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QUATERNARY DELAYED COHEN-GROSSBERG FUZZY NEURAL NETWORK SYNCHRONIZATION WITH NON-CONTINUOUS ACTIVATIONS

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ABSTRACT. This paper focuses on the finite-time and fixed-time for fuzzy quaternion-valued Cohen-Grossberg neural networks (FQVCGNNs) via non-continuous activations and time-varying delays. By designing Lyapunov functions and utilizing differential inequalities, several effective conditions are derived to ensure the finite-time and fixed-time synchronization of the addressed neural networks based on two different time-delayed feedback controllers. Moreover, a novel fixed-time convergence method is proposed to study synchronization in fixed-time of discontinuous delayed FQVCGNNs. In the end, two numerical examples with simulations are given to confirm the effectiveness of the synchronization criteria.

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1. INTRODUCTION

In the last few years, the research on recurrent neural networks (RNNs) has attracted attention (see [6, 4, 5, 7, 28, 9, 1, 2]). In 1996, Yang and Yang firstly put forward fuzzy cellular neural networks [26] based on the traditional cellular neural networks in the aim of coping with the uncertainty in human cognitive processes.

After, we have, the Cohen-Grossberg neural networks, firstly studied in Cohen and Grossberg, [10] have attracted increasing interest due to the potential applications in signal, image processing, optimizations, and control problems. Quaternions were firstly presented by Hamilton in 1853, [11].The difference between complex value and quaternion value is that a quaternions value contains a real part and three imaginary units. Furthermore, if quaternions are considered as a n-dimensional (multi-dimensional) real space, the quaternion represents a 4-dimensional space, which is a 2-dimensional space relative to the plural. In the present case, it generally lies

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in 3-dimensional and 4-dimensional data modeling (see [12, 13]). For that reason, a system with quaternion-valued parameters merit further investigation. It is well seen that time-varying delays are necessarily encountered in the signal transmission between the neurons.

For that reason, time delays and especially time-varying delays should be established into network modelling and neural networks (see [23, 25, 16, 29, 17]). Another area that has been well studied is synchronization control theory. Synchronization means that the states of the response-system converges those of the drive system. During the last years, several kinds of synchronization control of quaternion-valued Cohen-Grossberg neural networks have been studied (see [25, 16]). In reality, the initial conditions practical models can rarely be calculated. In 2012 and in [20], Polyakov focused on the fixed-time stabilization of linear control systems by using a nonlinear feedback design law. To our best knowledge, the control method has not been applied to FQVCGNNs yet. The purpose of this paper is to explore the finite-time and fixed-time synchronization of delayed FQVCGNNs with discontinuous activation. The main novelty is as follows.

(1) It is the first time that the quaternion and discontinuous activation are introduced into FCGNNs.
(2) A novel fixed-time convergence method is adopted to deal with the synchronization problem.
(3) To overcome the non-commutativity of quaternion, the non-decomposition is proposed.

The remainder of this document is ordered in the following manner. The section 2 gives a description of the system and some preliminary information. In the section 3, the availability of Filippov solutions for FQVCGNNs is considered. This section 4 discusses finite-time and fixed-time synchronization of non-continuous master-response systems with variable timeframes. A few numerical examples with simulations are given to check the efficiency of the results obtained in the section 5. Recently, in section 6, the conclusion has been reached. **Notations:** Letting $\mathbf{R}$ represents the one-dimensional real space, $\mathbf{C}$ represents the one-dimensional complex space, the n-dimensional quaternion space is denoted by $\mathcal{Q}^n$, the $n \times n$-dimensional quaternion space is denoted by $\mathcal{Q}^{n\times n}$. For $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^T \in \mathcal{Q}$, $\|\nu\|_1$ represents a vector norm defined by $\|\nu\|_1 = \sum_{i=1}^{n} |\nu_i|$ and $\mathcal{C}([-\widetilde{\tau}_1, 0], \mathcal{Q}^n)$ represents the Banach space of all continuous functions from $[-\widetilde{\tau}_1, 0] \to \mathcal{Q}^n$ and $\|.\|_p$ is an induced matrix norm on $\mathbf{R}^{n\times n}$ with $p = 1, 2, \infty$.

2. MODEL DESCRIPTION AND PRELIMINARIES

A proper quaternion, only called quaternion, can be given by

$$q = q^{(R)} + iq^{(I)} + jq^{(J)} + kq^{(K)} : \mathbf{R} \to \mathcal{Q},$$

where $q^{\iota} : \mathbf{R} \to \mathbf{R}, \iota = \{(R), (I), (J), (K)\}$, the derivative of the quaternion-valued function q is defined by $\dot{q}(t) = \dot{q}^{(R)}(t) + i\dot{q}^{(I)}(t) + j\dot{q}^{(J)}(t) + k\dot{q}^{(K)}(t)$. The model QVNNs with time-varying delays is introduced as follows:

$$\begin{aligned} \dot{z}_i(t) &= d_i(z_i(t))\Big[- c_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}(t)h_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}\vartheta_j + \bigwedge_{j=1}^{n} \mathcal{T}_{ij}\vartheta_j \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij}(t)h_j(z_j(t-\tau_1(t)) + \bigvee_{j=1}^{n} \beta_{ij}(t)h_j(z_j(t-\tau_1(t)) + \bigvee_{j=1}^{n} \mathcal{S}_{ij}\vartheta_j \\ &+ \mathcal{I}_i(t)\Big], \ i = 1, 2, \cdots, n \end{aligned} \tag{2.1}$$

where $z_i(.) \in \mathcal{Q}$ represents the state vector of the ith unit; $d_i(z_i(.)) \in \mathcal{Q}$ denotes an amplification function; $c_i(z_i(.)) \in \mathcal{Q}$ stands for the behaved function; $a_{ij}(.), b_{ij}(.) \in \mathcal{Q}$ are the time-varying connection weights; $\alpha_{ij}(.), \beta_{ij}(.) \in \mathcal{Q}$ is a fuzzy feedback MAX template and a fuzzy feedback MIN template, respectively; $mathcalT_{ij}$ and $mathcalS_{ij}$ are a MIN model of fuzzy feedback and a MAX model of fuzzy feedback.; $\bigwedge$ and $\bigvee$ represent the fuzzy AND and fuzzy OR operations; ϑ_j denotes bias of jth neuron; $h_j(z_j(.)) : \mathcal{Q} \to \mathcal{Q}$; $\tau_1(t)$ is the time-varying delays with $\widetilde{\tau}_1 = \sup_{t\in\mathbf{R}^+} |\tau_1(t)|\}$ and $\mathcal{I}_i(.) \in \mathcal{Q}$ corresponds the external input function. We will discuss the model (2.1) as the drive system and the corresponding response-system is given by:

$$\begin{aligned} \dot{w}_i(t) &= d_i(w_i(t))\Big[- c_i(w_i(t)) + \sum_{j=1}^{n} a_{ij}(t)h_j(w_j(t)) + \sum_{j=1}^{n} b_{ij}\vartheta_j + \bigwedge_{j=1}^{n} \mathcal{T}_{ij}\vartheta_j \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij}(t)h_j(w_j(t-\tau_1(t)) + \bigvee_{j=1}^{n} \beta_{ij}(t)h_j(w_j(t-\tau_1(t)) + \bigvee_{j=1}^{n} \mathcal{S}_{ij}\vartheta_j \\ &+ \mathcal{I}_i(t) + \mathcal{U}_i(t)\Big], \ i = 1, 2, \cdots, n \end{aligned} \tag{2.2}$$

where $w_i(.) \in \mathcal{Q}$ represents the state of the response-system of the ith unit, $\mathcal{U}_i(.)$ is the control input vector. The other notations are the same as those in system (2.1). The initial conditions of drive-response systems (2.1) and (2.2) are assumed to be:

$$z_i(s) = \varphi_i^z(s), w_i(s) = \varphi_i^w(s), i = 1, 2, \cdots, n, s \in [-\widetilde{\tau}_1, 0],$$

We have

Assumption. 1 Parameters in system (2.1) are all continuous.

Assumption. 2 For all $i = 1, 2, \cdots, n, h_i(.) : \mathcal{Q} \to \mathcal{Q}$ is piecewise continuous.

Assumption. 3 For every $j = 1, 2, cdots, n$, assume that there are positive constants L_{1j}, L_{2j},so that the $h_j(.) activation function in mathecal Q$ meets the following condition.

$$|\widetilde{\gamma}_j(t) - \gamma_j(t)| \leq L_{1j}|w_j(t) - z_j(t)| + L_{2j}, \ \forall z_j(t), w_j(t) \in \mathcal{Q} \tag{2.3}$$

with $\gamma_j(t) \in \overline{co}[h_j(z_j(t))], \widetilde{\gamma}_j(t) \in \overline{co}[h_j(w_j(t))]$ and $0 \in \overline{co}[h_j(0)]$.

Assumption. 4 The amplification functions $d_i(t)$ are continuous, bounded and there exist positive numbers $\underline{d}_i, \ \overline{d}_i$ such that $0 < \underline{d}_i \leq d_i(t) \leq \overline{d}_i$ for all $t \geq 0$ and $i = 1, 2, \cdots, n$.

Assumption. 5 For any $i = 1, 2, \cdots, n$, there exists a constant $\widehat{c}_i$ such that $(c_i(w) - c_i(z))(w - z) \geq \widehat{c}_i(w - z)^2$ for $z, w \in \mathcal{Q}$.

2.1. **Definition. 1.** [24] The drive-system (2.1) and the response-system (2.2) are deemed to be synchronized within a finite time, if a real number exists of $0 \leq \widetilde{\mathrm{T}} < +\infty$, dependent on initial system values (2.1) and (2.2) such as

$$\lim_{t \to t_0 + \widetilde{\mathrm{T}}} \|w_i^\iota(t) - z_i^\iota(t)\|_1 = 0 \text{ and } \|w_i^\iota(t) - z_i^\iota(t)\|_1 = 0, \forall t \geq t_0 + \widetilde{\mathrm{T}},$$

for $i = 1, 2, \cdots, n, t_0 \geq 0$ and $\iota = \{(R), (I), (J), (K)\}$. Moreover,

$$\widetilde{\mathrm{T}}_0 = \left\{ \widetilde{\mathrm{T}} \geq 0 : \ z(t) = w(t) \text{ for } t \geq t_0 + \widetilde{\mathrm{T}} \right\}.$$

2.2. **Definition. 2.** [24] (2.1) and (2.2) are described as fixed-time synchronized, if (2.1) and (2.2) are infinite time synchronization and the settlement time $\widetilde{\mathrm{T}}$ is globally limited, i.e.if there exists $\mathrm{T}_{\max} \in \mathbf{R}^+$, such that $\widetilde{\mathrm{T}} \leq \mathrm{T}_{\max}$.

2.3. **Definition. 3.** [19] Considering $\dot{z}(t) = h(t, z(t)), z(0) = z_0 \geq 0$ with discontinuous right-hand side, where $z(t) \in \mathcal{Q}^n$. A vector function $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))^T$ is said to be the Filippov solution of initial value in $[0, \widetilde{\mathrm{T}})$ with $\widetilde{\mathrm{T}} \geq 0$ if

- $z(t)$ is absolutely continuous,
- $z(t)$ satisfies the following differential inclusion:

$$\dot{z}(t) \in \mathcal{F}[h](t, z(t)), \text{ a.e.t } \in [0, \widetilde{\mathrm{T}}],$$

in which, the following set-valued map $\mathcal{F}[h](t, z(t))$ is defined as

$$\mathcal{F}[h](t, z(t)) = \bigcap_{\epsilon > 0} \bigcap_{\vartheta(N) = 0} \overline{co}\big[h(t, B(z, \epsilon) \setminus N)\big],$$

where $\overline{co}(\Theta)$ denotes the convex closure of set Θ, $B(z, \epsilon) = \big\{y : \|y - z\| \leq \epsilon\big\}$ and $\vartheta(N)$ corresponds the Lebesgue measure of set N.

We extend the theory of the Filippov solution to the following differential equation (2.1):

2.4. **Definition. 4.** [14] Considering the function $z : [-\widetilde{\tau}_1, \widetilde{\mathrm{T}}) \to \mathcal{Q}^n, \widetilde{\mathrm{T}} \in (0, +\infty]$ is a Filippov solution of the non-continuous system (2.1) over $[-\widetilde{\tau}_1,\ \widetilde{\mathrm{T}})$, if:

- $z(t)$ is continuous on $[-\widetilde{\tau}_1, \widetilde{\mathrm{T}})$ and absolutely continuous on $[0, \widetilde{\mathrm{T}})$,
- $z(t)$ satisfies:

$$\begin{aligned} \dot{z}_i(t) \ \in\ & d_i(w_i(t))\Big[- c_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\overline{co}[h_j(z_j(t))] + \sum_{j=1}^{n} b_{ij}\vartheta_j \\ & + \bigwedge_{j=1}^{n} \mathcal{T}_{ij}\vartheta_j + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\overline{co}[h_j(z_j(t-\tau_1(t))] + \bigvee_{j=1}^{n} \beta_{ij}(t)\overline{co}[h_j(z_j(t-\tau_1(t))] \\ & + \bigvee_{j=1}^{n} \mathcal{S}_{ij}\vartheta_j + \mathcal{I}_i(t)\Big], \end{aligned} \tag{2.4}$$

or equivalently, there exist measurable functions $\gamma_j(t) \in \overline{co}[h_j(z_j(t))]$ for a.e. $t \in [-\widetilde{\tau}_1, \widetilde{\mathrm{T}}), j = 1, 2, \cdots, n$ and

$$\begin{aligned} \dot{z}_i(t) \ =\ & d_i(w_i(t))\Big[- c_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) + \sum_{j=1}^{n} b_{ij}\vartheta_j + \bigwedge_{j=1}^{n} \mathcal{T}_{ij}\vartheta_j \\ & + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\gamma_j(t-\tau_1(t)) + \bigvee_{j=1}^{n} \beta_{ij}(t)\gamma_j(t-\tau_1(t)) + \bigvee_{j=1}^{n} \mathcal{S}_{ij}\vartheta_j \\ & + \mathcal{I}_i(t)\Big], \text{ for all } t \in [-\widetilde{\tau}_1, \widetilde{\mathrm{T}}), \end{aligned} \tag{2.5}$$

where the function $\gamma_j(t)$ is called measurable selection of the function $\mathcal{F}$. Evidently, for all the discontinuous function $h(.)$ obeys the **Assumption. 2**, the set-valued map

$$\begin{aligned} \dot{z}_i(t) \ \hookrightarrow\ & d_i(w_i(t))\Big[- c_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\overline{co}[h_j(z_j(t))] + \sum_{j=1}^{n} b_{ij}\vartheta_j \\ & + \bigwedge_{j=1}^{n} \mathcal{T}_{ij}\vartheta_j + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\overline{co}[h_j(z_j(t-\tau_1(t))] \\ & + \bigvee_{j=1}^{n} \beta_{ij}(t)\overline{co}[h_j(z_j(t-\tau_1(t))] + \bigvee_{j=1}^{n} \mathcal{S}_{ij}\vartheta_j + \mathcal{I}_i(t)\Big] \end{aligned}$$

has nonempty compact convex values. Furthermore, it is upper-semi continuous [15] and hence it is measurable. By the measurable selection theorem [8], if $z(t)$ is a solution of (2.1), then there exists a measurable function $\gamma_j(t) \in \overline{co}[h_j(z_j(t))]$ such that for a.e. $t \in [0, +\infty)$, the equation (2.5) is true.

2.5. **Lemma. 1.** [21] Suppose that a continuous, positive and definite function $V(t)$ satisfies the next differential inequality $\dot{V}(t) \leq -\varpi V^{\alpha}(t)$ for $t \geq t_0, V(t_0) \geq 0$ and $\varpi > 0, 0 < \alpha < 1$ are two constants. Then, for any given t_0, $V(t)$ satisfies the following inequality $V^{1-\alpha}(t) \leq V^{1-\alpha}(t_0) - \varpi(t-t_0)(1-\eta)$ for $t_0 \leq t \leq t_1$ and $V(t) = 0$ for $t \geq t_1$. Moreover, the time t_1 is given by

$$t_1 = t_0 + \frac{V^{1-\alpha}(t_0)}{\varpi(1-\alpha)}. \tag{2.6}$$

2.6. **Lemma. 2.** [23] Suppose that $V(t) : \mathbf{R}^n \to \mathbf{R}^+ \cup \{0\}$ is radially unbounded continuous and if there exists an undefined function $\text{II}_1(t)$ and a non positive function $\text{II}_2(t)$ such that $\dot{V}(t) \leq \text{II}_1(t)V^{\alpha}(t) + \text{II}_2(t)V^{\beta}(t)$ for $t \in [t_0, +\infty)$ where $0 < \alpha < 1, \beta > 1$ and $\text{II}_1(t), \text{II}_2(t)$ satisfy that

$$\int_{t_0}^{t} \text{II}_1^*(s)ds \leq \mathcal{N}, \int_{t_0}^{t} \text{II}_{1*}(s)ds \leq -\theta_1(t-t_0) + \mathcal{M}_1 \tag{2.7}$$

and

$$\int_{t_0}^{t} \text{II}_2(s)ds \leq -\theta_2(t-t_0) + \mathcal{M}_2, \tag{2.8}$$

for all $t \geq t_0$, where $\text{II}_1^*(s) = \text{II}_1(s) \bigvee 0, \text{II}_{1*}(s) = \text{II}_1(s) \bigwedge 0$ and $\mathcal{N}, \mathcal{M}_i, \theta_i$ $(i = 1, 2)$ are positive constants. Then, $V(t) = 0$ for all $t \geq \text{T}_{\max}$ and the settling time is

$$\text{T}_{\max} = t_0 + \frac{(1-\alpha)(\mathcal{N} + \mathcal{M}_1) + 1}{\theta_1(1-\alpha)} + \frac{(\beta-1)(\mathcal{N} + \mathcal{M}_2) + 1}{\theta_2(\beta-1)}. \tag{2.9}$$

2.7. **Lemma. 3.** [20] Suppose that z and w are two states of a system drive (2.1). Well, then, for any $i = 1, 2, \cdots, n$, the estimated following are true:

$$\begin{aligned} |\bigwedge_{i=1}^{n} \alpha_{ij}h_j(w_j) - \bigwedge_{j=1}^{n} \alpha_{ij}h_j(z_j)| &\leq \sum_{j=1}^{n} |\alpha_{ij}||h_j(w_j) - h_j(z_j)| \\ |\bigvee_{j=1}^{n} \beta_{ij}h_j(w_j) - \bigvee_{j=1}^{n} \beta_{ij}h_j(z_j)| &\leq \sum_{j=1}^{n} |\beta_{ij}||h_j(w_j) - h_j(z_j)|. \end{aligned} \tag{2.10}$$

2.8. **Lemma. 4.** [3] Let $z = (z_1, z_2, \cdots, z_n) \geq 0, 0 < \alpha < 1$ and $\beta > 1$. The following two inequalities hold

$$\sum_{i=1}^{n} z_i^{\alpha} \geq \left(\sum_{i=1}^{n} z_i\right)^{\alpha}, \sum_{i=1}^{n} z_i^{\beta} \geq n^{1-\beta}\left(\sum_{i=1}^{n} z_i\right)^{\beta}.$$

3. EXISTENCE OF FILIPPOV SOLUTIONS

Under certain conditions, in this section, we show that there exists at least one solution of the system (2.1) and (2.2), in the Filippov sense.

3.1. **Theorem.1.** Suppose $\mathcal{F}$ satisfies the growth condition (g.c.): there are two constants $\widetilde{k}_i, \widetilde{h}_i$, with $\widetilde{k}_i \geq 0$ such that

$$|\mathcal{F}(z_i)| = \sup_{\xi \in \mathcal{F}(z_i)} |\xi| \leq \widetilde{k}_i|z_i| + \widetilde{h}_i, i = 1, 2, \cdots, n. \tag{3.1}$$

So there is at least one solution for the system (2.1).

3.1.1. *Proof.* Based on the detailed discussions in Section 2, the set-valued map

$$\begin{aligned} z(t) &\hookrightarrow d(z(t))\big[-c(z(t)) + A(t)\mathcal{F}(z(t)) + \mathcal{B}\overline{\vartheta} + \mathcal{T}\overline{\vartheta} + \overline{\alpha}(t)\mathcal{F}(z(t-\tau_1(t))) \\ &+ \overline{\beta}(t)\mathcal{F}(z(t-\tau_1(t))) + \mathcal{S}\overline{\vartheta} + I(t)\big] \end{aligned}$$

Describe $\theta = \max\limits_{1\le i\le n}\left\{\max\limits_{-\widetilde{\tau}_1\le t\le 0}\theta_i(t)\right\}$. According to the inequality (3.1), for a.e. $t \in [0,+\infty)$ we obtain

$$\begin{aligned} (3.2)\ & \|d(z(t))\big[-c(z(t)) + A(t)\mathcal{F}(z(t)) + \mathcal{B}\overline{\vartheta} + \mathcal{T}\overline{\vartheta} + \overline{\alpha}(t)\mathcal{F}(z(t-\tau_1(t))) \\ + & \ \overline{\beta}(t)\mathcal{F}(z(t-\tau_1(t))) + \mathcal{S}\overline{\vartheta} + I(t)\big]\|_p \\ \le & \ \|d(z(t))\|_p\Big(\|c(z(t))\|_p + \|A(t)\|_p\|\mathcal{F}(z(t))\|_p + \|\mathcal{B}\overline{\vartheta}\|_p + \|\mathcal{T}\overline{\vartheta}\|_p \\ + & \ \|\overline{\alpha}(t)\|_p\|\mathcal{F}(z(t-\tau_1(t)))\|_p + \|\overline{\beta}(t)\|_p\|\mathcal{F}(z(t-\tau_1(t)))\|_p + \|\mathcal{S}\overline{\vartheta}\|_p + \|I(t)\|_p\Big) \\ \le & \ \|d(z(t))\|_p\|c(z(t))\|_p + \|d(z(t))\|_p\|A(t)\|_p(\mathcal{K}\|z(t)\|_p + \mathcal{H}) + \|d(z(t))\|_p|\overline{B}(t)\overline{\vartheta}\|_p \\ + & \ \|d(z(t))\|_p\|\mathcal{T}\overline{\vartheta}\|_p + \|d(z(t))\|_p\|\overline{\alpha}(t)\|_p(\mathcal{K}\|z(t-\tau_1(t))\|_p + \mathcal{H}) \\ + & \ \|d(z(t))\|_p\|\overline{\beta}(t)\|_p(\mathcal{K}\|z(t-\tau_1(t))\|_p + \mathcal{H}) + \|d(z(t))\|_p\|\mathcal{S}\overline{\vartheta}\|_p + \|d(z(t))\|_p\|I(t)\|_p \\ \le & \ \|d(z(t))\|_p\|c(z(t))\|_p + \|d(z(t))\|_p\|A(t)\|_p(\mathcal{K}\|z(t)\|_p + \mathcal{H}) + \|d(z(t))\|_p|\mathcal{B}\overline{\vartheta}\|_p \\ + & \ \|d(z(t))\|_p\|\mathcal{T}\overline{\vartheta}\|_p + \|d(z(t))\|_p\|\overline{\alpha}(t)\|_p(\widetilde{\theta}\mathcal{K} + \mathcal{K}\|z(t)\|_p + \mathcal{H}) \\ + & \ \|d(z(t))\|_p\|\overline{\beta}(t)\|_p(\widetilde{\theta}\mathcal{K} + \mathcal{K}\|z(t)\|_p + \mathcal{H}) + \|d(z(t))\|_p\|\mathcal{S}\overline{\vartheta}\|_p + \|d(z(t))\|_p\|I(t)\|_p \\ = & \ \big(\|d(z(t))\|_p\|A(t)\|_p\mathcal{K} + |d(z(t))\|_p\|\overline{\alpha}(t)\|_p\mathcal{K} + \|d(z(t))\|_p\|\overline{\beta}(t)\|_p\mathcal{K}\big)\,\|z(t)\|_p \\ + & \ \big(\|d(z(t))\|_p\|A(t)\|_p + \|d(z(t))\|_p\|\overline{\alpha}(t)\|_p + \|d(z(t))\|_p\|\overline{\beta}(t)\|_p\big)\,\mathcal{H} \\ + & \ \big(\|d(z(t))\|_p\|\overline{\alpha}(t)\|_p + \|d(z(t))\|_p\|\overline{\beta}(t)\|_p\big)\widetilde{\theta}\mathcal{K} \\ + & \ \big(\|d(z(t))\|_p\|c(z(t))\|_p + \|d(z(t))\|_p|\mathcal{B}\overline{\vartheta}\|_p + \|d(z(t))\|_p\|\mathcal{T}\overline{\vartheta}\|_p + \|d(z(t))\|_p\|\mathcal{S}\overline{\vartheta}\|_p \\ + & \ \|d(z(t))\|_p\|I(t)\|_p\big) \\ \le & \ \Psi\|z(t)\|_p + \widetilde{\Psi}. \end{aligned}$$

It follows that

$$\|z(t)\|_p \ \le \ \|z(0)\|_p + \|\int_0^t \dot{z}(s)ds\| \le \|z(0)\|_p + \widetilde{\Psi}t + \int_0^t \Psi\|z(s)\|_p ds. \tag{3.3}$$

According to the Gronwall inequality, one has

$$\|z(t)\|_p \ \le \ \Big(\|z(0)\|_p + \widetilde{\Psi}t\Big)\, e^{\Psi t}. \tag{3.4}$$

4. MAIN RESULTS

4.1. **Finite-time synchronization with non-continuous activation functions:** Define $r_i(t) = w_i(t) - z_i(t), i = 1, 2, \cdots, n$. At present, we propose $\mathcal{U}_i(t)$ as:

$$\begin{aligned}\mathcal{U}_i(t) &= -\zeta_i^1 r_i(t) - \zeta_i^2 sign(r_i(t)) - \zeta_i^3 sign(r_i(t))|r_i(t-\tau_1(t))| \\ &- \zeta_i^4 sign(r_i(t))|r_i(t)|^{\alpha},\end{aligned} \tag{4.1}$$

where $\zeta_i^1, \zeta_i^2, \zeta_i^3, \zeta_i^4 > 0$ and α satisfies $0 < \alpha < 1$.

4.2. **Theorem.2.** According to **Assumption.1**–**Assumption. 5**, if

$$\begin{cases}\sum_{j=1}^{n} |a_{ji}(t)|L_{1i} - \widehat{c}_i - \zeta_i^1 \leq 0, \\ \sum_{j=1}^{n} (|a_{ji}(t)|L_{2i} + |\alpha_{ji}(t)|L_{2i} + |\beta_{ji}(t)|L_{2i}) - \zeta_i^2 \leq 0, \\ \sum_{j=1}^{n} (|\alpha_{ji}(t)| + |\beta_{ji}(t)|)L_{1i} - \zeta_i^3 \leq 0, \\ i = 1, 2, \cdots, n.\end{cases} \tag{4.2}$$

Then, the systems (2.1) and (2.2) is synchronized under the controller (4.1) in finite time. Besides,

$$\widetilde{\mathrm{T}_0} \leq \frac{\|r(t_0)\|^{1-\alpha}}{\min\limits_{1\leq i\leq n}\{\zeta_i^4\}\underline{d}^{\alpha}(1-\alpha)}. \tag{4.3}$$

4.2.1. *Proof.* Constructing

$$V(t) = \sum_{i=1}^{n} SIGN(r_i(t)) \int_{z_i(t)}^{w_i(t)} \frac{ds}{d_i(s)}. \tag{4.4}$$

Obviously

$$\frac{1}{\overline{d}} \sum_{i=1}^{n} |r_i(t)| \leq V(t) \leq \frac{1}{\underline{d}} \sum_{i=1}^{n} |r_i(t)|, \tag{4.5}$$

where $\overline{d} = \max\left\{\overline{d}_i, i = 1, 2, \cdots, n\right\}$ and $\underline{d} = \min\left\{\underline{d}_i, i = 1, 2, \cdots, n\right\}$. Calculating the derivative of $V(t)$ and according to **Assumption. 1** – **Assumption. 5** and Lemma

2.7, we have

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{n} SIGN(r_i(t)) \left\{ \frac{\dot{w}_i(t)}{d_i(w_i(t))} - \frac{\dot{z}_i(t)}{d_i(z_i(t))} \right\}, \\
&= \sum_{i=1}^{n} SIGN(r_i(t)) \Big\{ - c_i(w_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{\gamma}_j(t) + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\widetilde{\gamma}_j(t-\tau_1(t)) \\
&+ \bigvee_{j=1}^{n} \beta_{ij}(t)\widetilde{\gamma}_j(t-\tau_1(t)) + \mathcal{U}_i(t) + c_i(z_i(t)) - \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) - \bigwedge_{j=1}^{n} \alpha_{ij}(t) \\
&\times \gamma_j(t-\tau_1(t)) - \bigvee_{j=1}^{n} \beta_{ij}(t)\gamma_j(t-\tau_1(t)), \\
&= \sum_{i=1}^{n} SIGN(r_i(t)) \Big\{ - c_i(w_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{\gamma}_j(t) + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\widetilde{\gamma}_j(t-\tau_1(t)) \\
&+ \bigvee_{j=1}^{n} \beta_{ij}(t)\widetilde{\gamma}_j(t-\tau_1(t)) - \zeta_i^1 r_i(t) - \zeta_i^2 SIGN(r_i(t)) - \zeta_i^3 SIGN(r_i(t)) \\
&\times |r_p(t-\tau_1(t))| - \zeta_i^4 SIGN(r_i(t))|r_i(t)|^{\alpha} + c_i(z_i(t)) - \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) \\
&- \bigwedge_{j=1}^{n} \alpha_{ij}(t)\gamma_j(t-\tau_1(t)) - \bigvee_{j=1}^{n} \beta_{ij}(t)\gamma_j(t-\tau_1(t)) \Big\}, \\
&= \sum_{i=1}^{n} SIGN(r_i(t)) \Big\{ - [c_i(w_i(t)) - c_i(z_i(t))] + \sum_{j=1}^{n} a_{ij}(t)[\widetilde{\gamma}_j(t) - \gamma_j(t)] \\
&+ \bigwedge_{j=1}^{n} \alpha_{ij}(t)[\widetilde{\gamma}_j(t-\tau_1(t)) - \gamma_j(t-\tau_1(t))] + \bigvee_{j=1}^{n} \beta_{ij}(t)[\widetilde{\gamma}_j(t-\tau_1(t)) \\
&- \gamma_j(t-\tau_1(t))] - \zeta_i^1 r_i(t) - \zeta_i^2 SIGN(r_i(t)) - \zeta_i^3 SIGN(r_i(t))|r_i(t-\tau_1(t))| \\
&- \zeta_i^4 SIGN(r_i(t))|r_i(t)|^{\alpha}, \\
&\leq -\sum_{i=1}^{n} |c_i(w_i(t)) - c_i(z_i(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t)||\widetilde{\gamma}_j(t) - \gamma_j(t)| + \sum_{i=1}^{n} |\bigwedge_{j=1}^{n} \alpha_{ij}(t) \\
&\times [\widetilde{\gamma}_j(t-\tau_1(t)) - \gamma_j(t-\tau_1(t))]| + \sum_{i=1}^{n} |\bigvee_{j=1}^{n} \beta_{ij}(t)[\widetilde{\gamma}_j(t-\tau_1(t)) - \gamma_j(t-\tau_1(t))]| \\
&- \sum_{i=1}^{n} \zeta_i^1 |r_i(t)| - \sum_{i=1}^{n} \zeta_i^2 - \sum_{i=1}^{n} \zeta_i^3 |r_i(t-\tau_1(t))| - \sum_{i=1}^{n} \zeta_i^4 |r_i(t)|^{\alpha} \\
&\leq -\sum_{i=1}^{n} \widehat{c}_i |r_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t)|[L_{1j}|r_j(t)| + L_{2j}] + \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_{ij}(t)|[L_{1j} \\
&\times |r_j(t-\tau_1(t))| + L_{2j}] + \sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}(t)|[L_{1j}|r_j(t-\tau_1(t))| + L_{2j}] - \sum_{i=1}^{n} \zeta_i^1 |r_i(t)| \\
&- \sum_{i=1}^{n} \zeta_i^2 - \sum_{i=1}^{n} \zeta_i^3 |r_i(t-\tau_1(t))| - \sum_{i=1}^{n} \zeta_i^4 |r_i(t)|^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left[\sum_{j=1}^{n}|a_{ji}(t)|L_{1i}-\widehat{c}_i-\zeta_i^1\right]|r_i(t)|+\sum_{i=1}^{n}\left[\sum_{j=1}^{n}(|\alpha_{ji}(t)|+|\beta_{ji}(t)|)L_{1i}-\zeta_i^3\right]\\
&\times |r_i(t-\tau_1(t))|+\sum_{i=1}^{n}\left[\sum_{j=1}^{n}(|a_{ji}(t)|L_{2i}+|\alpha_{ji}(t)|L_{2i}+|\beta_{ji}(t)|L_{2i})-\zeta_i^2\right]\\
&- \sum_{i=1}^{n}\zeta_i^4|r_i(t)|^\alpha \leq -\sum_{i=1}^{n}\zeta_i^4|r_i(t)|^\alpha, \qquad (4.6)
\end{aligned}
$$

Based on Lemma 2.8, it is easy to obtain that

$$
\sum_{i=1}^{n}|r_i(t)|^\alpha \;\geq\; \left[\sum_{i=1}^{n}|r_i(t)|\right]^\alpha, \qquad (4.7)
$$

which togethers with (4.6) and (4.5),

$$
\dot{V}(t) \;\leq\; -\min_{1\leq i\leq n}\{\zeta_i^4\}\left[[\sum_{i=1}^{n}|r_i(t)|\right]^\alpha \leq -\min_{1\leq i\leq n}\{\zeta_i^4\}\underline{d}^\alpha V^\alpha(t). \qquad (4.8)
$$

By the lemma 2.5, the finite-time synchronization between the systems (2.1) and (2.2) can be performed in the controller (4.1). The proof of the theorem 4.2 is well completed.

Remark.1

The model (2.1) is decomposed into the following cellular neural networks:

$$
\begin{aligned}
\dot{z}_i(t) &= -c_i(z_i(t))+\sum_{j=1}^{n}a_{ij}(t)h_j(z_j(t))+\sum_{j=1}^{n}b_{ij}\vartheta_j+\bigwedge_{j=1}^{n}\mathcal{T}_{ij}\vartheta_j\\
&+ \bigwedge_{j=1}^{n}\alpha_{ij}(t)h_j(z_j(t-\tau_1(t))+\bigvee_{j=1}^{n}\beta_{ij}(t)h_j(z_j(t-\tau_1(t))+\bigvee_{j=1}^{n}\mathcal{S}_{ij}\vartheta_j\\
&+ \mathcal{I}_i(t), \qquad (4.9)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{w}_i(t) &= -c_i(w_i(t))+\sum_{j=1}^{n}a_{ij}(t)h_j(w_j(t))+\sum_{j=1}^{n}b_{ij}\vartheta_j+\bigwedge_{j=1}^{n}\mathcal{T}_{ij}\vartheta_j\\
&+ \bigwedge_{j=1}^{n}\alpha_{ij}(t)h_j(w_j(t-\tau_1(t))+\bigvee_{j=1}^{n}\beta_{ij}(t)h_j(w_j(t-\tau_1(t))+\bigvee_{j=1}^{n}\mathcal{S}_{ij}\vartheta_j\\
&+ \mathcal{I}_i(t)+\mathcal{U}_i(t), \qquad (4.10)
\end{aligned}
$$

if $d_i(z_i(t)) = 1$ for all $t \geq 0$ and $i = 1, 2, \cdots, n$. In this case, **Assumption. 4** is true with $\overline{d}_i = \underline{d}_i = 1$ for $i = 1, 2, \cdots, n$. Hence, the following results are directly derived from Theorem 4.2.

4.3. **Corollary.** From **Assumption. 1** – **Assumption. 5**, if the conditions 4.2 are satisfied, so, the drive-response systems (4.9) and (4.10) are synchronized under the controller (4.1) in a finite time. In addition,

$$\widetilde{\mathrm{T}} \leq \frac{\|r(t_0)\|^{1-\alpha}}{\min\limits_{1\leq i\leq n}\{\zeta_i^4\}(1-\alpha)}. \tag{4.11}$$

4.4. **Fixed-time synchronization with non-continuous activation functions.** The delayed FQVCGNNs with non-continuous activation functions are investigated, in this subsection. So, the nonlinear controller is designed as follows:

$$\begin{aligned} \mathcal{U}_i(t) &= -\xi_i^1 r_i(t) - \xi_i^2 sign(r_i(t)) - \xi_i^3 sign(r_i(t))|r_i(t-\tau_1(t))| \\ &+ \amalg_1(t) sign(r_i(t))|r_i(t)|^\alpha + \amalg_2(t) sign(r_i(t))|r_i(t)|^\beta, \end{aligned} \tag{4.12}$$

where $\xi_i^1(t), \xi_i^2(t), \xi_i^3(t), \amalg_1(t)$ and $\amalg_2(t)$ are time-varying control gains to be determined later and $0 < \alpha < 1, \beta > 1$.

4.5. **Theorem.3.** Greater than **Assumption. 1** – **Assumption. 5**, if the following conditions (4.2) are verified and if there exists a positive constant $\mathcal{N}, \mathcal{M}_1, \mathcal{M}_2, \theta_1, \theta_2$ such that

$$\begin{aligned} \int_{t_0}^{t} \amalg_1^*(s)ds &\leq \mathcal{N}, \int_{t_0}^{t} \amalg_{1*}(s)ds \leq -\theta_1(t-t_0) + \mathcal{M}_1, \\ \int_{t_0}^{t} \amalg_2(s)ds &\leq -\theta_2(t-t_0) + \mathcal{M}_2. \end{aligned} \tag{4.13}$$

Then, the fixed-time synchronization between (2.1) and (2.2) can be achieved under controller (4.1). Moreover,

$$\begin{aligned} \mathrm{T}_{\max} &= t_0 + \frac{(1-\alpha)[n^{1-\alpha}\underline{d}^\alpha \mathcal{N} + \mathcal{M}_1] + 1}{\theta_1(1-\alpha)} \\ &+ \frac{(\beta-1)[n^{1-\alpha}\underline{d}^\alpha \mathcal{N} + n^{1-\beta}\underline{d}^\beta \mathcal{M}_2] + 1}{\theta_2 n^{1-\beta}\underline{d}^\beta(\beta-1)}. \end{aligned} \tag{4.14}$$

4.5.1. *Proof.* Choose the same Lyapunov function (4.4) in the proof of Theorem 4.2. We obtain

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{n} SIGN(r_i(t))\Big\{ -c_i(w_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{\gamma}_j(t) + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\widetilde{\gamma}_j(t-\tau_1(t)) \\
&+ \bigvee_{j=1}^{n} \beta_{ij}(t)\widetilde{\gamma}_j(t-\tau_1(t)) - \xi_i^1(t)r_i(t) - \xi_i^2(t)SIGN(r_i(t)) - \xi_i^3(t)SIGN(r_i(t)) \\
&\times |r_i(t-\tau_1(t))| + \text{II}_1(t)SIGN(r_i(t))|r_i(t)|^{\alpha} + \text{II}_2(t)SIGN(r_i(t))|r_i(t)|^{\beta} \\
&+ c_i(z_i(t)) - \sum_{j=1}^{n} a_{ij}(t)\gamma_j(t) - \bigwedge_{j=1}^{n} \alpha_{ij}(t)\gamma_j(t-\tau_1(t)) - \bigvee_{j=1}^{n} \beta_{ij}(t)\gamma_j(t-\tau_1(t))\Big\}, \\
&\le -\sum_{i=1}^{n} |c_i(w_i(t)) - c_i(z_i(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t)||\widetilde{\gamma}_j(t) - \gamma_j(t)| + \sum_{i=1}^{n} |\bigwedge_{j=1}^{n} \alpha_{ij}(t) \\
&\times [\widetilde{\gamma}_j(t-\tau_1(t)) - \gamma_j(t-\tau_1(t))]| + \sum_{i=1}^{n} |\bigvee_{j=1}^{n} \beta_{ij}(t)[\widetilde{\gamma}_j(t-\tau_1(t)) - \gamma_j(t-\tau_1(t))]| \\
&- \sum_{i=1}^{n} \xi_i^1|r_i(t)| - \sum_{i=1}^{n} \xi_i^2 - \sum_{i=1}^{n} \xi_i^3|r_i(t-\tau_1(t))| + \sum_{i=1}^{n} \text{II}_1(t)|r_i(t)|^{\alpha} \\
&+ \sum_{i=1}^{n} \text{II}_2(t)|r_i(t)|^{\beta}, \\
&\le -\sum_{i=1}^{n} \widehat{c}_i|r_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t)|[L_{1j}|r_j(t)| + L_{2j}] + \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_{ij}(t)|[L_{1j} \\
&\times |r_j(t-\tau_1(t))| + L_{2j}] + \sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}(t)|[L_{1j}|r_j(t-\tau_1(t))| + L_{2j}] - \sum_{i=1}^{n} \xi_i^1|r_i(t)| \\
&- \sum_{i=1}^{n} \xi_i^2 - \sum_{i=1}^{n} \xi_i^3|r_i(t-\tau_1(t))| + \sum_{i=1}^{n} \text{II}_1(t)|r_i(t)|^{\alpha} + \sum_{i=1}^{n} \text{II}_2(t)|r_i(t)|^{\beta},
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} |a_{ji}(t)|L_{1i} - \widehat{c}_i - \xi_i^1 \right] |r_i(t)| + \sum_{i=1}^{n} \left[\sum_{j=1}^{n} (|\alpha_{ji}(t)| + |\beta_{ji}(t)|)L_{1i} - \xi_i^3 \right] \\
&\times |r_i(t-\tau_1(t))| + \sum_{i=1}^{n} \left[\sum_{j=1}^{n} (|a_{ji}(t)|L_{2i} + |\alpha_{ji}(t)|L_{2i} + |\beta_{ji}(t)|L_{2i}) - \xi_i^2 \right] \\
&+ \sum_{i=1}^{n} \text{II}_1(t)|r_i(t)|^{\alpha} + \sum_{i=1}^{n} \text{II}_2(t)|r_i(t)|^{\beta}, \\
&\le \sum_{i=1}^{n} \text{II}_1|r_i(t)|^{\alpha} + \sum_{i=1}^{n} \text{II}_2|r_i(t)|^{\beta}, \\
&\le (\text{II}_1^*(t) + \text{II}_{1*}(t)) \sum_{i=1}^{n} |r_i(t)|^{\alpha} + \sum_{i=1}^{n} \text{II}_2(t)|r_i(t)|^{\beta}.
\end{aligned}
$$

According to Lemma 2.6 and Lemma 2.8, we have

$$\begin{aligned}\dot{V}(t) &\leq (\text{II}_1^*(t)+\text{II}_{1*}(t))\sum_{i=1}^{n}|r_i(t)|^{\alpha}+\sum_{i=1}^{n}\text{II}_2(t)|r_i(t)|^{\beta},\\ &\leq \left(\text{II}_1^*(t)n^{1-\alpha}+\text{II}_{1*}(t)\right)\left(\sum_{i=1}^{n}|r_i(t)|\right)^{\alpha}+\text{II}_2(t)n^{1-\beta}\left(\sum_{i=1}^{n}|r_i(t)|\right)^{\beta},\\ &= \left(\text{II}_1^*(t)n^{1-\alpha}+\text{II}_{1*}(t)\right)\underline{d}^{\alpha}V^{\alpha}(t)+\text{II}_2(t)n^{1-\beta}\underline{d}^{\beta}V^{\beta}(t).\end{aligned} \tag{4.15}$$

From Lemma 2.6, the fixed-time synchronization of drive-response systems (2.1) and (2.2) can be reached via controller (4.12). Furthermore,

$$\text{T}_{\max} = t_0+\frac{(1-\alpha)[n^{1-\alpha}\underline{d}^{\alpha}\mathcal{N}+\mathcal{M}_1]+1}{\theta_1(1-\alpha)}+\frac{(\beta-1)[n^{1-\alpha}\underline{d}^{\alpha}\mathcal{N}+n^{1-\beta}\underline{d}^{\beta}\mathcal{M}_2]+1}{\theta_2 n^{1-\beta}\underline{d}^{\beta}(\beta-1)}.$$

The proof of Theorem 4.5 is completed.

Remark.2

The lemma 2.6 degenerates into the traditional fixed-time convergence lemma used in [27, 22, 18], if the constructed Lyapunov function is differentiable and $athcalq_1(t) = -p_1, \text{II}_2(t) = -p_2$. The authors presented a new fixed time convergence lemma in [20], with $\dot{V}(t) \leq -p_1V^{\alpha}(t) - p_2V^{\beta}(t)$ for $t \geq t_0 \geq 0$. This new fixed-time convergence lemma is effective for the dynamics of discontinuous systems. That is, this lemma generalizes the existing fixed-time convergence lemmas in [27, 22, 18, 20].

Remark.3. In Theorem 4.2 and 4.5, the delayed FQVCGNNs is coped with its quaternion form without a decomposition. Moreover, it can be used to deal with the activations function or connection weights that cannot be explicitly expressed as real and imaginary parts. Then, a disadvantage of this approach is that the Hamilton rule has not been fully taken into consideration, thus the effect of the sign of each real and imaginary parts are neglected. Hence, non-decomposition method is proposed in our work.

5. **EXAMPLES**

Here, a number of simulation examples are illustrated. Firstly, we consider the simulation of Theorem 4.2 and in the second time, we consider the simulation of Theorem 4.5.

5.1. **Example 1.** The following delayed FQVCGNNs with non-continuous activation functions are investigated:

$$\begin{aligned}\dot{z}_i(t) &= d_i(z_i(t))\Big[-c_i(z_i(t)) + \sum_{j=1}^{n} a_{ij}(t)h_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}\vartheta_j + \bigwedge_{j=1}^{n}\mathcal{T}_{ij}\vartheta_j \\ &+ \bigwedge_{j=1}^{n}\alpha_{ij}(t)h_j(z_j(t-\tau_1(t)) + \bigvee_{j=1}^{n}\beta_{ij}(t)h_j(z_j(t-\tau_1(t)) + \bigvee_{j=1}^{n}\mathcal{S}_{ij}\vartheta_j \\ &+ \mathcal{I}_i(t)\Big], \quad (5.1)\end{aligned}$$

and

$$\begin{aligned}\dot{w}_i(t) &= d_i(w_i(t))\Big[-c_i(w_i(t)) + \sum_{j=1}^{n} a_{ij}(t)h_j(w_j(t)) + \sum_{j=1}^{n} b_{ij}\vartheta_j + \bigwedge_{j=1}^{n}\mathcal{T}_{ij}\vartheta_j \\ &+ \bigwedge_{j=1}^{n}\alpha_{ij}(t)h_j(w_j(t-\tau_1(t)) + \bigvee_{j=1}^{n}\beta_{ij}(t)h_j(w_j(t-\tau_1(t)) + \bigvee_{j=1}^{n}\mathcal{S}_{ij}\vartheta_j \\ &+ \mathcal{I}_i(t) + \mathcal{U}_i(t)\Big], \quad (5.2)\end{aligned}$$

where $i = 1, 2$ and the parameters are chosen to be:

$$\begin{aligned}d_1^\iota(z_1^\iota(t)) &= 0.5 + \frac{0.1}{1+(z_1^2(t))^\iota} \\ c_1^\iota(z_1^\iota(t)) &= 1.2z_1^\iota(t),\end{aligned}$$

$$\begin{aligned}d_2^\iota(z_2^\iota(t)) &= 0.5 + \frac{0.1}{1+(z_2^2(t))^\iota}, \\ c_2^\iota(z_2^\iota(t)) &= 1.4z_2^\iota(t),\end{aligned}$$

$$\begin{aligned}a_{11}(t) &= 0.2\sin(t) - 0.2\cos(t)i - 0.4\cos(t)j + 0.2\sin(t)k, \\ a_{12}(t) &= 0.3\cos(t) - 0.3\sin(t)i + 0.4\sin(t)j + 0.4\sin(t)k, \\ a_{21}(t) &= 0.3\cos(t) - 0.3\sin(t)i + 0.3\sin(t)j + 0.2\cos(t)k, \\ a_{22}(t) &= 0.2\sin(t) - 0.2\cos(t)i + 0.3\sin(t)j - 0.2\cos(t)k,\end{aligned}$$

$$\begin{aligned}\alpha_{11}(t) &= 0.3\cos(t) - 0.5\sin(t)i - 0.5\cos(t)j + 0.3\sin(t)k, \\ \alpha_{12}(t) &= 0.4\sin(t) - 0.3\sin(t)i - 0.4\cos(t)j + 0.5\cos(t)k, \\ \alpha_{21}(t) &= 0.7\sin(t) - 0.3\sin(t)i + 0.5\cos(t)j + 0.3\sin(t)k, \\ \alpha_{22}(t) &= 0.3\sin(t) - 0.2\cos(t)i + 0.2\sin(t)j + 0.9\cos(t)k,\end{aligned}$$

$$\begin{aligned}
\beta_{11}(t) &= 0.1\cos(t) - 0.3\sin(t)i - 0.3\cos(t)j + 0.1\sin(t)k,\\
\beta_{12}(t) &= 0.2\sin(t) - 0.1\sin(t)i - 0.2\cos(t)j + 0.3\cos(t)k,\\
\beta_{21}(t) &= 0.5\sin(t) - 0.1\sin(t)i + 0.3\cos(t)j + 0.1\sin(t)k,\\
\beta_{22}(t) &= 0.1\sin(t) - 0.1\cos(t)i + 0.1\sin(t)j + 0.7\cos(t)k,
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^{\iota}_{11} &= \mathcal{S}^{\iota}_{11} = b^{\iota}_{11} = \mathcal{T}^{\iota}_{12} = \mathcal{S}^{\iota}_{12} = b^{\iota}_{12} = 20,\\
\mathcal{T}^{\iota}_{21} &= \mathcal{S}^{\iota}_{21} = b_{21} = \mathcal{T}^{\iota}_{22} = \mathcal{S}^{\iota}_{22} = b^{\iota}_{22} = -20,\\
\mathcal{I}_1(t) &= 0.3\cos(t) - 0.2\sin(t)i + 0.2\cos(t)j + 0.2\sin(t)k,\\
\mathcal{I}_2(t) &= 0.2\cos(t) - 0.3\sin(t)i + 0.3\sin(t)j + 0.2\cos(t)k,\\
\tau_1(t) &= 0.5\sin(t) + 0.5, \widetilde{\tau}_1 = 1, \vartheta_1 = \vartheta_2 = 1,
\end{aligned}$$

for $\iota = \{(R), (I), (J), (K)\}$. The discontinuous activation functions is chosen as

$$\begin{aligned}
h_j(z_j) &= (z_j^{(R)} - 0.5)sign(z_j^{(R)}) + i(z_j^{(I)} - 0.25)sign(z_j^{(I)})\\
&+ j(z_j^{(J)} + 0.5)sign(z_j^{(J)}) + k(z_j^{(K)} - 0.25)sign(z_j^{(K)}).
\end{aligned}$$

It is not difficult to see that $h_j(z_j)$ is unbounded satisfying **Assumption. 3** with $L^{(\iota)}_{1j} = L^{(\iota)}_{2j} = 1$ for $j = 1, 2$ and $\iota = \{(R), (I), (J), (K)\}$. And, (0,0) is an discontinuous point with $\overline{co}[h_i^{(R)}(0)] = \overline{co}[h_i^{(J)}(0)] = [-0.5, 0.5]$ and $\overline{co}[h_i^{(I)}(0)] = \overline{co}[h_i^{(K)}(0)] = [-0.25, 0.25]$. Then, **Assumption. 2** is satisfied. Therefore, $0.5 \leq d_i(z) \leq 0.6$ and $\widehat{c}_1 = 1.2, \widehat{c}_2 = 1.4$. Then, the conditions in **Assumption. 4** – **Assumption. 5** are satisfied. We can construct a feedback controller in form of (4.1), where

$$\begin{aligned}
\zeta_1^{1(R)} &= 0.7, \zeta_1^{1(I)} = 0.8, \zeta_1^{1(J)} = 0.5, \zeta_1^{1(K)} = 0.9,\\
\zeta_2^{1(R)} &= 0.9, \zeta_2^{1(I)} = 1, \zeta_2^{1(J)} = 0.75, \zeta_2^{1(K)} = 0.85,\\
\zeta_1^{2(R)} &= 2.3, \zeta_1^{2(I)} = 1.8, \zeta_1^{2(J)} = 2.5, \zeta_1^{2(K)} = 1.4,\\
\zeta_2^{2(R)} &= 1.7, \zeta_2^{2(I)} = 1.5, \zeta_2^{2(J)} = 1.8, \zeta_2^{2(K)} = 3.2,\\
\zeta_1^{3(R)} &= 1.7, \zeta_1^{3(I)} = 1.3, \zeta_1^{3(J)} = 1.65, \zeta_1^{3(K)} = 0.85,\\
\zeta_2^{3(R)} &= 1.1, \zeta_2^{3(I)} = 0.75, \zeta_2^{3(J)} = 0.95, \zeta_2^{3(K)} = 2.45.
\end{aligned}$$

Moreover, we choose $\alpha = 0.5, \zeta_1^{4(R)} = \zeta_1^{4(I)} = \zeta_1^{4(J)} = \zeta_1^{4(K)} = 6.5$ and $\zeta_2^{4(R)} = \zeta_2^{4(I)} = \zeta_2^{4(J)} = \zeta_2^{4(K)} = 6$. From Theorem 4.2, the synchronization between two delayed FQVCGNNs (5.1) and (5.2) achieved in finite-time. Choosing the initial conditions as $z_1(0) = 10 - 0.4i + 0.5j + k, z_2(0) = -3 - 0.5i + 0.85j + k$ and $w_1(0) = -1.2 - 2.2i - 1.9j + 3.4k, w_2(0) = -6 - 2.7i + 1.8j + 2.2k$. The trajectories of the error states $r_1(t)$ and $r_2(t)$ are shown in figures 2 and 1, respectively. From the inequality (4.11), we have $\widetilde{\mathrm{T}}_0 \leq 2.3875$.

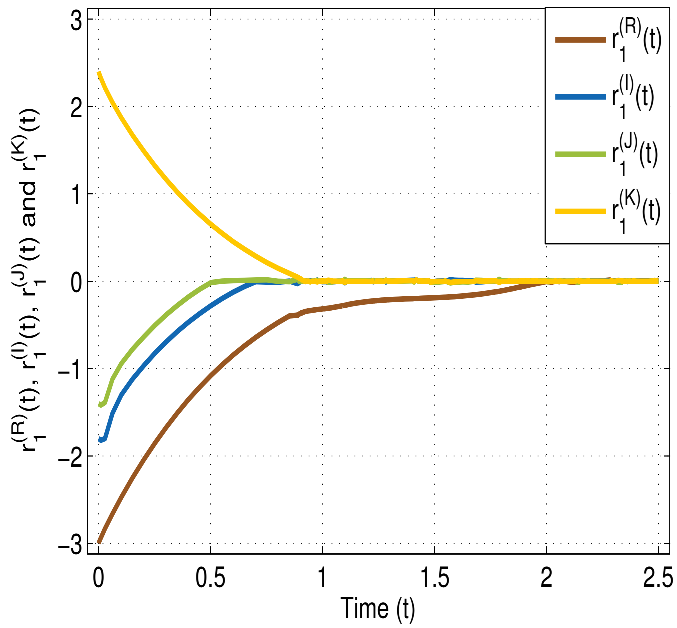

FIGURE 1. $r_1(t)$ under the controller (4.1) in Example 1.

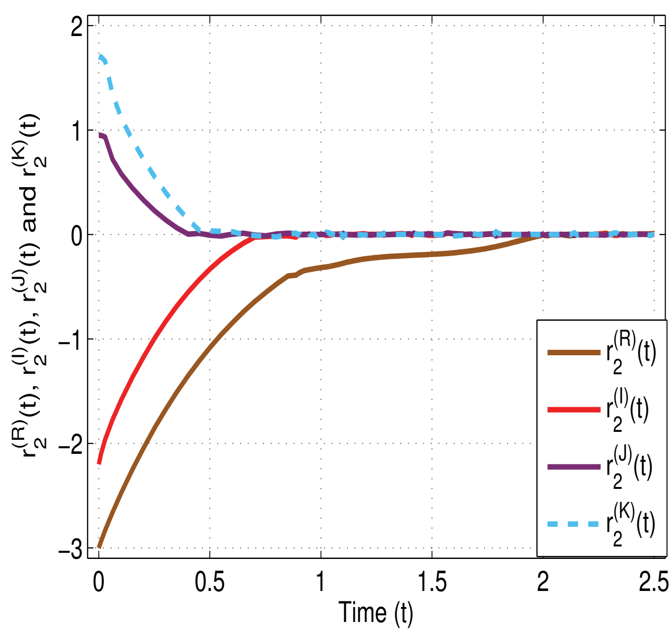

FIGURE 2. $r_2(t)$ under the controller (4.1) in Example 1.

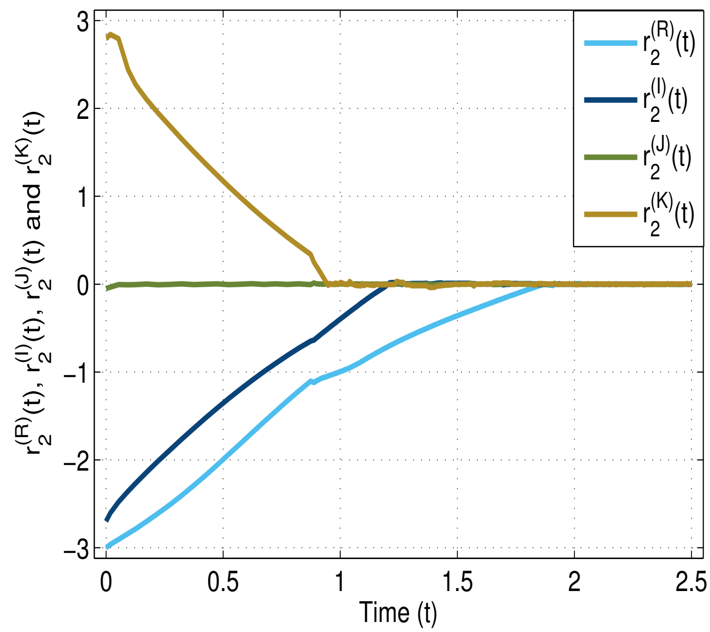

FIGURE 3. $r_1(t)$ under the controller (4.12) in Example 2.

5.2. **Example 2.** Lets consider the systems (5.1) and (5.2). Under the conditions (4.2) of the theorem 4.5, the parameters of the controller (4.1) are defined as follows:

$$\begin{aligned}
\zeta_1^{1(R)} &= 0.7, \zeta_1^{1(I)} = 0.8, \zeta_1^{1(J)} = 0.5, \zeta_1^{1(K)} = 0.9,\\
\zeta_2^{1(R)} &= 0.9, \zeta_2^{1(I)} = 1, \zeta_2^{1(J)} = 0.75, \zeta_2^{1(K)} = 0.85,\\
\zeta_1^{2(R)} &= 2.3, \zeta_1^{2(I)} = 1.8, \zeta_1^{2(J)} = 2.5, \zeta_1^{2(K)} = 1.4,\\
\zeta_2^{2(R)} &= 1.7, \zeta_2^{2(I)} = 1.5, \zeta_2^{2(J)} = 1.8, \zeta_2^{2(K)} = 3.2,\\
\zeta_1^{3(R)} &= 1.7, \zeta_1^{3(I)} = 1.3, \zeta_1^{3(J)} = 1.65, \zeta_1^{3(K)} = 0.85,\\
\zeta_2^{3(R)} &= 1.1, \zeta_2^{3(I)} = 0.75, \zeta_2^{3(J)} = 0.95, \zeta_2^{3(K)} = 2.45,
\end{aligned}$$

and chosen $\text{II}_1(t) = \dfrac{1}{1+t^2} - \dfrac{t}{2}|\cos(t)|, \text{II}_2(t) = -t|\sin(t)|$, with $\mathcal{N} = \dfrac{\pi}{2}, \mathcal{M}_1 = 1, \theta_1 = \dfrac{2}{3\pi}$ and $\mathcal{M}_2 = 2, \theta_2 = \dfrac{4}{3\pi}$. Based on Theorem 4.5, the synchronization between non-continuous drive-response systems FQVCGNNs (5.1) and (5.2) with time-varying delays, achieved in fixed-time. Choosing the initial conditions as $z_1(0) = 12 - 1.4i - 1.5j + 1.2k, z_2(0) = -5 - i + 2.85j + 0.7k$ and $w_1(0) = -1.5 - 3.2i - 2.9j + 3.4k, w_2(0) = -8 - 3.7i + 2.8j + 3.5k$. The trajectories of the error states $r_1(t)$ and $r_2(t)$ are shown in figures 3 and 4, respectively. Moreover, from the equality (4.14), we can estimate the settling $\text{T}_{\max} = 7.5697$.

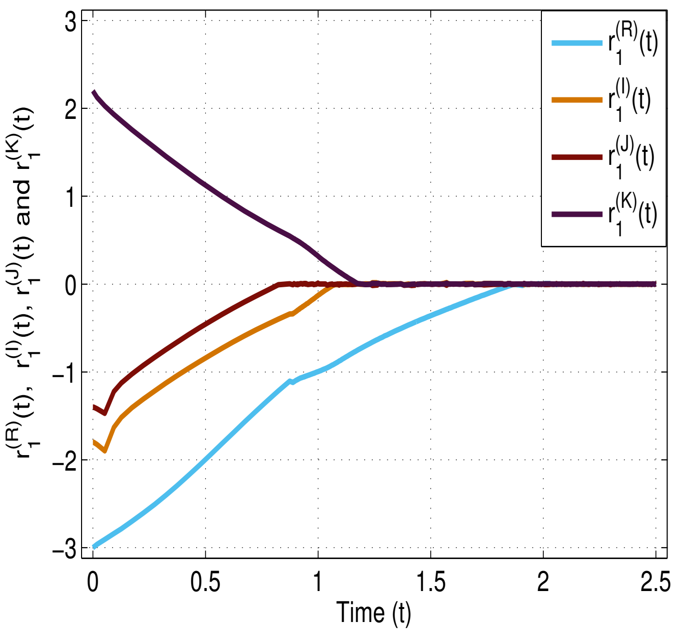

FIGURE 4. $r_2(t)$ under the controller (4.12) in Example 2.

6. CONCLUSIONS

In this paper, the discontinuous FQVCGNNs in finite-time and fixed-time synchronization are investigated. A novel fixed-time convergence theory is employed to solve the fixed-time synchronization of time-varying FQVCGNNs, that has a big advantage over traditional fixed time control methods. Due to the non-commutativity feature of quaternions, a non-decomposition method is proposed. Lastly, numerical examples are given to verify our results. Our future research will focus on the dynamical behavior of stochastic FQVCGNNs.

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