f)$ is a compact and minimal set of the shift dynamical system $(C(\mathbb{T} \times \mathcal{C}, \mathbb{R}^n), \mathbb{T}, \sigma)$ (see, for example, [17, Ch.I]).

Theorem 7.24. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *the function f is recurrent in time and $f(t,0)=0$ for any $t\in\mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.1) is uniformly stable and there exists a positive number a such that*
$$\lim_{t\to+\infty}|\varphi(t,u,f)|=0$$
for any $u\in B[0,a]:=\{u\in\mathcal{C}|\ \|u\|\le a\}$;
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g))\le\mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathcal{C}$, $g\in H(f)$ and $t\ge 0$.

Proof. Let $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ be the shift dynamical system on $Y=H(f)$. Note that the space Y is a compact and minimal set because the function f is recurrent in time (see Remark 7.23). Since the function f is r homogeneous of the degree zero, then by Corollary 7.13 the cocycle $\langle\mathcal{C},\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by the equation (7.1) is r homogeneous of the degree zero. To finish the proof of Theorem 7.24 it suffices to take Remark 7.16 into account and apply Theorem 6.3. □

7.1.2. *Neutral functional differential equations.* Let $\mathfrak{A}=\mathfrak{A}(\mathcal{C},\mathbb{R}^n)$ be the Banach space of all linear operators that act from $\mathcal{C}$ into $\mathbb{R}^n$ equipped with the operator norm. Now consider the neutral functional-differential equation

$$\frac{d}{dt}Du_t=F(t,u_t),\tag{7.13}$$

where $F\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ and the operator $D\in\mathfrak{A}$ is atomic at zero [23, p.49-50]. Like (7.1), equation (7.13) generates a cocycle $\langle\mathcal{C},\varphi,(Y,\mathbb{R},\sigma)\rangle$ over dynamical system $(Y,\mathbb{R},\sigma)$ with the fibre $\mathcal{C}$, where $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ is the shift dynamical system on $H(f)$.

Lemma 7.25. [16, Ch.XIII] *Let $H(f)$ be compact, and assume that the operator D is stable, that is, the zero solution of the homogeneous difference equation $Dy_t=0$ is uniformly asymptotically stable* [23, Ch.XII]. *Then the linear non-autonomous dynamical system $(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by the equation (7.13) is asymptotically compact.*

Along with the equation (7.13) let us consider the family of equations

$$\frac{d}{dt}Dv_t=g(t,v_t),\tag{7.14}$$

where $g\in H(f)$.

Recall that the function f is said to be *regular*, that is, for every equation (7.14) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

Remark 7.26. *Denote by $\tilde{\varphi}(t,u,f)$ the solution of equation (7.13) defined on $[-r,+\infty)$ (respectively, on $\mathbb{R}$) with the initial condition $u\in\mathcal{C}$. By $\varphi(t,u,f)$ we will denote below the trajectory of equation (7.13), corresponding to the solution $\tilde{\varphi}(t,u,f)$, i.e., a mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t,u,f)(s):=\tilde{\varphi}(t+s,u,f)$ for any $t\in\mathbb{R}_+$ (respectively, $t\in\mathbb{R}$) and $s\in[-r,0]$. Below we will use the notions of "solution" and "trajectory" for equation (7.13) as synonymous concepts.*

It is well-known [16] that the mapping $\varphi:\mathbb{R}_+\times\mathcal{C}\times H(f)\to\mathcal{C}$ possesses the following properties:

1. $\varphi(0,v,g)=v$ for any $v\in\mathcal{C}$ and $g\in H(f)$;

2. $\varphi(t+\tau,v,g)=\varphi(t,\varphi(\tau,v,g),\sigma(\tau,g))$ for any $t,\tau\in\mathbb{R}_+$, $v\in\mathcal{C}$ and $g\in H(f)$;
3. the mapping φ is continuous.

Thus the equation (7.13) generates a cocycle $\langle\mathcal{C},\varphi,(Y,\mathbb{R},\sigma)\rangle$ and a NDS $\langle(X,\mathbb{R}_+,\pi),\ (Y,\mathbb{R},\sigma),h\rangle$, where $Y:=H(f)$, $X:=\mathcal{C}\times Y$, $\pi:=(\varphi,\sigma)$ and $h:=pr_2:X\to Y$.

Denote by $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ the shift dynamical system on Y induced from $(C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n),\mathbb{R},\sigma)$, i.e., $\sigma(\tau,g)=g^{\tau}$ for $\tau\in\mathbb{R}$ and $g\in Y$. Then the equation (7.13) generates a cocycle $\langle\mathcal{C},\varphi,(Y,\mathbb{R},\sigma)\rangle$ and a NDS $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X:=\mathcal{C}\times Y$, $\pi:=(\varphi,\sigma)$ and $h=pr_2:X\to Y$.

Remark 7.27. Let F be a mapping from $H(f)\times\mathcal{C}\to\mathbb{R}^n$ defined by the equality

$$F(g,x)=g(0,x)$$

for any $(g,x)\in H(f)\times\mathcal{C}$, then F possesses the following properties:

1. F is continuous;
2. $F(g^{\tau},x)=g(\tau,x)$ for any $(g,x,\tau)\in H(f)\times\mathcal{C}\times\mathbb{R}$;
3. the equation (7.13) (and its H-class (7.14)) can be rewritten as follows

$$\frac{d}{dt}Du_t=F(\sigma(t,g),u_t)\ \ (g\in H(f)).$$

Example 7.28. Let $(Y,\mathbb{T},\sigma)$ be a dynamical system on the metric space Y and $\mathbb{T}=\mathbb{R}_+$ or $\mathbb{R}$. Consider a neutral functional-differential equation

$$\frac{d}{dt}Du_t=F(\sigma(t,y),u_t),\ \ (y\in Y) \tag{7.15}$$

where $F\in C(Y\times\mathcal{C},\mathbb{R}^n)$ is a regular function, i.e., for any $(u,y)\in\mathcal{C}\times Y$ there exists a unique solution $\varphi(t,u,y)$ of the equation (7.15) defined on $\mathbb{R}_+$ with initial data $\varphi(0,u,y)=u$. Then (see, for example, [16]) it is well defined the continuous mapping $\varphi:\mathbb{R}_+\times\mathcal{C}\times Y\to E$ satisfying the condition $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau\in\mathbb{R}_+$ and $(u,y)\in\mathcal{C}\times Y$. Then the triplet $\langle\mathcal{C},\varphi,(Y,\mathbb{T},\sigma)\rangle$ is a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre $\mathcal{C}$ generated by (7.15).

Lemma 7.29. *Assume that the function $F\in C(Y\times\mathcal{C},\mathbb{R}^n)$ and the operator $D\in\mathfrak{A}$ are r-homogeneous of the degree zero, then the cocycle φ generated by (7.15) is also r-homogeneous of the degree zero.*

Proof. Consider the function $\psi(t):=\mathbf{\Lambda}^r_{\mu}\varphi(t,u,y)$ for any $(\mu,t,u,y)\in(0,+\infty)\times\mathbb{R}_+\times\mathcal{C}\times Y$. Since the operator $D\in\mathfrak{A}$ is r-homogeneous of the degree zero, then we have

$$\tfrac{d}{dt}D\psi(t)=\tfrac{d}{dt}D\mathbf{\Lambda}^r_{\mu}\varphi(t,u,y)=\Lambda^r_{\mu}F(\sigma(t,y),\varphi(t,u,y))=$$
$$F(\sigma(t,y),\mathbf{\Lambda}^r_{\mu}\varphi(t,u,y))=F(\sigma(t,y),\psi(t))$$

for any $t\in\mathbb{R}_+$. Since $\psi(0)=\mathbf{\Lambda}^r_{\mu}u$, then we obtain $\psi(t)=\varphi(t,\mathbf{\Lambda}^r_{\mu}u,y)$, i.e., $\mathbf{\Lambda}^r_{\mu}\varphi(t,u,y)=\varphi(t,\mathbf{\Lambda}^r_{\mu}u,y)$ for any $(\mu,t,u,y)\in(0,+\infty)\times\mathbb{R}_+\times\mathcal{C}\times Y$. Lemma is proved. □

Corollary 7.30. *Assume that the function $f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ and the operator $D\in\mathfrak{A}$ are r homogeneous of the degree zero, then the cocycle $\langle\mathcal{C},\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by the equation (7.13) is also r homogeneous of the degree zero.*

Proof. This statement follows from Lemmas 7.29 and 7.9. □

Theorem 7.31. *Let $f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and $f(t,0)=0$ for any $t\in\mathbb{R}$;*

2. *the function f and the operator $D \in \mathfrak{A}$ are r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.13) is uniformly asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,g)) \le \mathcal{N}e^{-\nu t}\rho(u)$$

for any $u \in \mathcal{C}$, $g \in H(f)$ and $t \ge 0$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Denote by $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ the cocycle generated by the neutral functional-differential equation (7.13). Since the function f is r homogeneous of the degree zero, then by Corollary 7.30 the cocycle φ generated by the equation (7.13) is also r homogeneous of the degree zero. To finish the proof of Theorem 7.31 it suffices to take Remarks 7.15 and 7.16 into account and apply Theorem 3.28. □

Theorem 7.32. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *$f(t, 0) = 0$ for any $t \in \mathbb{R}$;*
2. *the operator $D \in \mathfrak{A}$ is r-homogeneous of the degree zero;*
3. *the function f is r homogeneous of the degree zero and Lagrange stable.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.13) is asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,g)) \le \mathcal{N}e^{-\nu t}\rho(u)$$

for any $u \in \mathcal{C}$, $g \in H(f)$ and $t \ge 0$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is compact because the function f is Lagrange stable. Since the function f is r homogeneous of the degree zero, then by Corollary 7.30 the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (7.13) is r homogeneous of the degree zero. To finish the proof of Theorem 7.32 it suffices to take Remark 7.16 into account and apply Theorem 5.7. □

Theorem 7.33. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *the function f is recurrent in time and $f(t, 0) = 0$ for any $t \in \mathbb{R}$;*
2. *the function f and the operator $D \in \mathfrak{A}$ are r-homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.13) is uniformly stable and there exists a positive number a such that*

$$\lim_{t\to+\infty} |\varphi(t,u,f)| = 0$$

for any $u \in B[0,a] := \{u \in \mathcal{C} \mid \|u\| \le a\}$;

2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,g)) \le \mathcal{N}e^{-\nu t}\rho(u)$$

for any $u \in \mathcal{C}$, $g \in H(f)$ and $t \ge 0$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is a compact and minimal set because the function f is recurrent in time (see Remark 7.23). Since the function f is r homogeneous of the degree zero, then by Corollary 7.30 the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (7.13) is also r homogeneous of the degree zero. To finish the proof of Theorem 7.33 it suffices to take Remark 7.16 into account and apply Theorem 6.3. □

7.2. **Semi-Linear Parabolic Equations.** Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the norm $|\cdot|^2 := \langle\cdot,\cdot\rangle$, and A be a self-adjoint operator with domain $D(A)$.

An operator is said (see, for example, [19, Ch.II]) to have a discrete spectrum if in the space H, there exists an orthonormal basis $\{e_k\}$ of eigenvectors, such that $\langle e_k, e_j\rangle = \delta_{kj}$, $Ae_k = \lambda_k e_k$ $(k, j = 1, 2, \dots)$ and $0 < \lambda_1 \le \lambda_2 \le \dots, \lambda_k \le \dots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

One can define an operator $f(A)$ for a wide class of functions f defined on the positive semi-axis as follows:

$$D(f(A)) := \{h = \textstyle\sum_{k=1}^{\infty} c_k e_k \in H : \ \sum_{k=1}^{\infty} c_k [f(\lambda_k)]^2 < +\infty\},$$
$$f(A)h := \textstyle\sum_{k=1}^{\infty} c_k f(\lambda_k) e_k, \quad h \in D(f(A)).$$

In particular, we can define operators A^α for all $\alpha \in \mathbb{R}$. For $\alpha = -\beta < 0$ this operator is bounded. The space $D(A^{-\beta})$ can be regarded as the completion of the space H with respect to the norm $|\cdot|_\beta := |A^{-\beta}\cdot|$.

The following statements hold [19, Ch.II]:

1. The space $\mathcal{F}_{-\beta} := D(A^{-\beta})$ with $\beta > 0$ can be identified with the space of formal series $\sum_{k=1}^{\infty} c_k e_k$ such that
$$\sum_{k=1}^{\infty} c_k \lambda_k^{-2\beta} < +\infty;$$
2. For any $\beta \in \mathbb{R}$, the operator A^β can be defined on every space $D(A^\alpha)$ as a bounded operator mapping $D(A^\alpha)$ into $D(A^{\alpha-\beta})$ such that
$$A^\beta D(A^\alpha) = D(A^{\alpha-\beta}), \ A^{\beta_1+\beta_2} = A^{\beta_1} A^{\beta_2}.$$
3. For all $\alpha \in \mathbb{R}$, the space $\mathcal{F} := D(A^\alpha)$ is a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_\alpha := \langle A^\alpha\cdot, A^\alpha\cdot\rangle$ and the norm $|\cdot|_\alpha := |A^\alpha\cdot|$.
4. The operator A with the domain $\mathcal{F}_{1+\alpha}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_\alpha$.
5. The embedding of the space $\mathcal{F}_\alpha$ into $\mathcal{F}_\beta$ for $\alpha > \beta$ is continuous, i.e., $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ and there exists a positive constant $C = C(\alpha, \beta)$ such that $|\cdot|_\beta \le C|\cdot|_\alpha$.
6. $\mathcal{F}_\alpha$ is dense in $\mathcal{F}_\beta$ for any $\alpha > \beta$.
7. Let $\alpha_1 > \alpha_2$, then the space $\mathcal{F}_{\alpha_1}$ is compactly embedded into $\mathcal{F}_{\alpha_2}$, i.e., every sequence bounded in $\mathcal{F}_{\alpha_1}$ is precompact in $\mathcal{F}_{\alpha_2}$.
8. The resolvent $\mathcal{R}_\lambda(A) := (A - \lambda I)^{-1}$, $\lambda \ne \lambda_k$ is a compact operator in each space $\mathcal{F}_\alpha$, where I is the identity operator.

Let (Y, ρ) be a compact complete metric space and $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y. Consider an evolutionary differential equation

$$u' + Au = F(\sigma(t, y), u) \quad (y \in Y) \tag{7.16}$$

in the separable Hilbert space H, where A is a linear (generally speaking unbounded) positive operator with discrete spectrum, and F is a nonlinear continuous mapping acting from $Y \times \mathcal{F}_\theta$ into H, $0 \le \theta < 1$, possessing the property

$$|F(y,u_1) - F(y,u_2)| \le L(r)|A^\theta(u_1 - u_2)|$$

for all $u_1, u_2 \in B_\theta[0,r] := \{u \in \mathcal{F}_\theta : \ |u|_\theta \le r\}$. Here $L(r)$ denotes the Lipschitz constant of F on the set $B_\theta(0,r)$.

A function $u : [0,a) \mapsto \mathcal{F}_\theta$ is said to be a weak solution (in $\mathcal{F}_\theta$) of the equation (7.16) passing through the point $x \in \mathcal{F}_\theta$ at the initial moment $t = 0$ (notation $\varphi(t,x,y)$) if $u \in C([0,T],\mathcal{F}_\theta)$ and satisfies the integral equation

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-\tau)A}F(\sigma(\tau,y),u(\tau))d\tau$$

for all $t \in [0,T]$ and $0 < T < a$.

According to (7.16) we can define an exponential operator e^{-tA}, $t \ge 0$, in the scale spaces $\{\mathcal{F}_\alpha\}$. Note some of its properties [19, Ch.II]:

a. For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator e^{-tA} maps $\mathcal{F}_\alpha$ into $\bigcap\limits_{\beta \ge 0} \mathcal{F}_\beta$ and

$$|e^{-tA}x|_\alpha \le e^{-\lambda_1 t}|x|_\alpha$$

for any $x \in \mathcal{F}_\alpha$.

b. $e^{-t_1A}e^{-t_2A} = e^{-(t_1+t_2)A}$ for any $t_1, t_2 \in \mathbb{R}_+$;

c.

$$|e^{-tA}x - e^{-\tau A}x|_\beta \to 0$$

as $t \to \tau$ for every $x \in \mathcal{F}_\beta$ and $\beta \in \mathbb{R}$;

d. For any $\beta \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_\beta, e^{-tA})$;

e.

$$|A^\alpha e^{-tA}h| \le \Big[\big(\tfrac{\alpha-\beta}{t}\big)^{\alpha-\beta} + \lambda_1^{\alpha-\beta}\Big]e^{-t\lambda_1}|A^\beta h|, \ \alpha \ge \beta$$
$$||A^\alpha e^{-tA}|| \le \big(\tfrac{\alpha}{t}\big)^\alpha e^{-\alpha}, \ t > 0, \ \alpha > 0.$$

Theorem 7.34. [10] *Let $x_0 \in \mathcal{F}_\theta$, $r > 0$ and the conditions listed above be fulfilled. Then, there exist positive numbers $\delta = \delta(x_0,r)$ and $T = T(x_0,r)$ such that the equation (7.16) admits a unique solution $\varphi(t,x,y)$ ($x \in B_\theta[x_0,\delta] = \{x \in \mathcal{F}_\theta \mid |x - x_0|_\theta \le \delta\}$) defined on the interval $[0,T]$ with the conditions: $\varphi(0,x,y) = x$, $|\varphi(t,x,y) - x_0|_\theta \le r$ for all $t \in [0,T]$ and the mapping $\varphi : [0,T] \times B[x_0,\delta] \times Y \to \mathcal{F}_\theta$ $((t,x,y) \mapsto \varphi(t,x,y))$ is continuous.*

A function $F \in C(Y \times \mathcal{F}_\theta, H)$ is said to be regular, if for any $u \in \mathcal{F}_\theta$ and $y \in Y$ there exists a unique solution $\varphi(t,u,y)$ of equation (7.16) passing through the point u at the initial moment $t = 0$, defined on $\mathbb{R}_+$.

Then it is well defined the continuous mapping $\varphi : \mathbb{R}_+ \times \mathcal{F}_\theta Y \to \mathcal{F}_\theta$ satisfying the condition $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau \in \mathbb{R}_+$ and $(u,y) \in \mathcal{F}_\theta \times Y$. Then the triplet $\langle \mathcal{F}_\theta, \varphi, (Y,\mathbb{T},\sigma)\rangle$ is a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre $\mathcal{C}$ generated by the equation (7.16).

Below everywhere we suppose that the function $F \in C(Y \times \mathcal{F}_\theta, H)$ is regular.

Lemma 7.35. [10] *Let $\langle \mathcal{F}_\theta, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be a cocycle generated by equation (7.16) and $M \subseteq X_\theta := \mathcal{F}_\theta \times Y$ positively invariant (with respect to skew-product dynamical system $(X, \mathbb{R}_+, \pi)$, where $\pi := (\varphi, \sigma)$) and bounded. Then, there exists a precompact set $K \subseteq X_\alpha$ ($\alpha \in (\theta, 1)$) such that*

$$\lim_{t\to+\infty} \beta(\pi(t, M), K) = 0,$$

where $\beta(A, B) := \sup_{a\in B} \rho_\alpha(a, B)$, $\rho_\alpha(a, B) := \inf_{b\in B} \rho_\alpha(a, b)$, $\rho_\alpha(a, b) := \rho(y_a, y_b) + |x_a - x_b|_\alpha$, $a := (x_a, y_a)$ and $b := (x_b, y_b)$.

Corollary 7.36. *Under the conditions of Lemma the cocycle $\langle \mathcal{F}_\theta, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ generated by the equation (7.16) (respectively, the skew-product dynamical system $(X_\alpha, \mathbb{R}_+, \pi)$ with $X_\alpha := \mathcal{F}_\theta \times Y$ and $\pi := (\varphi, \sigma)$) is asymptotically compact.*

Proof. This statement follows directly from Lemma 7.35 and corresponding definition of asymptotically compactness of dynamical systems. □

Lemma 7.37. *Assume that the following statements are fulfilled:*

1. $e^{-At}\lambda^\tau = \lambda^\tau e^{-At}$ *for any* $(t, \tau) \in \mathbb{R}_+ \times \mathbb{R}$;
2. $F(y, \lambda^\tau u) = \lambda^\tau F(u, y)$ *for any* $(\tau, y, u) \in \mathbb{R} \times Y \times \mathcal{F}_\theta$.

Then the cocycle $\langle \mathcal{F}_\theta, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ generated by the equation (7.16) is λ-homogeneous of the degree zero.

Proof. Since

$$\varphi(t, u, y) = e^{-At}u + \int_0^t e^{-A(t-\tau)}F(\sigma(\tau, y), \varphi(\tau, u, y))d\tau \tag{7.17}$$

then we have

$$\varphi(t, \lambda^\tau u, y) = e^{-At}\lambda^\tau u + \int_0^t e^{-A(t-\tau)}F(\sigma(\tau, y), \varphi(\tau\lambda^\tau, u, y))d\tau \tag{7.18}$$

for any $(t, \tau, u, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathcal{F}_\theta \times Y$.

On the other hand from (7.17) under the conditions of Theorem we have

$$\begin{aligned} \lambda^\tau\varphi(t, u, y) = \lambda^\tau e^{-At}u + \textstyle\int_0^t \lambda^\tau e^{-A(t-\tau)}F(\sigma(\tau, y), \varphi(\tau, u, y))d\tau = \\ e^{-At}\lambda^\tau u + \textstyle\int_0^t e^{-A(t-\tau)}F(\sigma(\tau, y), \lambda^\tau\varphi(\tau, u, y))d\tau \end{aligned} \tag{7.19}$$

for any $(t, \tau, u, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathcal{F}_\theta \times Y$.

From (7.18) and (7.19) we obtain

$$\varphi(t, \lambda^\tau u, y) = \lambda^\tau\varphi(t, u, y)$$

for any $(t, \tau, u, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathcal{F}_\theta \times Y$. Lemma is proved. □

Theorem 7.38. *Assume that the function $F \in C(Y \times \mathcal{F}_\theta, H)$ is λ-homogeneous of the degree zero, $e^{-At}\lambda^\tau = \lambda^\tau e^{-At}$ and $F(y, 0) = 0$ for any $(t, \tau, y)) \in \mathbb{R}_+ \times \mathbb{R} \times Y$.*

Then the following conditions are equivalent:

1. *the trivial solution of the equation (7.16) is uniformly asymptotically stable;*
2. *there are positive numbers $\mathcal{N}$ and ν such that $\rho_\theta(\varphi(t, u, y)) \le \mathcal{N}\rho_\theta(u)$ for any $(t, u, y) \in \mathbb{R}_+ \times E_\theta \times Y$, where ρ_θ is a λ-homogeneous norm on the Banach space $\mathcal{F}_\theta$.*

Proof. Denote by $\langle \mathcal{F}_\theta, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ the cocycle generated by the differential equation (7.16). Since the function F is λ-homogeneous of the degree zero and $A\lambda^\tau = \lambda^\tau A$ for any $\tau \in \mathbb{R}$, then by Lemma 7.37 the cocycle φ generated by the equation (7.16) is also λ-homogeneous of the degree zero. To finish the proof of Theorem 7.38 it suffices to take Remarks 7.15 and 7.16 into account and apply Theorem 3.28. □

Theorem 7.39. *Let $F \in C(Y \times \mathcal{F}_\theta, \mathcal{F}_\theta)$ be a regular function. Assume that the following conditions are fulfilled:*

1. $e^{-At}\lambda^\tau = \lambda^\tau e^{-At}$ *for any* $(t, \tau) \in \mathbb{R}_+ \times \mathbb{R}$*;*
2. $\mathcal{F}(y, 0) = 0$ *for any* $y \in Y$*;*
3. *the function F is λ-homogeneous of the degree zero and the space Y is compact.*

Then the following statements are equivalent:

1. *the trivial solution of equation (7.16) is asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho_\theta(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho_\theta(u)$$

for any $u \in \mathcal{F}_\theta$, $G \in H(F)$ and $t \geq 0$.

Proof. Since the space Y is compact and the function F is λ-homogeneous of the degree zero, then by Lemma 7.37 the cocycle $\langle \mathcal{F}_\theta, \varphi, (H(F), \mathbb{R}, \sigma)\rangle$ generated by the equation (7.16) is also λ-homogeneous of the degree zero. To finish the proof of Theorem 7.39 it suffices to take Remark 7.16 into account and apply Theorem 5.6. □

Theorem 7.40. *Let $F \in C(Y \times \mathcal{F}_\theta, \mathcal{F}_\theta)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *Y is a compact and minimal set;*
2. $F(y, 0) = 0$ *for any* $y \in Y$*;*
3. $e^{-At}\lambda^\tau = \lambda^\tau e^{-At}$ *for any* $(t, \tau) \in \mathbb{R}_+ \times \mathbb{R}$*;*
4. *the function F is λ- homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (7.16) is uniformly stable and there exists a positive number a such that*

$$\lim_{t \to +\infty} |\varphi(t, u, y_0)| = 0$$

for any $u \in B[0, a] := \{u \in \mathcal{F}_\theta |\ |u|_\theta \leq a\}$ and for some $y_0 \in Y$;

2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho_\theta(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho_\theta(u)$$

for any $u \in \mathcal{F}_\theta$, $y \in Y$ and $t \geq 0$.

Proof. Note that the space Y is a compact and minimal set. Since the function F is λ-homogeneous of the degree zero, then by Lemma 7.37 the cocycle $\langle \mathcal{F}_\theta, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ generated by the equation (7.16) is λ-homogeneous of the degree zero. To finish the proof of Theorem 7.40 it suffices to take Remark 7.16 into account and apply Theorem 6.3. □

Here is an example illustrating the results formulated in this Subsection.

Example 7.41. Denote by $C(\mathbb{R},\mathbb{R})$ the space of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ equipped with the distance

$$d(f_1,f_2)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{d_k(f_1,f_2)}{1+d_k(f_1,f_2)}$$

which generates the compact-open topology on $C(\mathbb{R},\mathbb{R})$. Let $(C(\mathbb{R},\mathbb{R}),\mathbb{R},\sigma)$ be the shift dynamical system [16, Ch.I] on the space $C(\mathbb{R},\mathbb{R})$. If $p\in C(\mathbb{R},\mathbb{R})$, then by $H(f)$ we denote the closure in $C(\mathbb{R},\mathbb{R})$ of the family of all translations of p, i.e., $H(p):=\overline{\{p^h|\ h\in\mathbb{R}\}}$ ($p^h(t):=p(t+h)$ for any $t\in\mathbb{R}$).

Consider the heat equation on the interval [0,1] with Dirichlet boundary condition:

$$\frac{\partial u(t,x)}{\partial t}=\frac{\partial^2 u(t,x)}{\partial x^2}+p(t)|u(t,x)| \tag{7.20}$$

on the interval [0,1] with Dirichlet boundary condition $u(t,0)=u(t,1)=0,\quad t>0$.

Along with equation (7.20) we consider its H-class, i.e., the family of equations

$$\frac{\partial u(t,x)}{\partial t}=\frac{\partial^2 u(t,x)}{\partial x^2}+q(t)|u(t,x)|\quad (q\in H(p)). \tag{7.21}$$

Let A be the operator defined by $A\varphi(x)=\varphi''(x)$ $(0<x<1)$, then $A:D(A)=H^2(0,1)\cap H_0^1(0,1)\to L^2(0,1)$ (for more details see [24, Ch.I]). Denote $H:=L^2(0,1)$ and the norm on H by $||\cdot||$. Then the equation (7.20) (respectively the family of equations (7.21)) can be written as an abstract evolution equation (respectively as a family of abstract equations)

$$y'(t)=\mathcal{A}y(t)+P(t,y(t))\quad (y'(t)=\mathcal{A}y(t)+Q(t,y(t))$$

on the Hilbert space H, where

$$y(t):=u(t,\cdot),\quad P(t,y(t)):=p(t)|y(t)|) \tag{7.22}$$
$$\text{(respectively }\ Q(t,y):=q(t)|y(t)|,\quad q\in H(p)).$$

Note that $\sigma(A)=\{-n^2\pi^2|\ n\in\mathbb{N}\}$ and the operator A generates a $\mathcal{C}^0$-semigroup $\{U(t)\}_{t\geq 0}=\{e^{At}\}_{t\geq 0}$ on H. Note that

$$Lip(F)\leq L:=\sup\{|p(t)|\ |\ t\in\mathbb{R}\}.$$

It is easy to check that

$$F(t,\mu y)=\mu F(t,y) \tag{7.23}$$

for any $(t,\mu,y)\in\mathbb{R}\times\mathbb{R}_+\times\mathcal{F}_{1/2}$.

According to [19, Ch.II] we have

$$\|F(t,y_1)-F(t,y_2)\|_{L^2(0,1)}\leq L\sqrt{2}\|A^{1/2}(y_1-y_2)\|_{L^2(0,1)} \tag{7.24}$$

for any $y_1,y_2\in L^2(0,1)$ and $t\in\mathbb{R}$.

By Theorem 7.34 (see also Theorem 2.4 from [19, Ch.II]) the equation (7.20) (and its H-class (7.21)) generates a cocycle $\langle\mathcal{F}_{1/2},\varphi,(Y,\mathbb{R},\sigma)\rangle$, where $Y=H(p)$, $(Y,\mathbb{R},\sigma)$ is the shift dynamical system on $H(p)$ and $\varphi:\mathbb{R}_+\times\mathcal{F}_{1/2}\times Y\to\mathcal{F}_{1/2}$ ($(t,u,q)\to\varphi(t,u,q)$ and $\varphi(t,u,q)$ is the unique solution of equation (7.21) with the initial data $\varphi(0,u,q)=u$ and $(u,q)\in\mathcal{F}_{1/2}\times H(p)$).

Lemma 7.42. *The cocycle $\langle\mathcal{F}_{1/2},\varphi,(Y,\mathbb{R},\sigma)\rangle$ generated by equation (7.20) is r-homogeneous ($r=1$) of the degree zero.*

Proof. This statement follows directly from the relation (7.23). □

Applying the results obtained in Sections 5 and 6 to the cocycle constructed above, we obtain the corresponding results for the equation (7.20). Let us formulate one of these results.

Theorem 7.43. *Assume that the function $p \in C(\mathbb{R}, \mathbb{R})$ is recurrent, i.e., the set $H(p)$ is a compact and minimal set of the shift dynamical system (Bebutov's dynamical system) $(C(\mathbb{R}, \mathbb{R}), \mathbb{R}, \sigma)$. Then the following statement are equivalent:*

1. *the trivial solution of the equation (7.20) is uniformly stable and there exists a positive number a such that*
$$\lim_{t \to +\infty} \|\varphi(t, u, p)\|_{\mathcal{F}_{1/2}} = 0$$
for any $u \in B_{1/2}[0, a] := \{u \in \mathcal{F}_{1/2} | \ \|u\|_{\mathcal{F}_{1/2}} \le a\}$;
2. *there exist positive numbers $\mathcal{N}$ and ν such that*
$$\|\varphi(t, u, q)\|_{\mathcal{F}_{1/2}} \le \mathcal{N} e^{-\nu t} \|u\|_{\mathcal{F}_{1/2}}$$
for any $(t, u, q) \in \mathbb{R}_+ \times \mathcal{F}_{1/2} \times H(p)$.

Proof. Consider the cocycle $\langle \mathcal{F}_{1/2}, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ generated by equation (7.20). By Lemma 7.42 this cocycle is r-homogeneous ($r = 1$) of the degree zero. By Corollary 7.36 the cocycle $\langle \mathcal{F}_{1/2}, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ is asymptotically compact. Finally under the conditions of Theorem 7.43 $Y = H(p)$ is a compact and minimal set. To finish the proof of this statement it suffices apply Theorem 7.40. □

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9. CONFLICT OF INTEREST

The author declare that he does not have conflict of interest.

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