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APPROXIMATION OF MULTIPLE TIME SEPARATING RANDOM FUNCTIONS BY NEURAL NETWORKS REVISITED

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ABSTRACT. Here we study the multivariate quantitaive approximation of multiple time separating random functions over a $\mathbb{R}^N, N \in \mathbb{N}$, by the normalized bell and squashing type multivariate neural network operators. Activation functions here are of compact support. These approximations are derived by establishing Jackson type multivariate inequalities involving the multivariate modulus of continuity of the engaged random function or its high order partial derivatives. The approximations are pointwise and with respect to the L_P norm. The feed-forward neural networks are with one hidden layer. We finish with a variety of interesting applications.

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1. Introduction

The neural network multivariate Cardaliaguet-Euvrard operators were first introduced and studied thoroughly in [2], where the authors among many other interesting things proved that these multivariate operators converge uniformly on compact, to the unit over continuous and bounded multivariate functions. The multivariate normalized "bell" and "squashing" type operators (1) and (15) were motivated and inspired by the "bell" and "squashing" functions of [2]. The work in [2] is qualitative where the used multivariate bell-shaped function is general. However, though the work of the first author is greatly motivated by [2], it is quantitative and the used multivariate "bell-shaped" and "squashing" functions are of compact support, see [1], ch.2. Here we use a set of multivariate inequalities giving close upper bounds to the errors in approximating the unit operator by the above multidimensional neural network induced operators, see [1], ch.2. These are mainly pointwise estimates involving the

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first multivariate modulus of continuity of the engaged multivariate continuous function or its partial derivatives of some fixed order. The above mentioned theory is applied to perform quantitative approximations of multiple time separating random functions by neural networks. We finish with several interesting applications.
Specific motivations come by:

1. stationary Gaussian processes with an explicit representation such as
$$X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},$$
where ξ_1, ξ_2 are independent random variables with the standard normal distribution, see [3],
2. the "Fourier model" of a stationary process, see [4].

2. Quantitative Convergence by Multivariate Neural Network Operators

Here we follow [1], ch.2.

Definition 2.1. A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.

Definition 2.2. A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,
$$t \to b(x_1, ..., t, ..., x_d)$$
is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 2.3. Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1) ... b(x_d)$ is a d-dimensional bell-shaped function.

Assumption 2.4. Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function or a uniformly continuous function.

In [1], ch.2, the first author studied the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the "normalized bell" multivariate neural network operators
$$M_n(f)(\overrightarrow{x}) :=$$
$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}, \tag{1}$$
where $0 < \alpha < 1$ and $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly M_n is a positive linear operator.

The terms in the ratio of multiple sums (1) can be nonzero iff simultaneously

$$\left| n^{1-\alpha}\left(x_i-\frac{k_i}{n}\right)\right| \le T_i, \quad \text{all } i=1,...,d,$$

i.e., $\left|x_i-\frac{k_i}{n}\right| \le \frac{T_i}{n^{1-\alpha}}$, all $i=1,...,d$, iff

$$nx_i - T_i n^{\alpha} \le k_i \le nx_i + T_i n^{\alpha}, \quad \text{all } i=1,...,d. \tag{2}$$

To have the order

$$-n^2 \le nx_i - T_i n^{\alpha} \le k_i \le nx_i + T_i n^{\alpha} \le n^2, \tag{3}$$

we need $n \ge T_i + |x_i|$, all $i=1,...,d$. So (3) is true when we take

$$n \ge \max_{i\in\{1,...,d\}} \left(T_i + |x_i|\right). \tag{4}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (3) it is enough to assume that $n \ge 2T^*$, where $T^* := \max\{T_1,...,T_d\} > 0$. Consider

$$\widetilde{I_i} := \left[nx_i - T_i n^{\alpha}, nx_i + T_i n^{\alpha}\right], \;\; i=1,...,d, \;\; n \in \mathbb{N}.$$

The length of $\widetilde{I_i}$ is $2T_i n^{\alpha}$. By Proposition 1 of [1], Ch.2, we get that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I_i} := card\,(k_i) \ge \max\left(2T_i n^{\alpha} - 1, 0\right)$, any $i \in \{1,...,d\}$. In order to have $card\,(k_i) \ge 1$, we need $2T_i n^{\alpha} - 1 \ge 1$ iff $n \ge T_i^{-\frac{1}{\alpha}}$, any $i \in \{1,...,d\}$.

Therefore, a sufficient condition in order to obtain the order (3) along with the interval $\widetilde{I_i}$ to contain at least one integer for all $i=1,...,d$ is that

$$n \ge \max_{i\in\{1,...,d\}} \left\{T_i + |x_i|, T_i^{-\frac{1}{\alpha}}\right\}. \tag{5}$$

Clearly as $n \to +\infty$ we get that $card\,(k_i) \to +\infty$, all $i=1,...,d$. Also notice that $card\,(k_i)$ equals to the cardinality of integers in $\left[\lceil nx_i - T_i n^{\alpha}\rceil, \left[nx_i + T_i n^{\alpha}\right]\right]$ for all $i=1,...,d$. Here, $[\cdot]$ denotes the integral part of the number while. $\lceil\cdot\rceil$ denotes its ceiling.

From now on, in this article we will assume (5). Furthermore it holds

$$\left(M_n(f)\right)(\overrightarrow{x}) = \frac{\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} f\left(\frac{k_1}{n},...,\frac{k_d}{n}\right)}{V(\overrightarrow{x})}. \tag{6}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)$$

all $\overrightarrow{x} := (x_1,...,x_d) \in \mathbb{R}^d$, where

$$V(\overrightarrow{x}) :=$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

Denote by $\|\cdot\|_\infty$ the maximum norm on $\mathbb{R}^d$, $d \geq 1$. So if $\left|n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i$, all $i = 1, ..., d$, we get that

$$\left\| \overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right\|_\infty \leq \frac{T^*}{n^{1-\alpha}},$$

where $\overrightarrow{k} := (k_1, ..., k_d)$.

Definition 2.5. Let $f : \mathbb{R}^d \to \mathbb{R}$. We call

$$\omega_1(f,h) := \sup_{\substack{\text{all } \overrightarrow{x}, \overrightarrow{y}: \\ \|\overrightarrow{x} - \overrightarrow{y}\|_\infty \leq h}} \left| f(\overrightarrow{x}) - f(\overrightarrow{y}) \right|, \tag{7}$$

where $h > 0$, the first modulus of continuity of f.

Here we present the first main result.

Theorem 2.6. *Let $\overrightarrow{x} \in \mathbb{R}^d$; then*

$$\left|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})\right| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{8}$$

Inequality (8) is attained by constant functions.

In case f is uniformly continuous, then inequality (8) gives $M_n(f)(\overrightarrow{x}) \to f(\overrightarrow{x})$, pointwise with rates, as $n \to +\infty$, where $\overrightarrow{x} \in \mathbb{R}^d$, $d \geq 1$.

Proof. see [1], ch.2. □

The second main result follows.

Theorem 2.7. *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous and bounded. Then,*

$$\left|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})\right| \leq \tag{9}$$

$$\left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\overrightarrow{x}) \right) \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (9) is attained by constant functions. Also, inequality (9) gives us with rates the pointwise convergence of $M_n(f) \to f$ over $\mathbb{R}^d$, as $n \to +\infty$.

Proof. see [1], ch.2. □

Corollary 2.8. *Here, additionally assume that* b *is continuous on* $\mathbb{R}^d$*. Let*

$$\Gamma := \prod_{i=1}^{d} [-\gamma_i, \gamma_i] \subset \mathbb{R}^d, \quad \gamma_i > 0,$$

and take

$$n \geq \max_{i \in \{1,...,d\}} \left(T_i + \gamma_i, T_i^{-\frac{1}{\alpha}} \right).$$

Consider $p \geq 1$*. Then,*

$$\|M_n f - f\|_{p,\Gamma} \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{10}$$

attained by constant functions. From (10), we get the L_p *convergence of* $M_n f$ *to* f *with rates.*

Corollary 2.9. *Same assumptions as in Corollary 2.8. Then*

$$\|M_n f - f\|_{p,\Gamma} \leq \left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left\| \left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f \right\|_{p,\Gamma} \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{11}$$

attained by constants. Here, from (11), we get again the L_p *convergence of* $M_n(f)$ *to* f *with rates.*

3. The Multivariate "Normalized Squashing Type Operators" and their Convergence to the Unit with Rates

We give the following definition

Definition 3.1. Let the non-negative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d} (-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [2], ch.2).

Example 3.2. Let $\widehat{S}$ as above when $d = 1$. Then,

$$S(\overrightarrow{x}) := \widehat{S}(x_1) ... \widehat{S}(x_d), \quad \overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d,$$

is a multivariate "squashing function".

Let $f : \mathbb{R}^d \to \mathbb{R}$ be either uniformly continuous or continuous and bounded function.

For $\overrightarrow{x} \in \mathbb{R}^d$, we define the multivariate "normalized squashing type operator",

$$L_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \dots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, \dots, \frac{k_d}{n}\right) S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{W\left(\overrightarrow{x}\right)}, \tag{12}$$

where $0 < \alpha < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1, \dots, d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\alpha}} \right\}, \tag{13}$$

and

$$W\left(\overrightarrow{x}\right) := \sum_{k_1=-n^2}^{n^2} \dots \sum_{k_d=-n^2}^{n^2} S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right). \tag{14}$$

Obviously L_n is a positive linear operator. It is clear that

$$\left(L_n(f)\right)\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^\alpha\right]} \frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{\Phi\left(\overrightarrow{x}\right)} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right), \tag{15}$$

where

$$\Phi\left(\overrightarrow{x}\right) := \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^\alpha\right]} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right). \tag{16}$$

Here, we study the pointwise convergence with rates of $\left(L_n(f)\right)\left(\overrightarrow{x}\right) \to f\left(\overrightarrow{x}\right)$, as $n \to +\infty$, $\overrightarrow{x} \in \mathbb{R}^d$.

This is given by the next result.

Theorem 3.3. *Under the above terms and assumptions, we find that*

$$\left|\left(L_n(f)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{17}$$

Inequality (17) is attained by constant functions. In case f is uniformly continuous, then (17) give us the pointwise convergence $L_n f \to f$ as $n \to \infty$.

We also give

Theorem 3.4. *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then,*

$$\left|\left(L_n(f)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \tag{18}$$

$$\left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f\left(\overrightarrow{x}\right) \right) \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha} : |\widetilde{\alpha}| = N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (18) is attained by constant functions. Inequality (18) gives us with rates the pointwise convergence of $L_n(f) \to f$ over $\mathbb{R}^d$, as $n \to +\infty$.

Note 3.5. We see that

$$M_n(1) = L_n(1) = 1.$$

4. Multiple Time Separating Random Functions

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\omega \in \Omega; Y_1, Y_2, \ldots, Y_m, m \in \mathbb{N}$, be real-valued random variables on Ω with finite expectations, and $h_1(t), h_2(t), \ldots h_m(t) : \mathbb{R}^N \to \mathbb{R}$. Clearly, then

$$Y(t, \omega) := \sum_{i=1}^{m} h_i(t) Y_i(\omega),\ t \in \mathbb{R}^N, \tag{19}$$

is a quite common time separating random function.
We can assume that $h_i \in C^r(\mathbb{R}^N)$. Consequently, we have that the expectation

$$(EY)(t) = \sum_{i=1}^{m} h_i(t) EY_i \in C\left(\mathbb{R}^N\right) \text{ or } C^r\left(\mathbb{R}^N\right). \tag{20}$$

A classical example of multiple time separating process is

$$\left(\sin\left(\prod_{j=1}^{N} t_j\right)\right) Y_1(\omega) + \left(\cos\left(\prod_{j=1}^{N} t_j\right)\right) Y_2(\omega),\ t_j \in \mathbb{R},$$

for $j = 1, \ldots, N$.
Notice that $\left|\sin\left(\prod_{j=1}^{N} t_j\right)\right| \le 1$ and $\left|\cos\left(\prod_{j=1}^{N} t_j\right)\right| \le 1$.
Another typical example is

$$\left(\sinh\left(\prod_{j=1}^{N} t_j\right)\right) Y_1(\omega) + \left(\cosh\left(\prod_{j=1}^{N} t_j\right)\right) Y_2(\omega),\ t_j \in \mathbb{R},\ \text{for } j = 1, \ldots, N. \tag{21}$$

In this article we will apply the main results of Sections 2 and 3, to $f(t) = (EY)(t)$. We will finish with several applications.

5. Main Results

We present the following stochastic approximation result.

Theorem 5.1. *Let $(EY)(t)$ as in (20), $t \in \mathbb{R}^N$, $N \ge 1$. Then,*

$$|(M_n(EY))(t) - (EY)(t)| \le \omega_1\left((EY), \frac{T^*}{n^{1-\alpha}}\right). \tag{22}$$

Inequality (22) is attained by constant functions.

In case $h_i, i = 1, \ldots, m$ are uniformly continuous, then inequality (22) gives $M_n(EY)(t) \to (EY)(t)$, pointwise with rates, as $n \to +\infty$, where $t \in \mathbb{R}^N$, $N \ge 1$.

Proof. From Theorem 2.6. □

The next Theorem it follows

Theorem 5.2. *Let $(EY)(t)$ similar to (20), $t=(t_1,t_2,\ldots,t_N)\in\mathbb{R}^N$,with $h_i\in C^K\left(\mathbb{R}^N\right)$, $K,N\in\mathbb{N}$, with $N\geq 1$, such that all of its partial derivatives $h_{i,\widetilde{\alpha}}$ of order K, $\widetilde{\alpha}:|\widetilde{\alpha}|=K$, are uniformly continuous or continuous and bounded, for every $i=1,\ldots,m$. Then,*

$$|(M_n(EY))(t)-(EY)(t)|\leq \left\{\sum_{j=1}^{K}\frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial t_i}\right|\right)^j(EY)(t)\right)\right\}+ \frac{(T^*)^K N^K}{K!n^{K(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=K}\omega_1\left((EY)_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right). \tag{23}$$

Inequality (23) is attained by constant functions. Inequality (23) gives us with rates the pointwise convergence of $M_n(EY)\to(EY)$ over $\mathbb{R}^N$, as $n\to+\infty$.

Proof. Notice that all of the partial derivatives of $(EY)_{\widetilde{\alpha}}$ of order K, $\widetilde{\alpha}:|\widetilde{\alpha}|=K$, are uniformly continuous or continuous and bounded. Hence the result is coming from Theorem 2.7. □

Corollary 5.3. *Here, additionally assume that b is continuous on $\mathbb{R}^N$. Let*

$$\Gamma:=\prod_{i=1}^{N}[-\gamma_i,\gamma_i]\subset\mathbb{R}^N,\quad \gamma_i>0,$$

and take

$$n\geq\max_{i\in\{1,\ldots,N\}}\left(T_i+\gamma_i,T_i^{-\frac{1}{\alpha}}\right).$$

Consider $p\geq 1$. Then,

$$\|M_n(EY)-(EY)\|_{p,\Gamma}\leq\omega_1\left((EY),\frac{T^*}{n^{1-\alpha}}\right)2^{\frac{N}{p}}\prod_{i=1}^{N}\gamma_i^{\frac{1}{p}}, \tag{24}$$

attained by constant functions. From (24), we get the L_p convergence of $M_n(EY)$ to (EY) with rates.

Proof. From Corollary 2.8. □

Corollary 5.4. *Same assumptions as in Corollary 5.3. Then*

$$\|M_n(EY)-(EY)\|_{p,\Gamma}\leq\left\{\sum_{j=1}^{K}\frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left\|\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial t_i}\right|\right)^j(EY)\right\|_{p,\Gamma}\right\}+$$

$$\frac{(T^*)^K N^K}{K!n^{K(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=K}\omega_1\left((EY)_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right)2^{\frac{N}{p}}\prod_{i=1}^{N}\gamma_i^{\frac{1}{p}}, \tag{25}$$

attained by constants. Here, from (25), we get again the L_p convergence of $M_n(EY)$ to (EY) with rates.

Proof. From Corollary 2.9. □

Next we give the next result.

Theorem 5.5. *Let $(EY)(t)$ similar to (20), $t=(t_1,t_2,\dots,t_N)\in\mathbb{R}^N$,with h_i are uniformly continuous or continuous and bounded, for every $i=1,\dots,m$. Then, under the terms of Definition 3.1, we find that*

$$|(L_n(EY))(t)-(EY)(t)|\le\omega_1\left((EY),\frac{T^*}{n^{1-\alpha}}\right). \tag{26}$$

Inequality (26) is attained by constant functions. In case $h_i, i=1,2,\dots,m$ are uniformly continuous, then (26) give as the pointwise convergence $L_nEY\to EY$ as $n\to\infty$.

Proof. Notice that (EY) is uniformly continuous or continuous and bounded. Hence the result is coming from Theorem 3.3. □

We also give

Theorem 5.6. *Let $(EY)(t)$ similar to (20), $t=(t_1,t_2,\dots,t_N)\in\mathbb{R}^N$,with $h_i\in C^K(\mathbb{R}^N)$, $K,N\in\mathbb{N}$, with $N\ge1$, such that all of its partial derivatives $h_{i,\widetilde{\alpha}}$ of order K, $\widetilde{\alpha}:|\widetilde{\alpha}|=K$, are uniformly continuous or continuous and bounded, for every $i=1,\dots,m$. Then,*

$$|(L_n(EY))(t)-(EY)(t)|\le \tag{27}$$
$$\left\{\sum_{j=1}^{K}\frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial t_i}\right|\right)^j(EY)(t)\right)\right\}+$$
$$\frac{(T^*)^K N^K}{K!n^{K(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=K}\omega_1\left((EY)_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (27) is attained by constant functions. Inequality (27) gives us with rates the pointwise convergence of $L_n(EY)\to(EY)$ over $\mathbb{R}^N$, as $n\to+\infty$.

Proof. Notice that all of the partial derivatives of $(EY)_{\widetilde{\alpha}}$ of order K, $\widetilde{\alpha}:|\widetilde{\alpha}|=K$, are uniformly continuous or continuous and bounded. Hence the result is coming from Theorem 3.4. □

6. **Applications**

For the next applications we consider (Ω,F,P) be a probability space and Y_1,Y_2 be real valued random variables on Ω with finite expectations. We consider the stochastic processes $Z_i(t,\omega)$ for $i=1,2,\dots,6$, where $t=(t_1,\dots,t_N)\in\mathbb{R}^{\mathbb{N}}$ and $\omega\in\Omega$ as follows:

$$Z_1(t,\omega)=\sin\left(\xi\sum_{j=1}^{N}t_j\right)Y_1(\omega)+\cos\left(\xi\sum_{j=1}^{N}t_j\right)Y_2(\omega), \tag{28}$$

where $\xi > 0$ is fixed;

$$Z_2(t,\omega) = sech\left(\mu\sum_{j=1}^{N} t_j\right) Y_1(\omega) + \tanh\left(\mu\sum_{j=1}^{N} t_j\right) Y_2(\omega), \tag{29}$$

where $\mu > 0$ is fixed.

Here $sechx := \frac{1}{\cosh\left(\sum_{j=1}^{N} x_j\right)} = \frac{2}{\exp\left(\sum_{j=1}^{N} x_j\right)+\exp\left(-\sum_{j=1}^{N} x_j\right)}, x = (x_1, \ldots, x_n) \in \mathbb{R}^{\mathbb{N}}$.

$$Z_3(t,\omega) = \frac{1}{1+\exp\left(-\ell_1\prod_{j=1}^{N} t_j\right)} Y_1(\omega) + \frac{1}{1+\exp\left(-\ell_2\prod_{j=1}^{N} t_j\right)} Y_2(\omega), \tag{30}$$

where $\ell_1, \ell_2 > 0$ are fixed;

The expectations of $Z_i, i = 1,2,3$ are

$$(EZ_1)(t) = \sin\left(\xi\sum_{j=1}^{N} t_j\right) E(Y_1) + \cos\left(\xi\sum_{j=1}^{N} t_j\right) E(Y_2), \tag{31}$$

$$(EZ_2)(t) = sech\left(\mu\sum_{j=1}^{N} t_j\right) E(Y_1) + \tanh\left(\mu\sum_{j=1}^{N} t_j\right) E(Y_2), \tag{32}$$

$$(EZ_3)(t) = \frac{1}{1+\exp\left(-\ell_1\prod_{j=1}^{N} t_j\right)} E(Y_1) + \frac{1}{1+\exp\left(-\ell_2\prod_{j=1}^{N} t_j\right)} E(Y_2), \tag{33}$$

For the next $(EZ_i)(t), i = 1,2,3$ are as defined in relations between (31) and (33) respectively.

We present the following result.

Proposition 6.1. Let $t \in \mathbb{R}^N$, $N \geq 1$, Then for $i = 1,2,3$,

$$\left|(M_n(EZ_i))(t) - (EZ_i)(t)\right| \leq \omega_1\left((EZ_i), \frac{T^*}{n^{1-\alpha}}\right). \tag{34}$$

Inequality (34) is attained by constant functions. Inequality (34) gives $M_n(EZ_i)(t) \to (EZ_i)(t)$, pointwise with rates, as $n \to +\infty$, where $t \in \mathbb{R}^N$, $N \geq 1$.

Proof. From Theorem 5.1. □

Next we present

Proposition 6.2. Let $K, N \in \mathbb{N}$ with $N \geq 1$, $t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N$, Then

$$\left|(M_n(EZ_1))(t) - (EZ_1)(t)\right| \leq \tag{35}$$

$$\left\{\sum_{j=1}^{K} \frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial t_i}\right|\right)^j (EZ_1)(t)\right)\right\} +$$

$$\frac{(T^*)^K N^K}{K! n^{K(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=K} \omega_1 \left((EZ_1)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Inequality (35) is attained by constant functions. Also, (35) gives us with rates the pointwise convergence of $M_n (EZ_1) \to (EZ_1)$ over $\mathbb{R}^N$, as $n \to +\infty$.

Proof. From Theorem 5.2. □

Corollary 6.3. *Here, additionally assume that b is continuous on $\mathbb{R}^N$. Let*

$$\Gamma := \prod_{i=1}^{N} [-\gamma_i, \gamma_i] \subset \mathbb{R}^N, \quad \gamma_i > 0,$$

and take

$$n \geq \max_{i \in \{1,\ldots,N\}} \left(T_i + \gamma_i, T_i^{-\frac{1}{\alpha}} \right).$$

Consider $p \geq 1$. Then for $j = 1, 2, 3$,

$$\left\| M_n (EZ_j) - (EZ_j) \right\|_{p,\Gamma} \leq \omega_1 \left((EZ_j), \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{N}{p}} \prod_{i=1}^{N} \gamma_i^{\frac{1}{p}}, \tag{36}$$

attained by constant functions. From (36), we get the L_p convergence of $M_n (EZ_j)$ to (EZ_j) with rates.

Proof. From Corollary 5.3. □

Corollary 6.4. *Same assumptions as in Corollary 6.3. Then*

$$\left\| M_n (EZ_1) - (EZ_1) \right\|_{p,\Gamma} \leq \left\{ \sum_{j=1}^{K} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left\| \left(\sum_{i=1}^{N} \left| \frac{\partial}{\partial t_i} \right| \right)^j (EZ_1) \right\|_{p,\Gamma} \right\} +$$

$$\frac{(T^*)^K N^K}{K! n^{K(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=K} \omega_1 \left((EZ_1)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{N}{p}} \prod_{i=1}^{N} \gamma_i^{\frac{1}{p}}, \tag{37}$$

attained by constants. Here, from (37), we get again the L_p convergence of $M_n (EZ_1)$ to (EZ_1) with rates.

Proof. From Corollary 5.4. □

Next we give the next result.

Proposition 6.5. Let $t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N$. Then, under the terms of Definition 3.1, we find that for $i = 1, 2, 3$

$$\left| (L_n (EZ_i)) (t) - (EZ_i) (t) \right| \leq \omega_1 \left((EZ_i), \frac{T^*}{n^{1-\alpha}} \right). \tag{38}$$

Inequality (38) is attained by constant functions.

Proof. From Theorem 5.5. □

We also give

Proposition 6.6. Let $K, N \in \mathbb{N}$, with $N \geq 1$, $t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N$. Then

$$|(L_n(EZ_1))(t) - (EZ_1)(t)| \leq \tag{39}$$

$$\left\{\sum_{j=1}^{K} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{N} \left|\frac{\partial}{\partial t_i}\right|\right)^j (EZ_1)(t)\right)\right\} +$$

$$\frac{(T^*)^K N^K}{K! n^{K(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=K} \omega_1\left((EZ_1)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (39) is attained by constant functions. Also, (39) gives us with rates the pointwise convergence of $L_n(EZ_1) \to (EZ_1)$ over $\mathbb{R}^N$, as $n \to +\infty$.

Proof. From Theorem 5.6. □

Proposition 6.7. Let $N \in \mathbb{N}$ with $N \geq 1$, $t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N$, Then for $\ell = 1, 2, 3$

$$|(M_n(EZ_\ell))(t) - (EZ_\ell)(t)| \leq \tag{40}$$

$$\left\{\frac{T^*}{n^{(1-\alpha)}} \left(\sum_{i=1}^{N} \left|\frac{\partial}{\partial t_i}(EZ_\ell)(t)\right|\right)\right\} +$$

$$\frac{T^* N}{n^{(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=1} \omega_1\left((EZ_\ell)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (40) is attained by constant functions. Also, (40) gives us with rates the pointwise convergence of $M_n(EZ_\ell) \to (EZ_\ell)$ over $\mathbb{R}^N$, as $n \to +\infty$.

Proof. From Proposition 6.2. □

Corollary 6.8. *Same assumptions as in Corollary 6.3. Then for* $\ell = 1, 2, 3$

$$\|M_n(EZ_\ell) - (EZ_\ell)\|_{p,\Gamma} \leq \left\{\frac{T^*}{n^{(1-\alpha)}} \left\|\sum_{i=1}^{N} \left|\frac{\partial}{\partial t_i}(EZ_\ell)\right|\right\|_{p,\Gamma}\right\} +$$

$$\frac{T^* N}{n^{(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=1} \omega_1\left((EZ_\ell)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right) 2^{\frac{N}{p}} \prod_{i=1}^{N} \gamma_i^{\frac{1}{p}}, \tag{41}$$

attained by constants. Here, from (41), we get again the L_p *convergence of* $M_n(EZ_\ell)$ *to* (EZ_ℓ) *with rates.*

Proof. From Corollary 6.4. □

Proposition 6.9. Let $N \in \mathbb{N}$, with $N \geq 1$, $t = (t_1, t_2, \ldots, t_N) \in \mathbb{R}^N$. Then for $\ell = 1, 2, 3$

$$\left|\left(L_n\left(EZ_\ell\right)\right)(t) - \left(EZ_\ell\right)(t)\right| \leq \left\{\frac{T^*}{n^{(1-\alpha)}}\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial t_i}\left(EZ_\ell\right)(t)\right|\right)\right\} + \frac{T^* N}{n^{(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=1} \omega_1\left(\left(EZ_\ell\right)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right). \tag{42}$$

Inequality (42) is attained by constant functions. Also, (42) gives us with rates the pointwise convergence of $L_n\left(EZ_\ell\right) \to \left(EZ_\ell\right)$ over $\mathbb{R}^N$, as $n \to +\infty$.

Proof. From Proposition 6.6. □

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