Neural, Parallel, and Scientific Computations 32 (2024) 1-15

NUMERICAL SOLUTION OF NONLINEAR INITIAL VALUE PROBLEM ON ITS INTERVAL OF EXISTENCE

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ABSTRACT. In general, the computational and numerical methods available in the literature for solving nonlinear differential equations with initial conditions provide only the local existence of the solution. In this work, we present a computational and numerical method for computing the solution for first-order nonlinear differential equations with initial conditions on its entire interval of existence. The interval of existence is guaranteed by upper and lower solutions and/or coupled lower and upper solutions. In this work, we have used the generalized quasilinearization method to construct our numerical and computational methods to compute the solution on its entire interval of existence. As an example, we have presented various numerical results relating to the Ricatti type of differential equations. Our work also includes examples from biological models, such as the logistic equation where the interval of existence is $[0, \infty)$.

AMS (MOS) Subject Classification. 34A08, 34A12.

Key Words and Phrases: Computaional method, Interval of existence, Generalized Quasilinearization method.

1. INTRODUCTION

Mathematical modeling in many branches of science and engineering leads to the qualitative and quantitative study of dynamic equations with initial and/or boundary conditions. See [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 19, 11, 12, 13, 15, 16, 17, 20, 23, 25] for some of them. However, computation of the explicit solution of a non-linear differential equation is rarely possible. Computing the explicit solution of the non-linear differential equation by numerical methods on its interval of existence is challenging. Several numerical methods are available in the literature to solve non-linear differential equations, but most of them provide only local existence. It is to be noted in computing the solution by iterative methods such as Picard's method, even though the iterates may exist for all time, the limit of the sequence of the iterates developed is not guaranteed to exist for all time. As an example, when we are computing the solution of $u' = u^2,\ u(0) = 1$, each of the Picard iterates are polynomial functions that exist for all time, but the limit of that sequence constructed is $u = \frac{1}{(1-t)}$,

Received January 12, 2024 ISSN 1061-5369 © Dynamic Publishers, Inc.
www.dynamicpublishers.org https://doi.org/10.46719/npsc2024.32.01 .

which exists only on the interval $[0,1)$. It is well known that the method of upper and lower solutions is a theoretical method of existence of solution, and it guarantees the interval of existence for the differential equations with initial conditions. The monotone method combined with the method of upper and lower solutions is a versatile tool that proves both a theoretical and a constructive method of proving the existence of minimal and maximal solutions on its interval of existence. See [8, 9, 11, 12, 13, 14, 22] for a monotone method for ordinary, partial, fractional, and non-linear functions with discontinuities. However, to construct the monotone iterates by using the monotone method, the non-linear function has to be an increasing function or can be made increasing by adding a linear function. Again, considering the example,$u' = u^2 + t,\ u(0) = 1$, it is easy to see that the lower solution can be chosen as $v_0 = 1$, and we cannot construct an upper solution easily to determine the interval of existence of the solution in this example. On the other hand, $v_0 = \frac{1}{(1-t)}$, is also a lower solution, which provides an upper bound for the interval of existence. In a simple logistic equation $u' = u - u^2,\ u(0) = u_0$, the upper and lower solutions are the equilibrium solutions. For example, $0 \leq u_0 \leq 1$, then it is easy to see that $v_0 = 0$, and $w_0 = 1$, are the lower and upper solutions, respectively which guarantees the interval of existence of the solution as $[0, \infty)$. Further, the solution is unique since it is easy to see that $u - u^2$, satisfies global uniqueness condition. Since the non-linear function $u - u^2$, is the sum of an increasing and decreasing function, the usual monotone method will not work. To extend the monotone method when the non-linear function is the sum of an increasing and decreasing function, the generalized monotone method was developed for scalar first-order ordinary differential equations with initial conditions. See [5, 23, 24, 25] and the references therein for some of the initial work for the generalized monotone method. The generalized monotone method can handle when the non-linear function is the sum of increasing and decreasing functions. In order to achieve this, if we use the natural upper and lower solutions, we need an extra assumption regarding the iterates to stay within the sector defined by means of lower and upper solutions. This extra assumption prevents the construction of the iterates on its entire interval of existence of the solution. See [25] for more details. However, if we use coupled lower and upper of Type 1, then we can develop iterates that converge to coupled minimal and maximal solutions of the non-linear scalar initial value problem when the non-linear function is the sum of increasing and decreasing functions. Further, in this situation, we do not need any extra assumptions. Unfortunately, computing coupled lower and upper solutions of Type 1 is nontrivial. See [2, 20, 21], which provides a method of computing coupled lower and upper solutions of Type 1 on small incremental intervals using the generalized monotone method and mixed generalized and quasilinearization method. In this work, we develop monotone iterates using natural upper and lower solutions and by generalized quasilinearization method. In order to explain what the generalized quasilinearization method, it is to be noted that the original quasilinearization method was developed when the non-linear function is either a convex function or a concave function. When the non-linear function is a convex function, we can start solving the linear initial value problem starting with a lower solution, which is monotonically increasing iterates, which converges quadratically to the unique solution of the non-linear problem on its interval of existence. Analogously, if the function is concave, one can start solving a linear initial value problem starting with an upper solution, which is a decreasing sequence that converges quadratically to the unique solution of the non-linear problem on its interval of existence. See for the [3, 4, 17] for the quasilinearization method applied for initial and boundary value problems. The generalized quasilinearization method is a useful tool when a non-linear function is the sum of convex and concave functions. Further, the increasing and decreasing linear iterates starting from the lower and upper solutions converge quadratically to the unique solution on its interval of existence, which is guaranteed by the lower and upper solutions. In this work, we have used the generalized quasilinearization method using the lower and upper solutions when the non-linear function is the sum of a convex and concave function. Further, each of the linear iterates developed exists on the entire interval of existence of the original non-linear problem. Further, each pair of increasing and decreasing iterates sandwiches the unique solution on its entire interval of existence. The sequences converge quadratically to the unique solution of the non-linear initial value problem on its whole interval of existence. We present several numerical examples of the Ricatti type of equations, especially with application to the biological models, namely the logistic equation. In a simple logistic model, the non-linear function can be seen as a sum of convex and concave functions. See [1, 6, 10, 19] for mathematical models arising in biological and infectious diseases.

2. **Preliminaries**

In this section, we recall the definitions and results we need to develop our main results. For this purpose, consider the first-order differential equation of the form.

$$u' = f(t,u) + g(t,u), \quad u(0) = u_0 \text{ on } [0,T] = J, \tag{2.1}$$

where f, g lie in $C(J \times \mathbb{R}, \mathbb{R})$, the space of continuous functions from $J \times \mathbb{R}$ to $\mathbb{R}$.

Definition 2.1. The functions v_0,$w_0 \in C^1(J, \mathbb{R})$ are called natural upper and lower solutions of (2.1) if

$$\begin{aligned} v_0' \leq& f(t,v_0) + g(t,v_0), \quad v_0(0) \leq u_0, \\ w_0' \geq& f(t,w_0) + g(t,w_0), \quad w_0(0) \geq u_0. \end{aligned}$$

Definition 2.2. The functions v_0,$w_0 \in C^1(J, \mathbb{R})$ are called coupled upper and lower solutions of (2.1) Type-I if

$$\begin{aligned} v_0' \leq& f(t, v_0) + g(t, w_0), \quad v_0(0) \leq u_0, \\ w_0' \geq& f(t, w_0) + g(t, v_0), \quad w_0(0) \geq u_0. \end{aligned}$$

Next, we provide a result which proves the existence of a solution of the non-linear initial value problem (2.1). For that purpose, we assume

$$f(t, u) + g(t, u) = F(t, u).$$

Theorem 2.3. *Let $v, w \in C^1[J, \mathbf{R}]$ be natural lower and upper solutions of* (2.1) *such that $v(t) \leq w(t)$ on J, and let $F \in C[\Omega, \mathbf{R}]$ where $\Omega = [(t, x) : v(t) \leq x \leq w(t), t \in J]$ then there exists a solution $u(t)$ of* (2.1) *satisfying $v(t) \leq u(t) \leq w(t)$ on J provided that $v(0) \leq u(0) \leq w(0)$.*

See [11] for proof. Note that the above result related to Theorem 2.3 is only theoretical. However, this theorem also provides the interval of existence and the solution exists on the common interval where the lower and upper solutions exist. If the non-linear problems have equilibrium solutions, computing natural lower and upper solutions is relatively easy, provided that the initial conditions lie within the equilibrium solutions.

Further, the next result proves the uniqueness of the solution of the initial value problem (2.1) when F satisfies one-sided Lipschitz condition.

Theorem 2.4. *Let $v, w \in C^1(J, \mathbf{R})$ be upper lower solutions of* (2.1) *respectively. Suppose that $F(t, x) - F(t, y) \leq L(x - y)$ whenever $x \geq y$, and $L > 0$ is a constant, then $v(0) \leq w(0)$ implies that $v(t) \leq w(t), t \in J$.*

See [11] for proof. Our following results provide a methodology to compute natural upper and lower solutions. This result is beneficial when the equilibrium solutions are not available as easily as lower and upper solutions.

Theorem 2.5. *Let $f_1, f_2, f \in C[J \times R, R]$ and*

$$f_1(t, u) \leq f(t, u) \leq f_2(t, u), \quad (t, u) \in J \times R. \tag{2.2}$$

Let v be soution of $v' = f_1(t, v), v(0) \leq u_0$ and $w(t)$ be the maximal solution of $w' = f_2(t, w), w(0) \geq u_0$ existing on J.Then v, w are lower and upper solutions of (2.2) *such that $v(t) \leq w(t)$, $t \in J$*

See [11, 12] for proof and other details.
Next, we provide a generalized monotone method result where $f(t, u)$ is non-decreasing and $g(t, u)$ is non-increasing respectively in the initial value problem (2.1). The following result is a generalized monotone method relative to natural lower and upper solutions of (2.1).

Theorem 2.6. *Let $v_0, w_0 \in C^1(J,R)$ be natural upper and lower solutions such that $v_0(t) \le w_0(t)$ on J , and assume that f, g are elements of $C(J \times R, R)$ such that $f(t,u)$ is nondecreasing in u and $g(t,u)$ is non-increasing in u on J.*
The, there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on J such that

$$v_n(t) \to v(t) \text{ and } w_n(t) \to w(t)$$

uniformly and monotonically, and (v, w) are coupled minimal and maximal solutions, respectively, to (2.1). That is, (v, w) satisfy on J the equations

$$v^{'} = f(t,v) + g(t,w),\ v(0) = u_0, \tag{2.3}$$

$$w^{'} = f(t,w) + g(t,v),\ w(0) = u_0, \tag{2.4}$$

on J , provided also that $v_0 \le v_1$ and $w_1 \le w_0$ on J.

See [25] for proof. It is to be noted that restrictions $v_0 \le v_1$ and $w_1 \le w_0$ may not hold for the entire interval where lower and upper solutions exist. This prevents the computation of further iterates on the whole interval where lower and upper solutions exist.

The following result does not need the extra assumptions as needed in the above result.

Theorem 2.7. *Let $v_0, w_0 \in C^1(J,R)$ be coupled upper and lower solutions of type I such that $v_0(t) \le w_0(t)$ on J , and assume that f, g are elements of $C(J \times R, R)$ such that $f(t,u)$ is nondecreasing in u and $g(t,u)$ is nonincreasing in u on J.*
There exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on J such that

$$v_n(t) \to v(t) \text{ and } w_n(t) \to w(t)$$

uniformly and monotonically, and (v, w) are coupled minimal and maximal solutions, respectively, to (2.1). That is, (v, w) satisfy on J the equations

$$v^{'} = f(t,v) + g(t,w),\ v(0) = u_0, \tag{2.5}$$

$$w^{'} = f(t,w) + g(t,v),\ w(0) = u_0. \tag{2.6}$$

Here the iterative scheme is given on J by

$$v_{n+1}^{'} = f(t,v_n) + g(t,w_n),\ v_{n+1}(0) = u_0, \tag{2.7}$$

$$w_{n+1}^{'} = f(t,w_n) + g(t,v_n),\ w_{n+1}(0) = u_0. \tag{2.8}$$

See [25] for proof and other details. However, computing coupled lower-upper solutions of type I is not trivial. In [2, 20, 21] they have computed the iterates initially using natural lower and upper solutions and also using coupled lower and upper solutions on a small incremental level. Now consider an example

$$u' = u - u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0, T].$$

Using Theorem 2.7, $v_1(t) = \frac{1}{2} - t$ and $w_1(t) = \frac{1}{2} + t$, where the natural lower and upper solutions respectively are $v_0 = 0$ and $w_0 = 1$. One can observe that $v_1 \geq v_0$ and $w_1 \leq w_0$ for $t \in [0, \frac{1}{2}]$ only. In order to solve this issue, we recall the generalized quasilinearization method relative to the initial value problem (2.1). However, we assume that f, g are elements of $C(J \times R, R)$ such that $f(t,u)$ is convex in u and $g(t,u)$ is concave in u on J, to use generalized monotone method. In addition, we use the natural lower and upper solutions of (2.1). Note that, in our example. we can consider $f(t,u) = u$, is convex in u, and $g(t,u) = -u^2$ is concave in u. Further, the iterates developed are solutions of linear initial value problems with variable coefficients compared with the monotone method or generalized monotone method, which requires linear equations with constant coefficients. Also, the iterates stay within the lower and upper solution in the entire common interval of existence defined by lower and upper solutions. In addition, the linear iterates converge quadratically to the unique solution of the initial value problem (2.1).

The following result is precisely generalized quasilinearization method for the nonlinear problem (2.1) using natural lower and upper solutions.

Theorem 2.8. *Assume that,*

1. $v_0, w_0 \in C^1[J, \mathbf{R}], v_0(t) \leq w_0(t)$ *on* J *with* $v_0(t)$ *and* $w_0(t)$ *are natural lower and upper solutions for* (2.1). *That is*
$$v_0^{'} \leq f(t,v_0) + g(t,v_0), \quad v_0(0) \leq u_0,$$
$$w_0^{'} \geq f(t,w_0) + g(t,w_0), \quad w_0(0) \geq u_0, \quad t \in J = [0,T].$$
2. $f, g \in C[\Omega, \mathbf{R}], f_u, g_u, f_{uu}, g_{uu}$ *exists, are continuous and satisfy*
$$f_{uu}(t,u) \geq 0, \quad g_{uu}(t,u) \leq 0 \quad \text{for } (t,u) \in \Omega.$$

Then there exists monotone sequences $\{v_n(t)\}, \{w_n(t)\}$ *which converge uniformly to the unique solution of* (2.1) *and the convergence is quadratic.*

This is Theorem 1.3.1 of [16]. See [16] for a detailed proof.
In order to provide a proof that can be easily applied to our main result namely the numerical application of solving Ricatti type of equations, and general logistic equations, we use the iterative scheme which is given by,

$$v_{n+1}^{'} = f(t,v_n) + f_u(v_n)(v_{n+1} - v_n) + g(t,v_n) + g_u(w_n)(v_{n+1} - v_n), \tag{2.9}$$
$$w_{n+1}^{'} = f(t,w_n) + f_u(v_n)(w_{n+1} - w_n) + g(t,w_n) + g_u(w_n)(w_{n+1} - w_n), \tag{2.10}$$

for $n = 0, 1, 2, ...$ Using the convexity and concavity of $f(t,u)$ and $g(t,u)$ respectively, we can prove that

$$v_0 \leq v_1 \leq v_2 \leq v_3 .... \leq v_n \leq u \leq w_n \leq .... \leq w_2 \leq w_1 \leq w_0,$$

on J. In addition, we can prove that the pair (v_n, w_n) will be natural lower and upper solutions of (2.1) for all n. Further the sequences $\{v_n(t)\}, \{w_n(t)\}$ converges uniformly and quadratically to the unique solution of (2.1) on J.

Remark: Theorem 2.8 is not only a theoretical but also a constructive method. In the main result, we develop a working formula to get the numerical results using Theorem 2.8 on its interval of existence.

3. **Main Results**

It is well known that if the natural lower and upper solutions of (2.1), say v, w exist on $J = [0, t]$ such that $v \leq w$, on J, then one can prove that there exists a solution $u(t)$ of the nonlinear initial value problem on J, provided $v(0) \leq u_0 \leq w(0)$. See [11, 12] for proof and other details. The method of proving existence by upper and lower solutions is only theoretical. The generalized monotone method of proving the existence of (2.1) is both theoretical and computational. See [2, 20, 21] for numerical and computational application of the generalized monotone method and a combination of generalized monotone method and generalized quasilinearization by constructing coupled lower and upper solutions on finite intervals. In this work, we develop the computational and numerical application of Theorem 2.8 for (2.1), when $f(t, u)$ is convex in u, and $g(t, u)$ is concave in u. Initially, we develop a working formula for our numerical method using Theorem 2.8, relative to the nonlinear initial value problem (2.1).

Consider the iterations provided in the brief proof of Theorem 2.8 as follows:

$$v'_{n+1} = f(t, v_n) + f_u(v_n)(v_{n+1} - v_n) + g(t, v_n) + g_u(w_n)(v_{n+1} - v_n), \tag{3.1}$$

$$w'_{n+1} = f(t, w_n) + f_u(v_n)(w_{n+1} - w_n) + g(t, w_n) + g_u(w_n)(w_{n+1} - w_n). \tag{3.2}$$

These iterations can be rearranged as:

$$v'_{n+1} + (-f_u(v_n) - g_u(w_n))v_{n+1} = f(v_n) + g(v_n) + p_n(t)v_n, \tag{3.3}$$

$$w'_{n+1} + (-f_u(v_n) - g_u(w_n))w_{n+1} = f(w_n) + g(w_n) + p_n(t)w_n. \tag{3.4}$$

Or we can rewrite the above iterates as,

$$v'_{n+1} + p_n(t)v_{n+1} = q_n(t), \tag{3.5}$$

$$w'_{n+1} + \bar{p}_n(t)w_{n+1} = \bar{q}_n(t), \tag{3.6}$$

which are linear in v_{n+1} and w_{n+1}.
The coefficients $p_n(t), q_n(t), \bar{p}_n(t)$, and $\bar{q}_n(t)$ are given by:

$$\begin{aligned}
p_n(t) =& - f_u(v_n) - g_u(w_n),\\
q_n(t) =& f(v_n) + g(v_n) + p_n(t)v_n,\\
\bar{p}_n(t) =& - f_u(v_n) - g_u(w_n),\\
\bar{q}_n(t) =& f(w_n) + g(w_n) + \bar{p}_n(t)w_n.
\end{aligned}$$

Note that once we decompose the right-hand side of problem into $f(t,u)$ and $g(t,u)$, we use values of $p_n(t), q_n(t), \bar{p}_n(t)$, and $\bar{q}_n(t)$ into linear differential equations in v_{n+1} and w_{n+1} to solve them. Also note that $p_n(t) = \bar{p}_n(t)$. Since our objective is to solve the Ricatti type of ordinary differential equation, we start with a special form of Ricatti type of nonlinear function. Consider

$$u' = a(t)u - b(t)u^2 + c(t), \qquad u(0) = u_0, \quad t \in [0,T], \tag{3.7}$$

where $a(t) \geq 0$, and $b(t) \geq 0$. We choose, $f(t) = a(t)u + c(t)$ and $g(t) = -b(t)u^2$. Then $p_n(t), q_n(t), \bar{p}_n(t)$, and $\bar{q}_n(t)$ are given by

$$\begin{aligned}
p_n(t) =& - a(t) + 2b(t)w_n,\\
q_n(t) =& a(t)v_n + c(t) - b(t)v_n^2 + (-a(t) + 2b(t)w_n)v_n,\\
\bar{p}_n(t) =& - a(t) + 2b(t)w_n,\\
\bar{q}_n(t) =& a(t)w_n + c(t) + b(t)w_n^2 + (-a(t) + 2b(t)w_n)w_n.
\end{aligned}$$

Next, we develop numerical results as an application of both the theoretical and computational results of Theorem 2.8 for several examples. We take a simple logistic equations and develop the iterates using the formulas provided above together with the corresponding lower and upper solutions. All the numerical simulations are done using Runge Kutta fourth order method (ode45), as implemented in Matlab.
The Table 1 provides the initial conditions, the coefficients $a(t), b(t)$, and $c(t)$, the lower solutions, upper solutions, and the exact solutions of all the examples that we present here.

Examples	$u(0)$	$a(t)$	$b(t)$	$c(t)$	Lower Sol. v_0	Upper Sol. w_0	Exact Sol. $u(t)$
1	$\frac{1}{2}$	1	-1	0	$\frac{1}{2}$	1	$\frac{1}{1+e^{-t}}$
2	$\frac{3}{2}$	1	-1	0	1	$\frac{3}{2}$	$\frac{3}{3-e^{-t}}$
3	$\frac{1}{2}$	-1	1	0	0	1	$\frac{1}{1+e^{t}}$
4	$\frac{1}{2}$	$\frac{1}{1+t}$	$-\frac{1}{1+t}$	0	$\frac{1}{2}$	1	$\frac{t+1}{t+2}$

TABLE 1. Lower, Upper, & Exact Solutions and respective initial conditions

Example 3.1. Consider the following nonlinear problem (a simple logistic equation):

$$u' = u - u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0, \infty). \tag{3.8}$$

Here, we choose $f = u$, and $g = -u^2$, since $f_{uu} \geq 0$ and $g_{uu} \leq 0$ implying f is convex and g is concave. Clearly, $v_0(t) = 0.5$ and $w_0(t) = 1$ are natural lower and upper solutions respectively. Then using the iterations as in the brief proof of Theorem 2.8, we get $v_1(t) = \frac{3}{4} - \frac{e^{-t}}{4}$ and$w_1(t) = 1 - \frac{e^{-t}}{2}$. It is easy to see that $v_1(t) \geq v_0(t) = 1/2$, and $w_1(t) \leq w_0(t) = 1$, for all $t \in [0, \infty)$. Then by using Theorem 2.8 we will have

$$\frac{1}{2} = v_0(t) \leq v_1(t) = \frac{3}{4} - \frac{e^{-t}}{4} \leq u \leq w_1(t) = 1 - \frac{e^{-t}}{2} \leq w_0(t) = 1.$$

The graphical iterations of the simple logistic equation (3.8) have been presented in Figure 1, when the initial condition is such that $0 < u(0) < 1$. In this example, the actual solution of (3.8) can be computed analytically. From the graph, it is easy to

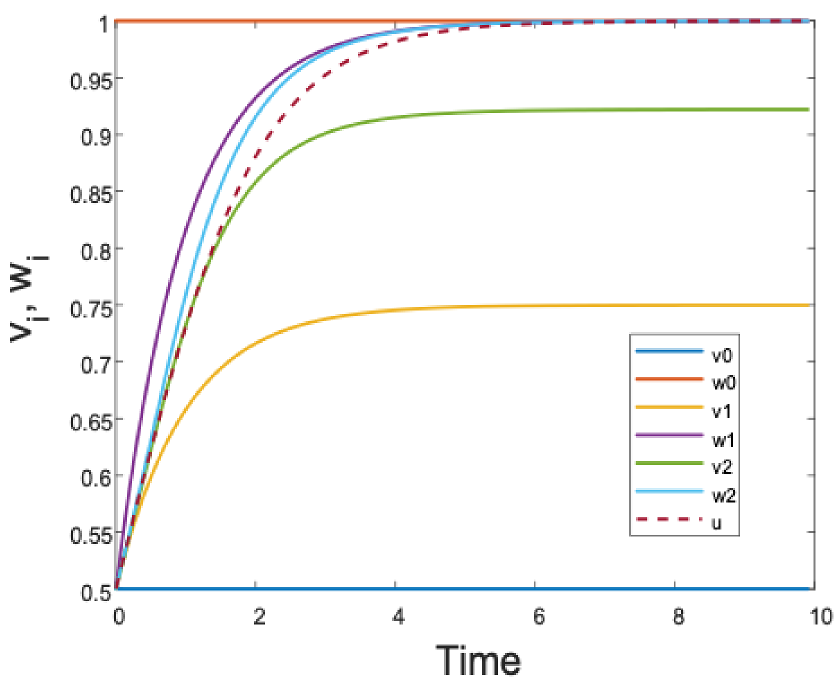

FIGURE 1. Graph for Example 3.1.

see that the analytical solution is sandwiched between the second lower and second upper iterates. In addition, the iterates converge quadratically to the analytical solution. Also, the numerical and graphical solution justifies that the equilibrium solution $u(t) = 1$, is uniformly asymptotically stable. The Table 2 illustrates the quadratic convergence of the iterates to an analytic (or exact) solution. The total number of time steps taken, in all examples solved, are 100.

Example 3.2. Consider the following equation :

$$u' = u - u^2, \; u(0) = \frac{3}{2}, \quad t \in [0, \infty). \tag{3.9}$$

time step	v_k	v_{k+1}	w_k	w_{k+1}	$e_k = w_k - v_k$	$e_{k+1} = w_{k+1} - v_{k+1}$	$\frac{e_{k+1}}{e_k^2}$
10	0.6485	0.7206	0.7969	0.7360	0.1484	0.0154	0.6992
25	0.7273	0.8952	0.9547	0.9457	0.2274	0.0505	0.9765
45	0.7469	0.9326	0.9939	0.9936	0.2470	0.0610	0.9998
70	0.7497	0.9371	0.9995	0.9995	0.2498	0.0624	0.9999

TABLE 2. Proof of quadratic convergence for Example 3.1.

In this example, only the initial condition has changed compared with Example 3.1. That is $f(u) = u$ and $g(u) = -u^2$ as before. However, it is easy to see that, $v_0(t) = 1$ and $w_0(t) = \frac{3}{2}$ are natural lower and upper solutions respectively. The graph of the

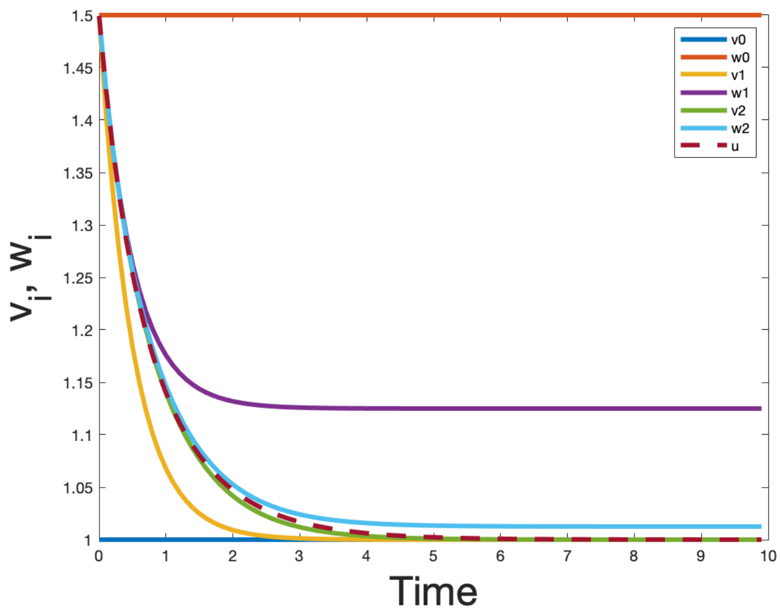

FIGURE 2. Graph for problem Example 3.2.

first three iterations, along with the analytical solution of Example 3.2, is presented in figure 2. In this case, also, the analytical solution can be computed, and it is $u(t) = \frac{3}{3-e^{-t}}$. Since the equilibrium solution exists for all time, each of the increasing and decreasing iterates also exists for all time. In addition, the equilibrium solution is also asymptotically stable. The Table 3 illustrates the quadratic convergence of the iterates to an analytic (or exact) solution.

Example 3.3. Consider

$$u' = u^2 - u, \quad u(0) = \frac{1}{2}, \quad t \in [0, \infty). \tag{3.10}$$

In this example $f(u) = u^2$ and $g(u) = -u$, and $f_u = 2u$, $g_u = -1$. It is easy to observe tha, $v_0 = 0$ and $w_0 = \frac{1}{2}$ are lower and upper solutions. Also in this example we have

time step	v_k	v_{k+1}	w_k	w_{k+1}	$e_k = w_k - v_k$	$e_{k+1} = w_{k+1} - v_{k+1}$	$\frac{e_{k+1}}{e_k^2}$
10	1.0825	1.1574	1.1869	1.1631	0.1044	0.0057	0.5229
25	1.0041	1.0253	1.1281	1.0369	0.1240	0.0116	0.7544
45	1.0001	1.0021	1.1251	1.0145	0.1250	0.0124	0.7936
70	1.0000	1.0001	1.1250	1.0126	0.1250	0.0125	0.8000

TABLE 3. Proof of quadratic convergence for Example 3.2.

$$\begin{aligned} p_n(t) =& -2v_n + 1 = \bar{p}_n(t), \\ q_n(t) =& v_n^2 - v_n + (-2v_n + 1)v_n, \\ \bar{q}_n(t) =& w_n^2 - w_n + (-2v_n + 1)w_n. \end{aligned}$$

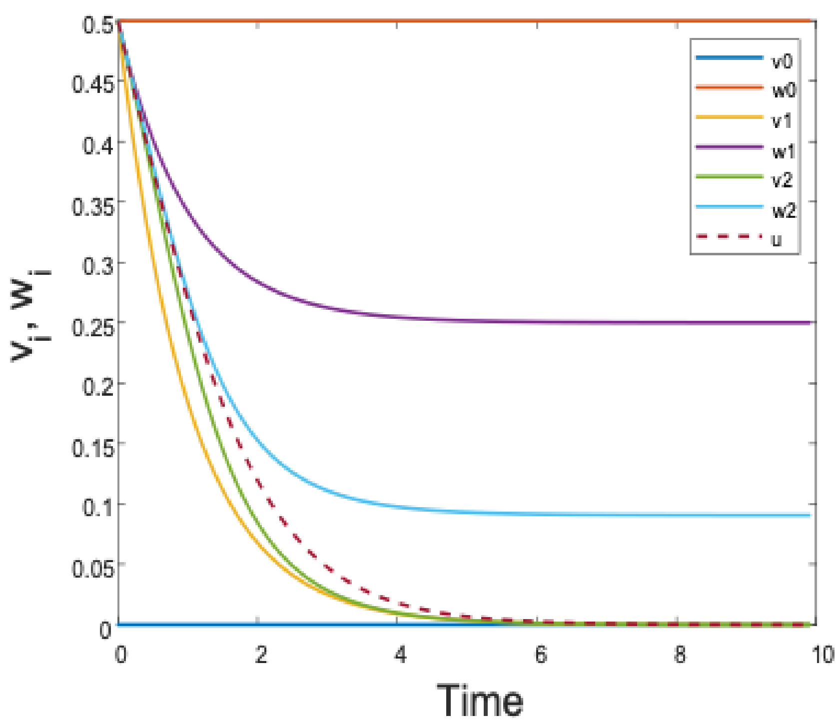

FIGURE 3. Graph for Example 3.3.

The graph of Example 3.3. has been presented in figure 3. In addition, the equilibrium solution $u = 0$ is also asymptotically stable. The Table 4 illustrates the quadratics convergence of the iterates to analytic (or exact) solution.

Example 3.4. Consider

$$u' = \frac{u}{1+t} - \frac{u^2}{1+t}, \quad u(0) = \frac{1}{2}, \qquad t \in [0, \infty). \tag{3.11}$$

time step	v_k	v_{k+1}	w_k	w_{k+1}	$e_k = w_k - v_k$	$e_{k+1} = w_{k+1} - v_{k+1}$	$\frac{e_{k+1}}{e_k^2}$
10	0.2031	0.2640	0.3515	0.2794	0.1484	0.0154	0.6992
25	0.0453	0.0543	0.2727	0.1048	0.2274	0.0505	0.9766
45	0.0061	0.0058	0.2531	0.0674	0.2470	0.0610	0.9998
75	0.0003	0.0003	0.2502	0.0627	0.2499	0.0624	0.9992

TABLE 4. Proof of quadratic convergence for Example 3.3.

In this example $f(u) = \frac{u}{1+t}$ and $g(u) = -\frac{u^2}{1+t}$, and $f_u = \frac{1}{1+t}$, $g_u = -\frac{2u}{1+t}$. It is easy to observe that, $v_0 = \frac{1}{2}$ and $w_0 = 1$ are lower and upper solutions. Also in this example we have

$$p_n(t) = \frac{-1 + 2w_n}{(1+t)} = \bar{p}_n(t),$$
$$q_n(t) = \frac{v_n - v_n^2 + (-1 + 2w_n)v_n}{(1+t)},$$
$$\bar{q}_n(t) = \frac{w_n - w_n^2 + (-1 + 2w_n)w_n}{(1+t)}.$$

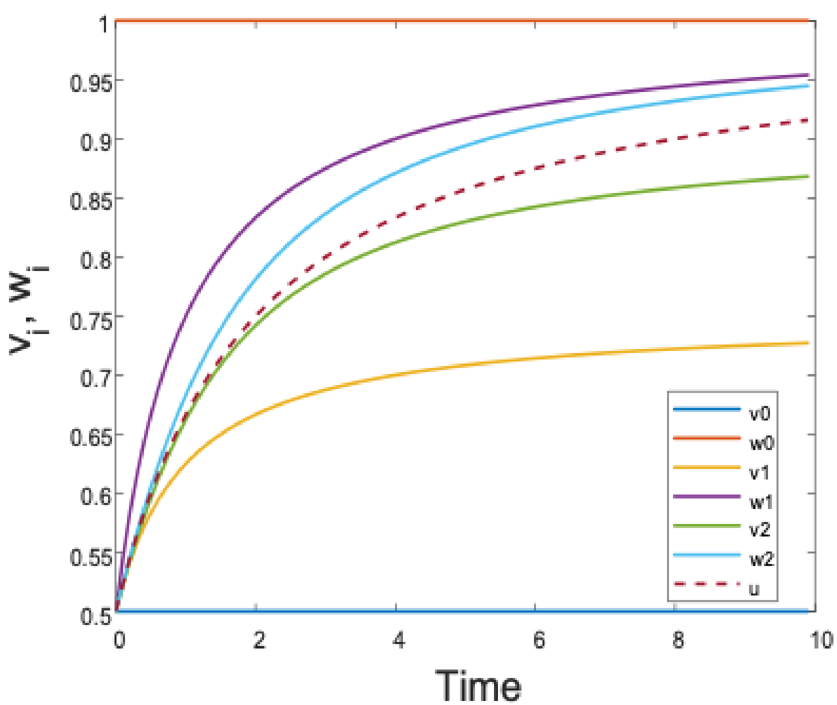

FIGURE 4. Graph for Example 3.4.

The Table 5 illustrates the quadratics convergence of the iterates to analytic (or exact) solution.

time step	v_k	v_{k+1}	w_k	w_{k+1}	$e_k = w_k - v_k$	$e_{k+1} = w_{k+1} - v_{k+1}$	$\frac{e_{k+1}}{e_k^2}$
10	0.6185	0.6618	0.7370	0.6696	0.1185	0.0078	0.5554
25	0.6765	0.7815	0.8530	0.8070	0.1765	0.0255	0.8185
45	0.7037	0.8433	0.9074	0.8813	0.2037	0.0380	0.9158
70	0.7184	0.8760	0.9367	0.9216	0.2183	0.0456	0.9568
95	0.7260	0.8923	0.9519	0.9420	0.2259	0.0497	0.9739

TABLE 5. Proof of quadratic convergence for Example 3.4.

4. **Conclusion**

The method of upper and lower solution provides the existence of a solution of the first-order nonlinear initial value problem on the common interval of existence of the upper and lower solutions. It is to be noted that computing the natural lower and upper solutions are relatively easy. In this work, we have combined natural lower and upper solutions together with the generalized quasilinearization method to obtain the unique solution of the nonlinear problem by computational and numerical methods. The method of generalized quasilinearization is useful when the nonlinear function is the sum of a convex and concave function. The methods developed in our work are very suitable for the general logistic equation and several biological models. In addition, our method is suitable for the Ricatti type of nonlinearities, especially when the nonlinearities are the sum of a convex and concave function. The unique solution can be computed on the entire interval of existence by increasing and decreasing sequences of upper and lower solutions, which sandwiches the unique solution on its interval of existence. The iterates are solutions of linear equations with variable coefficients. As a byproduct, it provides the stability result of the equilibrium solution. To the best of our knowledge, the computational monotone iterates exist for all time, which are defined by the natural lower and upper solutions for the scalar initial value problem when the nonlinear function is the sum of convex and concave functions. The problem is open for reaction-diffusion equation and Caputo fractional differential equations with initial conditions. It is open for a system of ordinary differential equations with initial conditions, such as SIR models of infectious diseases and predator-prey models.

Acknowledgement: Acknowledgement: The authors would like to thank Md Nafiz Musarrat, a graduate student from the Electrical and Computer Engineering department at UL Lafayette, for helping in the numerical (programming) and graphical work of this article.

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