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STABILITY AND BOUNDEDNESS THEOREMS OF SOLUTIONS OF CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we discuss certain conditions for uniform asymptotic stability and uniform ultimate boundedness of solutions to two systems of Aizermann-type of differential equations by means of second method of Lyapunov. In achieving our goal, some Lyapunov functions are constructed to serve as basic tools. The stability results in this paper, extend some stability results for some Aizermann-type of differential equations found in literature. Also, we prove some new results on uniform boundedness and uniform ultimate boundedness of solutions of systems of equations studied. Finally, we construct two numerical examples to show how valid our results are.

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1. Introduction

The systems of interest in this paper are the following

$$\dot{X} = F(X) + BY + P_1(t, X, Y), \quad \dot{Y} = CX + G(Y) + P_2(t, X, Y), \tag{1.1}$$

and

$$\dot{X} = AX + F(Y) + P_1(t, X, Y),\ \dot{Y} = CX + G(Y) + P_2(t, X, Y), \tag{1.2}$$

where $t \in \mathcal{R}^+ = [0, \infty)$; $A,\ B$ and C are real $n \times n$ constant symmetric matrices, $F,\ G : \mathcal{R}^n \to \mathcal{R}^n$ are C^1 functions satisfying $F(0) = 0 = G(0)$ and $P_1, P_2 : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$. Systems (1.1) and (1.2) are n-dimmensionl analogue of some Aizermann differential equations considered in [7], [11] and [13] when $P_1 = 0$ and $P_2 = 0$.

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The stability in the large of the systems

$$\dot{x} = ax + f_1(y),\ \ \dot{y} = f_2(x) + cy \tag{1.3}$$

$$\dot{x} = f_1(x) + ay,\ \ \dot{y} = bx + f_2(y)$$

where a, b, c are some positive constants, was first studied by Krasovskii[11] using the direct method of Lyapunov. Later Mufti [13] used geometrical approach to study system (1.3) and some others related systems, for asymptotic stability in the large of the zero solution. Mufti [14] also examined the stability property of another Aizermann equation of the form

$$\dot{x} = ay + xf(y),\ \ \dot{y} = bx + yg(x), \tag{1.4}$$

where a and b are some positive constants. Stability property of the zero solution of the corresponding n-dimensional analogue of equation (1.4) was later considered by Ezeilo [7]. Adeyanju *et al.*([2], [3], [4]) also established some sets of conditions that ensured the stability of the zero solution and boundedness of all solutions to the Aizermann systems of differential equations

$$\dot{X} = F(X) + BY + P_1(t, X, Y),\ \ \dot{Y} = G(X) + DY + P_2(t, X, Y),$$

and

$$\dot{X} = F(X) + H(Y) + P_1(t, X, Y),\ \ \dot{Y} = CX + DY + P_2(t, X, Y),$$

where $t \in \mathcal{R}^+ = [0, \infty)$, $X,\ Y\ \in \mathcal{R}^n$; $B,\ C$ and D are real $n \times n$ constant symmetric matrices, $F, G, H : \mathcal{R}^n \to \mathcal{R}^n$ are one time continuously differentiable functions(C^1) satisfying $F(0) = G(0) = H(0) = 0$ and $P_1, P_2 : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$.

It is worthy of mentioning here that, the results contained in this work are among few attempts in extending some known Aizermann scalar differential equations to their corresponding n-dimensional analogue after those in ([2], [3], [7]) as far as we know of literature available to us.

Thus, our motivation for this paper comes from the papers of Krasovskii [11], Mufti [13] and Ezeilo [7]. In [7] the stability of system similar to systems (1.1) and (1.2) was considered using second method of Lyapunov. Our goal is to also use this second method of Lyapunov to study stability, boundedness and ultimate boundedness of solutions of systems (1.1) and (1.2) which until now, remained open problems in literature(see, Ezeilo [7]).

2. Definitions and Preliminary Results

Below are some definitions and standard algebraic results needed to establish the proofs to our main results.

Definition 2.1 ([20])**.** The zero solution of equation (1.1) or (1.2) is stable, if given $\epsilon > 0$ and $t_0 \in I$, there exists a $\delta(t_0, \epsilon) > 0$, such that whenever, $\|X_0\| < \delta(t_0, \epsilon)$,

$$\|X(t; t_0, X_0)\| < \epsilon \text{ for all } t \geq t_0.$$

Definition 2.2 ([20])**.** The zero solution of equation (1.1) or (1.2) is uniformly stable, if it is stable and the δ in the definition of stability above is independent of t_0.

Definition 2.3 ([20])**.** The zero solution of equation (1.1) or (1.2) is asymptotically stable, if it is stable and in addition, there exists an $\alpha \in [t_1, t_2],\ t_0 \leq t_1 \leq t_2 \leq t$ such that if $\|X_0\| < \delta(t_0, \alpha)$, we have

$$\|X(t; t_0, X_0)\| \to 0 \text{ as } t \to \infty.$$

Definition 2.4 ([20])**.** The zero solution of equation (1.1) or (1.2) is uniformly asymptotically stable, if it is uniformly stable and if there is a $\delta > 0$ and $T(\epsilon)$, such that whenever $\|X_0\| < \delta$, we have

$$\|X(t; t_0, X_0)\| < \epsilon \text{ for all } t \geq t_0 + T(\epsilon).$$

Definition 2.5 ([20])**.** The solution of equation (1.1) or (1.2) is bounded if there exists a $\beta > 0$, such that $\|X(t; t_0, X_0)\| < \beta$ for all $t \geq t_0$, where β may depend on each solution.

Definition 2.6 ([20])**.** The solution of equation (1.1) or (1.2) is **u**niformly bounded if, for any $\alpha > 0$ and $t_0 \in I$, there exists a $\beta(\alpha) > 0$ such that if $X_0 \in S_\alpha$, then $\|X(t; t_0, X_0)\| < \beta(\alpha)$ for all $t \geq t_0$, where α is the length of interval.

Definition 2.7 ([20])**.** The solution of equation (1.1) or (1.2) is ultimately-bounded for bound M, if there exits an $M > 0$ and for every solution $X(t, t_0, X_0)$ of (1.5.1), there exists a $T = T(\alpha, X)$, such that

$$\|X(t; t_0, X_0)\| < M$$

for all $t \geq t_0 + T$.

Definition 2.8 ([20])**.** The solution of equation (1.1) or (1.2) is **u**niformly ultimately bounded for bound M, if there exists an $M > 0$ and if for any $\alpha > 0$ and $t_0 \in I$ there exists a $T(\alpha) > 0$ such that $X_0 \in S_\alpha$ implies that

$$\|X(t; t_0, X_0)\| < M$$

for all $t \geq t_0 + T(\alpha)$.

Lemma 2.9. *([1], [6], [10], [15], [17])*
Let A be a real $n \times n$-constant symmetric matrix and

$$\delta_a \leq \lambda_i(A) \leq \Delta_a, \ (i = 1, 2, ..., n),$$

where δ_a and Δ_a are positive constants representing the least and greatest eigenvalues of matrix A respectively.
Then,

$$\delta_a \langle X, X \rangle \leq \langle AX, X \rangle \leq \Delta_a \langle X, X \rangle.$$

Lemma 2.10. *([7])*
Let $H : \mathcal{R}^n \to \mathcal{R}^n$ be of class C^1 and suppose that $H(0) = 0$.

(i) *Then for any $X \in \mathcal{R}^n$,*

$$H(X) = \int_0^1 J_h(sX) X ds,$$

where $J_h(X)$ is the Jacobian matrix of $H(X)$;

(ii) *Let $J_h(X)$ be symmetric and commutes with a certain real $n \times n$ symmetric matrix E. Then*

$$\frac{d}{dt} \int_0^1 \langle EH(sX), X \rangle ds = \langle EH(X), \dot{X} \rangle,$$

for any real differentiable vector $X = X(t) \in \mathcal{R}^n$.

Lemma 2.11. *([5], [8], [9])*
Let A, B be any two real $n \times n$ symmetric positive definite matrices. Then, for $(i, j, k = 1, 2, \ldots, n)$,

(i) *the eigenvalues $\lambda_i(AB)$ of the product matrix AB are real and satisfy*

$$\min_{1 \leq j,k \leq n} \lambda_j(A) \lambda_k(B) \leq \lambda_i(AB) \leq \max_{1 \leq j,k \leq n} \lambda_j(A) \lambda_k(B);$$

(ii) *the eigenvalues $\lambda_i(A + B)$ of the sum of matrices A and B are real and satisfy*

$$\{\min_{1 \leq j \leq n} \lambda_j(A) + \min_{1 \leq k \leq n} \lambda_k(B)\} \leq \lambda_i(A + B) \leq \{\max_{1 \leq j \leq n} \lambda_j(A) + \max_{1 \leq k \leq n} \lambda_k(B)\}.$$

Lemma 2.12. ***(LaSalle's invariance principle)*** [18]
If V is a Lyapunov function on a set G and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \geq 0$, then $\omega(\phi) \neq 0$ is contained in the largest invariant subset of $E \equiv \{\psi \in G^* : V(\dot{\psi}) = 0\}$, where G^* is the closure of set G and ω denote the omega limit set of a solution.

3. **Formulation of Main Results for system** (1.1)

In this section, we state and provide the proofs of our main results regarding the system (1.1).

Theorem 3.1. *Let $J_f(X)$ and $J_g(Y)$ denote the Jacobian matrices $\frac{\partial f_i}{\partial x_j}$, $\frac{\partial g_i}{\partial y_j}$ of $F(X)$ and $G(Y)$ respectively. Suppose further that:*

(i) *the matrices $B, J_f(X), J_g(Y)$ are all symmetric and negative definite, while, C is symmetric and positive definite such that for some positive constants $\delta_1, \delta_2, \delta_3, \delta_4, \Delta_1, \Delta_2, \Delta_3$ and Δ_4, we have*

$$\delta_1 \le \lambda_i(C) \le \Delta_1,$$
$$\delta_2 \le \lambda_i(-B) \le \Delta_2,$$
$$-\Delta_3 \le \lambda_i(CJ_f(X)) \le -\delta_3,$$
$$\delta_4 \le \lambda_i(BJ_g(Y)) \le \Delta_4,$$

$(i, j = 1, 2, \ldots, n)$,

(ii) *matrix C commutes with matrix $J_f(X)$ while matrix B commutes with matrix $J_g(Y)$,*

(iii) *$P_1(t, X, Y) = 0$ and $P_2(t, X, Y) = 0$.*

Then, the trivial solution of system (1.1) *is asymptotically stable.*

Theorem 3.2. *Suppose that assumptions in (iii) of Theorem* (3.1) *are replaced by*

(iii) *$\|P_1(t, X, Y)\| \le \phi(t)$, $\|P_2(t, X, Y)\| \le \theta(t)$, for all $t \ge 0, X, Y \in \mathcal{R}^n$, $\max \phi(t) < \infty$, $\max \theta(t) < \infty$ and $\phi(t)$, $\theta(t) \in L^1(0, \infty)$, where $L^1(0, \infty)$ is the space of integrable Lebesgue functions.*

Then, solutions of system (1.1) *are bounded.*

Theorem 3.3. *Further to the assumptions (i)-(ii) of Theorem* (3.1)*, let*

(iii) *$\|P_1(t, X, Y)\| \le \phi_1(t)\{1 + \|X\|\}$, $\|P_2(t, X, Y)\| \le \theta_1(t)\{1 + \|Y\|\}$, for all $t \ge 0, X, Y \in \mathcal{R}^n$, $\max \phi_1(t) < \infty$, $\max \theta_1(t) < \infty$ and $\phi_1(t), \theta_1(t) \in L^1(0, \infty)$, where $L^1(0, \infty)$ is the space of integrable Lebesgue functions.*

Then, any solution $(X(t), Y(t))$ of system (1.1) *with the initial condition*

$$X(0) = X_0, Y(0) = Y_0$$

satisfies

$$\|X(t)\| \le K_{11}, \ \|Y(t)\| \le K_{11}$$

for all $t \ge 0, X, Y \in \mathcal{R}^n$, where $K_{11} > 0$ is a constant depending on B, C, $\theta_1(t)$, $\phi_1(t)$, X_0, Y_0, and on the functions $P_1(t, X, Y)$, and $P_2(t, X, Y)$.

Theorem 3.4. *Under the assumptions of Theorem* (3.2) *or Theorem* (3.3)*, all the solutions of system* (1.1) *are uniform-ultimately bounded.*

4. Proof of Main Results on system (1.1)

We shall make use of the continuously differentiable scalar function $V = V(X, Y)$ defined below to establish the proofs of our main results.

$$2V(X,Y) = \langle X, CX\rangle - \langle BY, Y\rangle, \tag{4.1}$$

where matrices B, C are as defined in Theorem (3.1). The following Lemmas are essential to prove our main results.

Lemma 4.1.
Suppose, under the assumptions of Theorem (3.1) *there exist some constants K_1 and K_2 both positive such that the function V defined by equation* (4.1), *satisfies*

$$K_1\{\|X(t)\|^2 + \|Y(t)\|^2\} \leq 2V(X,Y) \leq K_2\{\|X\|^2 + \|Y\|^2\} \tag{4.2}$$

and

$$V(X(t), Y(t)) \to +\infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty.$$

Furthermore, there exists a positive constant K_3 such that for any solution $(X(t), Y(t))$ of (1.1) *we have*

$$\dot{V} \leq -K_3\{\|X(t)\|^2 + \|Y(t)\|^2\},$$

for all $t \geq 0, X, Y \in \mathcal{R}^n$.

Proof. It is obvious from equation (4.1) that $V(X,Y) = 0$ when $X(t) = Y(t) = 0$. By applying assumptions in (i) of Theorem (3.1) and Lemma (2.9) to the terms contained in (4.1) , we can always find some positive constants δ_1 and δ_2 such that

$$\langle CX, X\rangle \geq \delta_1\|X\|^2$$

and

$$-\langle BY, Y\rangle \geq \delta_2\|Y\|^2.$$

Thus,

$$2V \geq \delta_1\|X\|^2 + \delta_2\|Y\|^2.$$

If we take $K_1 = \min\{\delta_1, \delta_2\}$, then, we obtain,

$$2V \geq K_1\{\|X\|^2 + \|Y\|^2\}. \tag{4.3}$$

for all $t \geq 0, X, Y \in \mathcal{R}^n$. Thus, it follows from (4.3) that $V(X,Y) = 0$ if and only if $\|X\|^2 + \|Y\|^2 = 0$ and $V(X,Y) > 0$ if and only if $\|X\|^2 + \|Y\|^2 \neq 0$, which now implies that

$$V(X,Y) \to \infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty.$$

We also proceed to get the upper bound for the function $V(X,Y)$. From the assumptions listed in (i) of Theorem (3.1) and Lemma (2.9), we have

$$2V \leq \Delta_1\langle X, X\rangle + \Delta_2\langle Y, Y\rangle.$$

On setting

$$K_2 = \max\{\Delta_1, \Delta_2\},$$

we obtain

$$2V \le K_2\{\|X\|^2 + \|Y\|^2\}. \tag{4.4}$$

Thus, inequality (4.2) follows immediately on combining inequalities (4.3) and (4.4).

In what follows, we obtain the derivative of V with respect to t along the solution path of (1.1) such that it satisfies

$$\dot{V}|_{(1.1)} \equiv \frac{d}{dt}V(X,Y)|_{(1.1)} \le -K_4$$

provided that $\|X\|^2 + \|Y\|^2 \le K_5$, both K_4 and K_5 are some positive constants. The derivative of V along (1.1) is

$$\begin{aligned}\dot{V}|_{(1.1)} &= \langle CX, F(X)\rangle - \langle BY, G(Y)\rangle \\ &= \int_0^1 \langle CX, J_f(s_1X)X\rangle ds_1 - \int_0^1 \langle BY, J_g(s_1Y)Y\rangle ds_2.\end{aligned}$$

In view of the assumptions listed (i) of Theorem (3.1) and Lemma (2.9), there exist some positive constants δ_3 and δ_4 such that:

$$\langle CX, J_f(s_1X)X\rangle \le -\delta_3\|X\|^2$$

and

$$-\langle BY, J_g(s_1Y)Y\rangle \le -\delta_4\|Y\|^2.$$

Therefore,

$$\begin{aligned}\dot{V}|_{(1.1)} &\le -\{\delta_3\|X\|^2 + \delta_4\|Y\|^2\}\int_0^1 ds_1, \\ &\le -\{\delta_3\|X\|^2 + \delta_4\|Y\|^2\}.\end{aligned}$$

Let $K_3 = \min\{\delta_3,\ \delta_4\}$, then we get,

$$\begin{aligned}\dot{V}|_{(1.1)} &\le -K_3\{\|X\|^2 + \|Y\|^2\} \\ &\le -K_4\end{aligned} \tag{4.5}$$

for all $t \ge 0, X, Y \in \mathcal{R}^n$, where $K_4 > 0$ is constant. This completes the proof of Lemma (4.1). □

Lemma 4.2. *Suppose under the assumptions of Theorem* (3.2) *there exists some positive constants K_3 and K_6 such that for any solution (X, Y) of the system* (1.1), *the function V defined by equation* (4.1), *satisfies*

$$\dot{V}|_{(1.1)} \le -K_3\{\|X\|^2+\|Y\|^2\}+\Delta_c(1+\|X\|^2)\|P_1(t,X,Y)\|+\Delta_2(1+\|Y\|^2)\|P_2(t,X,Y)\|,$$

for all $t \ge 0, X, Y \in \mathcal{R}^n$.

Proof. If we follow the same pattern used in the proof of Lemma (4.1), but in this case, $P_1(t,X,Y) \neq 0$, and $P_2(t,X,Y) \neq 0$, we obtain

$$\begin{aligned} \dot{V}|_{(1.1)} &\leq -K_3\{\| X \|^2 + \| Y \|^2\} + \langle CX, P_1(t,X,Y)\rangle - \langle BY, P_2(t,X,Y)\rangle \\ &\leq -K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_c\|X\|\|P_1(t,X,Y)\| + \Delta_2\|Y\|\|P_2(t,X,Y)\|. \end{aligned} \tag{4.6}$$

By applying the following identities

$$\|X\| \leq 1 + \|X\|^2 \text{ and } \|Y\| \leq 1 + \|Y\|^2$$

in (4.6), we obtain

$$\dot{V}|_{(1.1)} \leq -K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_c\{1+\|X\|^2\}\|P_1(t,X,Y)\| + \Delta_2\{1+\|Y\|^2\}\|P_2(t,X,Y)\|,$$

for all $t \geq 0, X, Y \in \mathcal{R}^n$.
This completes the proof of Lemma (4.2). □

Proof Theorem (3.1)
It is clear from inequalities (4.3) and (4.5) of the proof of Lemma (4.1) that the trivial/zero solution of the system (1.1) is stable.

Consider the set W define by

$$W = \{(X,Y) : \dot{V}|_{(1.1)}(X,Y) = 0\}.$$

By using LaSalle's invariance principle, we observe that $(X,Y) \in W$ implies that $X = Y = 0$. Hence, this shows that the largest invariant set contained in W is $(0,0) \in W$. Therefore, we conclude that the zero solution of the system (1.1) is asymptotically stable and this completes the proof of Theorem (3.1).

Corollary 4.3. *As a corollary to Theorem* (3.1)*, in view of inequalities* (4.3)*,* (4.4) *and* (4.5) *of Lemma* (4.1)*,*
(i) the trivial solution of the system (1.1) *is uniformly stable.*
(ii) and the conclusion of the proof of Theorem (3.1) *based on LaSalle's invariance principle, the trivial solution is uniformly asymptotically stable.*

Proof of Theorem (3.2)
From the conclusion of Lemma (4.2), we have

$$\begin{aligned} \dot{V}|_{(1.1)} &\leq -K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_c\{1+\|X\|^2\}\|P_1(t,X,Y)\| + \Delta_2\{1+\|Y\|^2\}\|P_2(t,X,Y)\| \\ &\leq \Delta_2\theta(t) + \Delta_c\phi(t) + \Delta_c\phi(t)\|X\|^2 + \Delta_2\theta(t)\|Y\|^2. \end{aligned} \tag{4.7}$$

From inequality (4.3), we have that

$$\|Y\|^2 \leq \|Y\|^2 + \|X\|^2 \leq 2K_1^{-1}V \text{ and } \|X\|^2 \leq \|Y\|^2 + \|X\|^2 \leq 2K_1^{-1}V.$$

On putting these into (4.7), we obtain

$$\dot{V}|_{(1.1)} \leq \Delta_2\theta(t) + \Delta_c\phi(t) + 2K_1^{-1}\{\Delta_2\theta(t) + \Delta_c\phi(t)\}V. \tag{4.8}$$

Letting $\theta_6(t) = \Delta_2\theta(t) + \Delta_c\phi(t)$ in (4.8) and integrate between 0 to t, $(t > 0)$, gives

$$V(t) \leq V(0) + \Delta_2 \int_0^t \theta_6(s)ds + 2\Delta_2 K_1^{-1} \int_0^t \theta_6(s)V(s)ds.$$

Setting

$$W_1 = V(0) + \Delta_2 \int_0^\infty \theta_6(s)ds \text{ and } W_2 = 2\Delta_2 K_1^{-1},$$

then,

$$V(t) \leq W_1 + W_2 \int_0^\infty V(s)\theta_6(s)ds.$$

On applying Gronwall-Bellman inequality [16], we have

$$V(t) \leq W_1 \exp(W_2 \int_0^\infty \theta_6(s)ds) \leq K_8, \tag{4.9}$$

where K_8 is a positive constant.

Thus, using estimate (4.3) in (4.9), one can easily conclude that all solutions of system (1.1) are bounded and this completes the proof of Theorem (3.2).

Corollary 4.4. *Under the assumptions of Theorem* (3.2) *all solutions of system* (1.1) *are uniformly bounded.*

Proof of Theorem (3.3)

The proof of this theorem is as follows. From the proof of Theorem (3.2), we know that

$$\dot{V}|_{(1.1)} \leq -K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_2\|Y\|\|P_2(t, X, Y)\| + \Delta_c\|X\|\|P_1(t, X, Y)\|.$$

Using the assumption (iii) of Theorem (3.3), we obtain

$$\begin{aligned}\dot{V}|_{(1.1)} &\leq \Delta_2\theta_1(t)\|Y\|\{1 + \|Y\|\} + \Delta_c\phi_1(t)\|X\|\{1 + \|X\|\} \\ &\leq \Delta_2\theta_1(t)\{\|Y\| + \|Y\|^2\} + \Delta_c\phi_1(t)\{\|X\| + \|X\|^2\}.\end{aligned}$$

Applying these two identities

$$\|Y\| \leq 1 + \|Y\|^2 \text{ and } \|X\| \leq 1 + \|X\|^2,$$

we have

$$\dot{V}|_{(1.1)} \leq \Delta_2\theta_1(t) + \Delta_c\phi_1(t) + 2\Delta_2\theta_1(t)\|Y\|^2 + 2\Delta_c\phi_1(t)\|X\|^2, \tag{4.10}$$

for all $t \geq 0, X, Y \in \mathcal{R}^n$. Now, from the inequality (4.3), we have

$$\|Y\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2K_1^{-1}V(t, X, Y)$$

and

$$\|X\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2K_1^{-1}V(t, X, Y).$$

On applying these facts in (4.10), we get

$$\dot{V}|_{(1.1)} \leq \Delta_2\theta_1(t) + \Delta_c\phi_1(t) + 4K_1^{-1}\{\Delta_2\theta_1(t) + \Delta_c\phi_1(t)\}V(X, Y)$$

$$\dot{V}|_{(1.1)} \leq \theta_6(t) + 4K_1^{-1}\theta_6(t)V(X, Y), \tag{4.11}$$

where $\theta_6 = \Delta_2\theta_1(t) + \Delta_c\phi_1(t)$. The integration of both sides of (4.11) between 0 to t, $(t > 0)$, gives

$$V(t) \leq V(X(0), Y(0)) + \int_0^t \theta_6(s)ds + 4K_1^{-1}\int_0^t \theta_6(s)V(s)ds.$$

Letting

$$W_3 = V(X(0), Y(0)) + \int_0^\infty \theta_6(s)ds \text{ and } W_4 = 4K_1^{-1},$$

then,

$$V(t) \leq W_3 + W_4\int_0^\infty V(s)\theta_6(s)ds.$$

By applying Gronwall-Bellman inequality [16], we have

$$V(t) \leq W_3 \exp(W_4\int_0^\infty \theta_6(s)ds) \leq K_9 \tag{4.12}$$

where $K_9 > 0$ is a constant. On using (4.3) in the inequality (4.12), we obtain

$$\|X\|^2 + \|Y\|^2 \leq 2K_9K_1^{-1} = K_{10}$$

and this implies

$$\|X\|^2 \leq K_{10} \text{ and } \|Y\|^2 \leq K_{10}.$$

This completes the proof of Theorem (3.3).

Proof of Theorem (3.4)**.**
From Lemma (4.2) we have that the derivative $\dot{V}|_{(1.1)}$ of the function V defined in (4.1), satisfied

$$\dot{V}|_{(1.1)} \leq -K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_2\|Y\|\|P_2(t, X, Y)\| + \Delta_c\|X\|\|P_1(t, X, Y)\|.$$

From the assumption (iii) of Theorem (3.3), we obtain

$$\begin{aligned}\dot{V}|_{(1.1)} \leq& - K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_c\phi_1(t)\|X\|\{1 + \|X\|\} + \Delta_2\theta_1(t)\|Y\|\{1 + \|Y\|\}\\ \leq& - K_3\{\| X \|^2 + \| Y \|^2\} + \Delta_c\phi_1(t)\{\|X\| + \|X\|^2\} + \Delta_2\theta_1(t)\{\|Y\| + \|Y\|^2\}.\end{aligned}$$

Now, suppose $K_{11} = \max\{\Delta_c, \Delta_2\}$ and $0 \leq \alpha_1 = \max\{\theta_1(t), \phi_1(t)\}$. Then, we obtain

$$\dot{V}|_{(1.1)} \leq -K_3\{\| X \|^2 + \| Y \|^2\} + \alpha_1K_{11}\{\|X\| + \|Y\|\} + K_{11}\alpha_1\{\|X\|^2 + \|Y\|^2\}.$$

On using the fact that

$$\{\|X\| + \|Y\|\} \leq 2^{\frac{1}{2}}\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}},$$

we have

$$\dot{V}|_{(1.1)} \leq -\{K_3 - K_{11}\alpha_1\}\{\|X\|^2 + \|Y\|^2\} + 2^{\frac{1}{2}}K_{11}\alpha_1\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}.$$

By letting $\alpha_2 = \frac{1}{2}(K_3 - K_{11}\alpha_1) > 0$, $\alpha_1 < K_3K_{11}^{-1}$ and $\alpha_3 = 2^{\frac{1}{2}}\alpha_1 K_{11}$, we have

$$\dot{V}|_{(1.1)} \leq -2\alpha_2\{\|X\|^2 + \|Y\|^2\} + \alpha_3\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}. \tag{4.13}$$

If we choose $(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}} \geq \alpha_4 = 2\alpha_3\alpha_2^{-1}$, then the inequality (4.13) implies that

$$\dot{V}|_{(1.1)} \leq -\alpha_2\{\|X\|^2 + \|Y\|^2\}. \tag{4.14}$$

Then, if we choose $(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}} \geq \max\{\alpha_2^{-\frac{1}{2}}, \alpha_4\}$ in (4.14), we obtain,

$$\dot{V}|_{(1.1)} \leq -1.$$

The conclusion of the proof of Theorem (3.4) follows exactly the Yoshizawa techniques employed in [19] or Meng [12]. Following the approach used in [12] or [19], we can establish that for any solution $(X(t), Y(t))$ of the system (1.1), we ultimately have

$$\|X(t)\|^2 + \|Y(t)\|^2 \leq K_{12},$$

for some positive constant K_{12}. This means that, for any solution $(X(t), Y(t))$ of system (1.1), we cannot have

$$\|X(t)\|^2 + \|Y(t)\|^2 \geq \alpha_4^2, \tag{4.15}$$

for all $t \geq 0$. But suppose on the contrary that (4.15) was true for all $t \geq 0$. Then, by (4.14), we should have

$$\dot{V}|_{(1.1)} \leq -\alpha_2\alpha_4^2 < 0 \text{ for all } t \geq 0,$$

which clearly means that $V(X(t), Y(t)) \to -\infty$ as $t \to \infty$. This contradicts the conclusion of Lemma (4.2) that V is non-negative. Thus, there exists a $t_1 \geq 0$ such that

$$\|X(t_1)\|^2 + \|Y(t_1)\|^2 < \alpha_4^2. \tag{4.16}$$

In view of the conclusion of Lemma (4.1), there exists a constant $\alpha_5 > \alpha_4$ such that

$$\max_{\|X\|^2+\|Y\|^2=\alpha_4^2} V(X, Y) < \min_{\|X\|^2+\|Y\|^2=\alpha_5^2} V(X, Y). \tag{4.17}$$

Then, it will be proven that any solution $(X(t),\ Y(t))$ of (1.1) satisfying (4.16) must necessarily satisfy

$$\|X\|^2 + \|Y\|^2 < \alpha_5^2, \text{ for } t \geq t_1, \tag{4.18}$$

thereby validating our claim.

Let's assume that (4.18) is not true. Then in view of (4.16) there exist t_2 and t_3, $t_1 < t_2 < t_3$, such that

$$\|X(t_2)\|^2 + \|Y(t_2)\|^2 = \alpha_4^2 \tag{4.19}$$

$$\|X(t_3)\|^2 + \|Y(t_3)\|^2 = \alpha_5^2 \tag{4.20}$$

and such that

$$\alpha_4^2 \leq (\|X(t)\|^2 + \|Y(t)\|^2) \leq \alpha_5^2 \tag{4.21}$$

for $t_2 \leq t \leq t_3$. By (4.14), inequality (4.21) implies that $V(t_2) > V(t_3)$ and this contradicts the claim that $V(t_2) < V(t_3)$ $(t_2 < t_3)$ which is obtained from (4.17), (4.19) and (4.20). Hence, any solution $(X(t), Y(t))$ of (1.1) must satisfy (4.18). This completes the proof of Theorem (3.4)

5. **Formulation of Main Results for system** (1.2)

The main results for system (1.2) are given below.

Theorem 5.1. *Let $J_f(Y)$, $J_g(Y)$ denote the Jacobian matrices $\frac{\partial f_i}{\partial y_j}$, $\frac{\partial g_i}{\partial y_j}$ of $F(Y)$ and $G(Y)$ respectively, $(i, j = 1, 2, ..., n)$. Suppose further that:*

(i) *the matrices A, $J_f(Y)$, $J_g(Y)$ are all symmetric and negative definite, while, C is symmetric and positive definite such that for some positive constants $\delta_c, \delta_f, \delta_5, \delta_6, \Delta_c, \Delta_f, \Delta_5$ and Δ_6, we have*
$$\begin{aligned} \delta_c \leq \lambda_i(C) &\leq \Delta_c, \\ \delta_f \leq \lambda_i(-J_f(Y)) &\leq \Delta_f, \\ -\Delta_5 \leq \lambda_i(AC) &\leq -\delta_5, \\ -\Delta_6 \leq \lambda_i(-J_f(Y)J_g(Y)) &\leq -\delta_6, \end{aligned}$$
$(i = 1, 2, \ldots, n)$.

(ii) *the matrix A commutes with matrix C, while matrix $J_f(Y)$ commutes with matrix $J_g(Y)$,*

(iii) *$P_1(t, X, Y) = 0$ and $P_2(t, X, Y) = 0$.*

Then, the trivial solution of system (1.2) *is asymptotically stable.*

Theorem 5.2. *Let the assumption (iii) of Theorem* (5.1) *be replaced by*

(iii) *$\|P_1(t, X, Y)\| \leq \phi_2(t)$, $\|P_2(t, X, Y)\| \leq \theta_2(t)$ for all $t \geq 0$, $\max \phi_2(t) < \infty$, $\max \theta_2(t) < \infty$ and $\phi_2(t)$, $\theta_2(t) \in L^1(0, \infty)$, where $L^1(0, \infty)$ is the space of integrable Lebesgue functions.*

Then, solutions of system (1.2) *are bounded.*

Theorem 5.3. *Further to the assumptions (i), (ii) of Theorem* (5.1), *let*

(vi) $\|P_1(t,X,Y)\| \leq \phi_3(t)\{1+\|X\|\}$, $\|P_2(t,X,Y)\| \leq \theta_3(t)\{1+\|Y\|\}$ *for all* $t \geq 0$, $\max \theta_3(t) < \infty$, $\max \phi_3(t) < \infty$ *and* $\theta_3(t)$, $\phi_3(t) \in L^1(0,\infty)$, *where* $L^1(0,\infty)$ *is the space of integrable Lebesgue functions.*

Then, any solution $(X(t), Y(t))$ *of system* (1.2) *with the initial condition*

$$X(0) = X_0, Y(0) = Y_0$$

satisfies

$$\|X(t)\| \leq D, \ \|Y(t)\| \leq D$$

for all $t \geq 0$ *where* $D > 0$ *depends on* A, C, $\phi_3(t), \theta_3(t), t_0$, X_0, Y_0, *and on the functions* $P_1(t,X,Y)$ *and* $P_2(t,X,Y)$.

Theorem 5.4. *Under the assumptions of Theorem* (5.2) *or Theorem* (5.3), *all the solutions of system* (1.2) *are uniform-ultimately bounded.*

6. **Proof of Main Results on system** (1.2)

The main tool in the proofs of our theorems is the Lypunov function $V = V(X,Y)$ defined by

$$2V(X,Y) = \langle X, CX\rangle - 2\int_0^1 \langle F(sY), Y\rangle ds, \tag{6.1}$$

where C is as defined in Theorem (5.1).

Lemma 6.1. *Suppose, under the assumptions of Theorem* (5.1) *there exist constants* D_1 *and* D_2 *both positive such that the function* V *defined by equation* (6.1), *satisfies*

$$D_1\{\|X\|^2 + \|Y\|^2\} \leq 2V(X,Y) \leq D_2\{\|X\|^2 + \|Y\|^2\}$$

and

$$V(X,Y) \to +\infty \ \text{as} \ \|X\|^2 + \|Y\|^2 \to \infty.$$

Furthermore, there exists a positive constant D_3 *such that for any solution* (X,Y) *of* (1.2), *we have*

$$\dot{V} \leq -D_3\{\|X\|^2 + \|Y\|^2\},$$

for all $t \geq 0, X(t), Y(t)) \in \mathcal{R}^n$.

Proof. Clearly, for $X(t) = Y(t) = 0$, $t \geq 0$, $V(X,Y) = 0$. Using Lemma (2.10) in (6.1), we have

$$2V(t,X,Y) = \langle X, CX\rangle - 2\int_0^1 \langle F(sY), Y\rangle ds$$

$$= \langle X, CX\rangle - 2\int_0^1 \int_0^1 \langle J_f(s_1 s_2 Y)Y, Y\rangle s_1 ds_1 ds_2.$$

On applying assumptions of Theorem (5.1) and Lemma (2.9), we have

$$\langle -J_f(s_1s_2Y)Y, Y\rangle \geq \delta_f\|Y\|^2$$

and

$$\langle X, CX\rangle \geq \delta_c\|X\|^2.$$

Thus,

$$\begin{aligned}\int_0^1\int_0^1 \langle -J_f(s_1s_2Y)Y, Y\rangle s_1 ds_1 ds_2 &\geq \delta_f\|Y\|^2\int_0^1\int_0^1 s_1 ds_1 ds_2\\ &= \frac{1}{2}\delta_f\|Y\|^2\end{aligned}$$

and hence,

$$2V(X,Y) \geq \delta_c\|X\|^2 + \delta_f\|Y\|^2.$$

Thus, for some constant $D_1 = \min\{\delta_f, \delta_c\}$, we obtain

$$2V(X,Y) \geq D_1(\|X\|^2 + \|Y\|^2) \tag{6.2}$$

for all $t \geq 0, X, Y \in \mathcal{R}^n$. It then follows from (6.2) that $V(X,Y) = 0$ if and only if $\|X\|^2 + \|Y\|^2 = 0$ and $V(X,Y) > 0$ if and only if $\|X\|^2 + \|Y\|^2 \neq 0$, which now implies that

$$V(X,Y) \to \infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty.$$

Similarly, by the assumptions of Theorem (5.1) and Lemma (2.9), we have

$$\langle -J_f(s_1s_2Y)Y, Y\rangle \leq \Delta_f\|Y\|^2$$

and

$$\langle X, CX\rangle \leq \Delta_c\|X\|^2.$$

Therefore,

$$\begin{aligned}\int_0^1\int_0^1 \langle -J_f(s_1s_2Y)Y, Y\rangle s_1 ds_1 ds_2 &\leq \Delta_f\|Y\|^2\int_0^1\int_0^1 s_1 ds_1 ds_2,\\ &= \frac{1}{2}\Delta_f\|Y\|^2.\end{aligned}$$

Hence,

$$2V(X,Y) \leq \Delta_f\|Y\|^2 + \Delta_c\|X\|^2.$$

Thus, for some constant $D_2 = \max\{\Delta_f, \Delta_c\}$, we have

$$2V(X,Y) \leq D_2(\|X\|^2 + \|Y\|^2)$$

for all $t \geq 0,\ X,\ Y$. Therefore,

$$D_1\{\|X\|^2 + \|Y\|^2\} \leq 2V(t) \leq D_2\{\|X\|^2 + \|Y\|^2\}.$$

We now proceed to obtain the derivative of V with respect to t along the solution path of the system (1.2) such that it satisfies

$$\dot{V}|_{(1.2)} \equiv \frac{d}{dt}V(X,Y)|_{(1.2)} \leq -D_3$$

provided that $\|X\|^2+\|Y\|^2 \leq D_4$, both D_3 and D_4 are some positive constants. From the Lyapunov function defined in (6.1), we obtain the derivative $\dot{V}|_{(1.2)}$ as

$$\dot{V}|_{(1.2)} = \langle AX, CX\rangle - \langle F(Y), G(Y)\rangle$$

$$= \langle AX, CX\rangle - \int_0^1\int_0^1 \langle J_f(s_1Y), J_g(s_2Y)Y\rangle ds_1 ds_2.$$

From the assumptions of Theorem (5.1) and Lemmas (2.9) - (2.11), we have

$$\dot{V}|_{(1.2)} \leq -\delta_5\|X\|^2 - \delta_6\|Y\|^2.$$

Thus, there exists a constant $D_3 = \min\{\delta_5, \delta_6\} > 0$ such that

$$\dot{V}|_{(1.2)} \leq -D_3\{\|X\|^2 + \|Y\|^2\}$$

for all $t \geq 0, X, Y \in \mathcal{R}^n$. This completes the proof of Lemma (6.1). □

Proof of Theorem (5.1)**.**
From the proof of Lemma (6.1), it is established that the trivial/zero solution of the system (1.2) is stable. Finally, we apply LaSalle's invariance principle to conclude the proof of the theorem as follows.
Consider the set W defined by

$$W = \{(X,Y) : \dot{V}(X,Y) = 0\}.$$

By using LaSalle's invariance principle, we observe that $(X,Y) \in W$ implies that $X = Y = 0$. Hence, this shows that the largest invariant set contained in W is $(0,0) \in W$. Therefore, we conclude that the zero solution of the system (1.2) is asymptotically stable and this completes the proof of Theorem (5.1).

Corollary 6.2. *As a corollary to Theorem* (5.1)*,*
(i) the trivial solution of system (1.2) *is uniformly stable;*
(ii) the trivial solution is uniformly asymptotically stable.

Proof of Theorem (5.2)
We have from the proof of Lemma (6.1), but now $P_i(t,X,Y) \neq 0$, $(i = 1,2)$ that

$$\dot{V}|_{(1.2)} \leq -D_3\{\|X\|^2 + \|Y\|^2\} + \langle CX, P_1(t,X,Y)\rangle - \langle F(Y), P_2(t,X,Y)\rangle.$$

But

$$\begin{aligned}-\langle F(Y), P(t,X,Y)\rangle \leq & |\langle F(Y), P(t,X,Y)\rangle| \\ \leq & \int_0^1 |\langle J_f(s_1 Y)Y, P(t,X,Y)\rangle| ds_1 \\ \leq & \Delta_f \|Y\| \|P(t,X,Y)\|\end{aligned}$$

and

$$\begin{aligned}\langle CX, P_1(t;X,Y)\rangle &\leq |\langle CX, P_1(t,X,Y)\rangle| \\ &\leq \Delta_c \|X\| \|P_1(t,X,Y)\|.\end{aligned}$$

Therefore,

$$\dot{V}|_{(1.2)} \leq \Delta_c \|X\| \|P_1(t,X,Y)\| + \Delta_f \|Y\| \|P_2(t,X,Y)\|.$$

Using the inequalities

$$\|X\| \leq 1 + \|X\|^2 \text{ and } \|Y\| \leq 1 + \|Y\|^2$$

in $\dot{V}|_{(1.2)}$, we have

$$\begin{aligned}\dot{V}|_{(1.2)} &\leq \Delta_c \phi_2(t)\{1 + \|X\|^2\} + \Delta_f \theta_2(t)\{1 + \|Y\|^2\} \\ &\leq \Delta_c \phi_2(t) + \Delta_f \theta_2(t) + \Delta_c \phi_2(t)\|X\|^2 + \Delta_f \theta_2(t)\|Y\|^2.\end{aligned}$$

From the inequality (6.2), we have the following facts,

$$\|X\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2D_1^{-1} V(X,Y)$$

and

$$\|Y\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2D_1^{-1} V(X,Y).$$

Thus,

$$\dot{V}|_{(1.2)} \leq \Delta_c \phi_2(t) + \Delta_f \theta_2(t) + 2D_1^{-1}\{\Delta_c \phi_2(t) + \Delta_f \theta_2(t)\} V(X,Y)$$

$$\dot{V}|_{(1.2)} \leq \theta_7(t) + 2D_1^{-1} \theta_7(t) V(X,Y) \tag{6.3}$$

where $\theta_7(t) = \Delta_c \phi_2(t) + \Delta_f \theta_2(t)$.

Integrating both sides of (6.3) between 0 to t, $(t > 0)$, produces

$$V(t) \leq V(0) + \int_0^t \theta_7(s) ds + 2D_1^{-1} \int_0^t \theta_7(s) V(s) ds.$$

Suppose we let

$$W_5 = V(0) + \int_0^\infty \theta_7(s) ds \text{ and } W_6 = 2D_1^{-1},$$

then,

$$V(t) \leq W_5 + W_6 \int_0^\infty V(s) \theta_7(s) ds.$$

By applying Gronwall-Bellman inequality [16], we have

$$V(t) \leq W_5 \exp(W_6 \int_0^\infty \theta_7(s) ds) \leq D_8,$$

where $D_8 > 0$ is a constant. From the estimate (6.2) and the assumptions on $\theta_2(t)$, we conclude that all solutions of (1.2) are bounded and this completes the proof of Theorem (5.2).

Corollary 6.3. *Under the assumptions of Theorem* (5.2) *all the solutions of system* (1.2) *are uniformly bounded.*

Proof of Theorem (5.3)

In proving this theorem, we follow the same pattern as in the proof of Theorem (5.2). We know from the proof of Theorem (5.2) that

$$\begin{aligned}\dot{V}|_{(1.2)} \leq &\Delta_c\|X\|\|P_1(t,X,Y)\| + \Delta_f\|Y\|\|P_2(t,X,Y)\| \\ \leq &\Delta_c\phi_3(t)\|X\|\{1+\|X\|\} + \Delta_f\theta_3(t)\|Y\|\{1+\|Y\|\} \\ \leq &\Delta_c\phi_3(t)\{\|X\|+\|X\|^2\} + \Delta_f\theta_3(t)\ \{\|Y\|+\|Y\|^2\}.\end{aligned}$$

Using the inequalities

$$\|X\| \leq 1+\|X\|^2 \text{ and } \|Y\| \leq 1+\|Y\|^2$$

in $\dot{V}|_{(1.2)}$, we have

$$\dot{V} \leq \Delta_c\phi_3(t) + \Delta_f\theta_3(t) + 2\Delta_c\phi_3(t)\|X\|^2 + 2\Delta_f\theta_3(t)\|Y\|^2.$$

From the inequality (6.2), we have the following facts,

$$\|X\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2D_1^{-1}V(X,Y)$$

and

$$\|Y\|^2 \leq \|X\|^2 + \|Y\|^2 \leq 2D_1^{-1}V(X,Y).$$

Thus,

$$\dot{V} \leq \Delta_c\phi_3(t) + \Delta_f\theta_3(t) + 2D_1^{-1}\{\Delta_c\phi_3(t) + \Delta_f\theta_3(t)\}V(X,Y)$$

$$\dot{V}|_{(1.2)} \leq \theta_7(t) + 2D_1^{-1}\theta_7(t)V(X,Y) \tag{6.4}$$

where $\theta_7(t) = \Delta_c\phi_3(t) + K_{20}\theta_3(t)$.

Integrating both sides of (6.4) between 0 to t, $(t > 0)$, produces

$$V(t) \leq V(0) + \Delta_f\int_0^t \theta_7(s)ds + 2\Delta_f D_1^{-1}\int_0^t \theta_7(s)V(s)ds.$$

Suppose we let

$$W_7 = V(0) + \Delta_f\int_0^\infty \theta_7(s)ds \text{ and } W_8 = 2\Delta_f D_1^{-1},$$

then,

$$V(t) \leq W_5 + W_6\int_0^\infty V(s)\theta_7(s)ds.$$

By applying Gronwall-Bellman inequality [16], we have

$$V(t) \leq W_5 \exp(W_6 \int_0^{\infty} \theta_7(s)ds) \leq D_9. \tag{6.5}$$

where D_9 is a positive constant. On using (6.2) in the inequality (6.5), we obtain

$$\|X\|^2 + \|Y\|^2 \leq 2D_9 D_1^{-1} = D_{10}$$

and this implies

$$\|X\|^2 \leq D_{10} \text{ and } \|Y\|^2 \leq D_{10}.$$

The proof of Theorem (5.3) is now complete.

Before we provide the proof of Theorem (5.4), the following lemma is essential.

Lemma 6.4. *Suppose that, under the assumptions of Theorem* (5.3) *there exists a positive constant,* D_{11} *such that for any solution* (X, Y) *of the system* (1.2)*, the function* V *defined by equation* (6.1)*, satisfies*

$$\begin{aligned}\dot{V}|_{(1.2)} \leq & - D_4\{\|X(t)\|^2 + \|Y(t)\|^2\} + \Delta_c(1 + \|X\|^2)\|P_1(t, X, Y)\| \\ & + D_{11}(1 + \|Y\|^2)\|P_2(t, X, Y)\|\end{aligned}$$

for all $t \geq 0, X(t), Y(t) \in \mathcal{R}^n$.

Proof. If we follow the same argument as in the proof of Lemma (6.1), but in this case, $P_i(t, X, Y) \neq 0,\ (i = 1, 2)$, we obtain

$$\begin{aligned}\dot{V}|_{(1.2)} \leq & - D_4\{\|X\|^2 + \|Y\|^2\} + \langle CX, P_1(t, X, Y)\rangle - \langle F(Y), P_2(t, X, Y)\rangle \\ \leq & - D_4\{\| X \|^2 + \| Y \|^2\} + \Delta_c\|X\|\|P_1(t, X, Y)\| + \Delta_f\|Y\|\|P_2(t, X, Y)\|.\end{aligned}$$

By applying the inequalities

$$\|Y\| \leq 1 + \|Y\|^2 \text{ and } \|X\| \leq 1 + \|X\|^2$$

in the above, we obtain

$$\begin{aligned}\dot{V}|_{(1.2)} \leq & - D_4\{\| X \|^2 + \| Y \|^2\} + \Delta_c\{1 + \|X\|^2\}\|P_1(t, X, Y)\| \\ & + \Delta_f\{1 + \|Y\|^2\}\|P_2(t, X, Y)\| \\ = & - D_4\{\| X \|^2 + \| Y \|^2\} + \Delta_c\{1 + \|X\|^2\}\|P_1(t, X, Y)\| \\ & + D_{11}\{1 + \|Y\|^2\}\|P_2(t, X, Y)\|,\end{aligned}$$

where $D_{11} = \Delta_f$, for all $t \geq 0, X, Y \in \mathcal{R}^n$.
This completes the proof of Lemma (6.4).

□

Proof of Theorem (5.4)
From Lemma (6.4), we have that the derivative $\dot{V}$ of the function V defined in (6.1), satisfied

$$\dot{V}|_{(1.2)} \leq -D_4\{\| X \|^2 + \| Y \|^2\} + D_{11}\|Y\|\|P_2(t,X,Y)\| + \Delta_c\|X\|\|P_1(t,X,Y)\|.$$

From the hypothesis (vi) of Theorem (5.4), we obtain,

$$\dot{V}|_{(1.2)} \leq -D_4\{\| X \|^2 + \| Y \|^2\} + D_{11}\theta_3(t)\|Y\|\{1+\|Y\|\} + \Delta_c\phi_3(t)\|X\|\{1+\|X\|\},$$
$$\leq -D_4\{\| X \|^2 + \| Y \|^2\} + \Delta_c\phi_3(t)\{\|X\| + \|X\|^2\} + D_{11}\theta_3(t)\{\|Y\| + \|Y\|^2\}.$$

Now, suppose $D_{12} = \max\{\Delta_c; D_{11}\}$ and $0 \leq \alpha_5 = \max\{\theta_3(t); \phi_3(t)\}$. Then, we obtain

$$\dot{V}|_{(1.2)} \leq -D_4\{\| X \|^2 + \| Y \|^2\} + \alpha_5 D_{12}\{\|X\| + \|Y\|\} + D_{12}\alpha_5\{\|X\|^2 + \|Y\|^2\}.$$

Using the following inequalities in the above

$$\{\|X\| + \|Y\|\} \leq 2^{\frac{1}{2}}\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}},$$

we have,

$$\dot{V}|_{(1.2)} \leq -\{D_4 - D_{12}\alpha_5\}\{\|X\|^2 + \|Y\|^2\} + 2^{\frac{1}{2}}D_{12}\alpha_5\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}.$$

Taking $\alpha_6 = \frac{1}{2}(D_4 - D_{12}\alpha_5)$, $\alpha_5 < D_4 D_{12}^{-1}$ and $\alpha_7 = 2^{\frac{1}{2}}\alpha_5 D_{12}$, we have

$$\dot{V}|_{(1.2)} \leq -2\alpha_6\{\|X\|^2 + \|Y\|^2\} + \alpha_7\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}. \tag{6.6}$$

If we choose $(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}} \geq \alpha_8 = 2\alpha_7\alpha_6^{-1}$, then the inequality (6.6) implies that

$$\dot{V}|_{(1.2)} \leq -\alpha_6\{\|X\|^2 + \|Y\|^2\}. \tag{6.7}$$

Suppose we take $(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}} \geq \max\{\alpha_6^{-\frac{1}{2}}, \alpha_8\}$, then (6.7) becomes

$$\dot{V}|_{(1.2)} \leq -1.$$

The conclusion of the proof of the theorem follows exactly as in ([12],[19]) or the proof of our Theorem (5.4).

7. Examples

We provide in this section, two examples to show the correctness of our main results.

Example 7.1. First, let $P_1(t,X,Y) \equiv 0, P_2(t,X,Y) \equiv 0$ in (1.1) and $n = 2$ such that

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \dot{Y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix},$$

$$F(X) = \begin{pmatrix} \tan^{-1} x_1 - 1.01x_1 \\ -0.1x_2 \end{pmatrix}, G(Y) = \begin{pmatrix} \sin y_1 - 2y_1 \\ \sin y_2 - 2y_2 \end{pmatrix}, B = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1.1 \end{pmatrix}.$$

From the above, the following systems of first order differential equations are obtained.

$$\begin{aligned}
\dot{x_1} &= \tan^{-1} x_1 - 1.01x_1 - 2y_1, \\
\dot{x_2} &= -0.1x_2 - y_2, \\
\dot{y_1} &= x_1 + \sin y_1 - 2y_1, \\
\dot{y_2} &= 1.1x_2 + \sin y_2 - 2y_2.
\end{aligned}$$

The Jacobian matrices of vectors $F(X)$ and $G(Y)$ are respectively

$$J_f(X) = \begin{pmatrix} \frac{1}{1+x_1^2} - 1.01 & 0 \\ 0 & -0.1 \end{pmatrix} \text{ and } J_g(Y) = \begin{pmatrix} \cos y_1 - 2 & 0 \\ 0 & \cos y_2 - 2 \end{pmatrix}.$$

Also, the product matrices $J_f(X)J_g(Y) = J_g(Y)J_f(X)$ and $BC = CB$ are

$$J_f(X)J_g(Y) = \begin{pmatrix} \frac{\cos y_1 - 2}{1+x_1^2} - 1.01\cos y_1 + 2.02 & 0 \\ 0 & -0.1\cos y_2 + 0.2 \end{pmatrix} = J_g(Y)J_f(X),$$

and

$$BC = \begin{pmatrix} -2 & 0 \\ 0 & -1.1 \end{pmatrix} = CB.$$

It easy to show by some elementary calculations that the eigenvalues of matrices $B, C, J_f(X)$, and $J_g(Y)$ satisfy:

$$\delta_b = -2 \leq \lambda_i(B) \leq \Delta_b = -1,$$

$$\delta_c = 1 \leq \lambda_i(C) \leq \Delta_c = 1.1,$$

$$\delta_f = -1.01 \leq \lambda_i(J_f(X)) \leq \Delta_f = -0.01,$$

$$\delta_g = -3 \leq \lambda_i(J_g(Y)) \leq \Delta_g = -1.$$

Therefore, matrices $B, J_f(X), J_g$ are symmetric and negative definite while matrix C is symmetric and positive definite. Hence, all the conditions of Theorem (3.1) hold.

Example 7.2. In addition to Example (7.1), let,

$$P_1(t, X, Y) = \frac{1}{[t^2 + (x_1 + x_2)^2 + (y_1 + y_2)^2 + 1]^2}$$

and

$$P_2(t, X, Y) = \frac{2}{[e^t + \sin^2(x_1 + x_2) + \sin^2(y_1 + y_2) + 1]^2}.$$

Then,

$$\begin{aligned}\|P_1(t,X,Y)\| &= \frac{1}{t^2+(x_1+x_2)^2+(y_1+y_2)^2+1},\\ &\leq \frac{1}{(t^2+1)},\\ &= \phi(t)\\ &\leq 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\|P_2(t,X,Y)\| &= \frac{\sqrt{2}}{e^t+\sin^2(x_1+x_2)+\sin^2(y_1+y_2)+1},\\ &\leq \frac{\sqrt{2}}{(e^t+1)},\\ &= \theta(t),\\ &\leq \sqrt{2}.\end{aligned}$$

Also, all the conditions of Theorem (3.2) are satisfied by this example.

REFERENCES

[1] A. A. Adeyanju and C. Tunc, Uniform-ultimate boundedness, stability and periodicity of solutions to certain differential equation of third order, *Montes Taurus J. Pure Appl. Math.*, 6(**2**): 1–14, 2024.

[2] A. A. Adeyanju, M. O. Omeike, J. O. Adeniran, and B. S. Badmus, Stability and boundedness of solutions of certain Aizermann differential equations, *Acta Universitatis Apulensis,* 72:131–153,2022. doi: 10.17114/j.aua.2022.72.09.

[3] A. A. Adeyanju, M. O. Omeike, J. O. Adeniran, and B. S. Badmus, On the stability and boundedness of solutions of Aizermann vector differential equations, *Journal of the Nigerian Mathematical Society*, 42(**2**), 169–179, 2023.

[4] A. A. Adeyanju, Some Qualitative Properties of Solutions of Certain Aizermann Differential Equations, *Ph.D. Thesis, Federal University of Agriculture, Abeokuta,* 2021.

[5] A. U. Afuwape and M. O. Omeike, Convergence of solutions of certain system of third order nonlinear ordinary differential equations, *Annals of differential equations*, 21(*4*), 533–540, 2005.

[6] J. O. C. Ezeilo and H. O. Tejumola, Further results for a system of third-order differential equations, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e. Naturali. Rendiconti Lincei*, 58: 143–151, 1975.

[7] J. O. C. Ezeilo, On the stability of solutions of certain systems of ordinary differential equations, *Annali di Matematica Pura ed Applicata*, 73, 17–26, 1966.

[8] J. O. C. Ezeilo and H. O. Tejumola, Boundedness and periodicity of solutions of a certain system of third-order nonlinear differential equations, *Annali di Matematica Pura ed Applicata*, 74:, 283-316, 1966.

[9] J. O. C. Ezeilo, Stability results for the solutions of some third and fourth order differential equations, *Annali di Matematica Pura ed Applicata*, 66: 233–249, 1964.

[10] J. O. C. Ezeilo, An elementary proof of a boundedness theorem for a certain third order differential equation, *Journal of London Mathematical Society* 38: 11–16, 1963.

[11] N. N. Krasovskii, On stability of solutions of systems of two differential equations, *Prikl. Mat. Mekh.,* 17(**6**): 651–672, 1953.

[12] F. W. Meng, Ultimate boundedness results for a certain system of third-order nonlinear differential equations, *Journal of Mathematics and Applications* 177: 496–509, 1993.

[13] I. H. Mufti, Stability in the Large of Autonomous Systems of two differential equations, *Archive for Rational Mechanics and Analysis*, 6:115–132, 1960.

[14] I. H. Mufti, Stability in the Large of Systems of two equations, *Archive for Rational Mechanics and Analysis*, 7:119–134, 1961.

[15] M. O. Omeike, O. O. Oyetunde, and A. L. Olutimo, Boundedness of solutions of certain system of second-order ordinary differential equations, *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, 53:107-115, 2014.

[16] M. R. M. Rao, Ordinary Differential Equations, *Affiliated East West Private Limited, London*, 1980.

[17] C. Tunç, On the stability and boundedness of solutions of nonlinear vector differential equations of third-order, *Nonlinear Analysis*, 70: 2232-2236, 2009.

[18] C. Tunç, S. A. Mohammed, On the qualitative properties of differential equations of third order with retarded argument, *Proyecciones*, 66: 325-347, 2014.

[19] T. Yoshizawa, Asymptotic behaviour of solutions of non-autonomous system near sets, *Journal of Mathematics. Kyoto University*, 1: 303-323, 1962.

[20] T. Yoshizawa, Stability theory by Liapunov's second method, *The Mathematical Society of Japan*, 1966.